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ZEROS OF PADÉ APPROXIMANTS FOR SOME CLASSES OF FUNCTIONS

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Abstract. In the present paper, we deal with functions $f(z) := \sum_{n=0}^{\infty} a_n z^n$ whose coefficients satisfy a special smoothness condition. Theorems concerning the asymptotic behaviour as $n \to \infty$, m - fixed, of the normalized in an appropriate way Padé approximants $\pi_{n,m}$ are provided. As a consequence, results concerning the limiting distribution of the zeros are deduced.

Let

$$f(z) := \sum_{j=0}^{\infty} a_j z^j \tag{1}$$

be a function with $a_j \neq 0$ for all nonnegative integers j ($j \in \mathbb{N}$) large enough. We set

$$\eta_j := a_{j+1} \cdot a_{j-1}/a_j^2, \qquad j = j_0, j_1, \dots .$$

The basic assumption throughout the present work is that

$$\eta_j \to 1, \text{ as } j \to \infty. \tag{2}$$

This kind of asymptotic behaviour of the Maclaurin coefficients has been introduced and studied by D. Lubinsky in [4]. More precisely, he considers a large class of functions for which the number 1 in (2) is replaced by a number $\eta, \eta \neq \infty$. In [1] theorems resulting from this smoothness condition with respect to Toeplitz determinants and the uniform convergence of the row in the table of classical Padé approximants are proved. Therefore, in what follows condition (2) will be called "Lubinsky's smoothness condition for $\eta = 1$".

Further, we assume that the numbers η_j tend to 1 in a prescribed "smooth way", namely there exist complex numbers $\{c_i\}_{i=1}^{\infty}$ with $c_1 \neq 0$ such that for each positive

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integer $N, N > 1$, the representation

$$\eta_n = 1 + c_1/n + \sum_{i=2}^{N} c_i/n^i + o(n^{-N}). \tag{3}$$

holds. Important functions which satisfy Lubinsky's smoothness condition for $\eta = 1$ and to which the presented considerations in this paper may be applied are the exponential function (see [6])

$$f(z) = \exp z = \sum_{j=0}^{\infty} z^j/j!$$

and the Mittag-Leffler function of order $\lambda, \lambda > 0$, (see [4])

$$f(z) = \sum_{j=0}^{\infty} z^j/\Gamma(1 + j/\lambda), \qquad \lambda > 0.$$

Let now m be a fixed positive integer. In our further considerations, we shall assume that f is holomorphic at the zero (in a neighbourhood) and is not a rational function having less or equal than m finite poles (multiplicities included) in $\mathbb{C}$ (we write $f \notin \mathcal{R}_m$).

For each $n, n \in \mathbb{N}$, let $\pi_{n,m}(= \pi_{n,m}(f))$ be the Padé approximant to the function f of order (n, m). Recall that $\pi_{n,m} = p/q, \deg p \leq n, \deg q \leq m, q \not\equiv 0$, where the polynomials p and q are determined by the condition $(f \cdot q - p)(z) = O(z^{n+m+1})$. For each pair (n, m) the function $\pi_{n,m}$ always exists and is uniquely determined (see, for example, [5]). We set

$$\pi_{n,m} = P_{n,m}/Q_{n,m},$$

where $Q_{n,m}(0) = 1$ and both polynomials $P_{n,m}$ and $Q_{n,m}$ do not have a common divisor.

Let $D(n, m) = det\{a_{n-j+k}\}_{j,k=1}^{m}$ be the Toeplitz determinant formed from the Maclaurin coefficients of the function f. From the nonrationality of f, it follows that the sequence Λ of those positive integers n for which $D(n, m) \cdot D(n, m + 1) \neq 0$, is infinite (see, [5], [1]) and the equality $\pi_{n,m} \equiv \pi_{k(n),m}$, where $k(n) := \max\{k, k \leq n, k \in \Lambda\}$ is valid. Without losing the generality we shall assume that $\Lambda \equiv \mathbb{N}$. In this case there holds (see [1])

$$Q_{n,m}(z) = 1 + \ldots + z^m \cdot (-1)^m D(n + 1, m)/D(n, m)$$

and

$$P_{n,m}(z) = z^m \cdot D(n, m + 1)/D(n, m) + \ldots + d_{n,m}.$$

Denote by $R_{n,m}(u)$ the numerator of the rational Padé function associated with f and normalized as follows:

$$R_{n,m}(u) := \frac{Q_{n,m}(ua_n/a_{n+1})}{(ua_n/a_{n+1})^n \cdot D(n, m + 1)/D(n, m)}.$$

In [3], theorems concerning the asymptotic behaviour as $n \to \infty$ of the sequence $R_{n,m}(u)$ in the case when the numbers η_n satisfy Lubinsky's smoothness condition for an arbitrary number $\eta, \eta \neq \infty$ are proved.

In the present paper, we confine ourselves at the case when (3) holds. Of basic importance for the forthcoming considerations is that (see [4])

$$Q_{n,m}(ua_n/a_{n+1}) \to \ (1-u)^m \text{ as } n \to \infty$$

In [3], the following theorem is established:

THEOREM 1. *Let $m \in \mathbb{N}$ be fixed and $f \notin \mathcal{R}_m$. Assume that $a_j \neq 0$ for j large enough; assume, further that η_n admits the expansion* (3) *with $\eta = 1$, $c_1 \neq O$ and $|\eta_n| \leq 1$ for all $n \in \mathbb{N}$ sufficiently large.*

Then

$$R_{n,m}(u) \to \frac{u^m}{(u-1)^{m+1}} \text{ as } n \to \infty$$

uniformly inside $\{u, |u| > 1\}$.

As usual, "uniformly inside" means an uniform convergence on compact subsets in the metric of Chebyshev.

From Theorem 1, we have

COROLLARY 1 (see [3]). *With the assumptions of Theorem* 1, *for each fixed $m \in \mathbb{N}$ and any positive ε, the Padé approximant $\pi_{n,m}(z)$ has no zeros in $|z| > |a_n/a_{n+1}| \cdot (1+\varepsilon)$ for n sufficiently large.*

The next result provides more precise information concerning the behaviour of the zeros of the sequence of the normalized Padé approximant $R_{n,m}(u)$ as $n \to \infty$ for the special case when the first coefficient c_1 in (3) is a real negative number.

THEOREM 2 (see [3]). *If $c_1 < 0$, then $u = 1$ is a limit point of zeros of $\{R_{n,m}(u)\}_{n=1}^{\infty}$.*

Set $\mathcal{A}_n(\varepsilon) := \{z, (1-\varepsilon)|a_n/a_{n+1}| \leq |z| \leq (1+\varepsilon)|a_n/a_{n+1}|\}$.

Combining Theorem 2 and Theorem 1, we come to

COROLLARY 2 (see [3]). *In the conditions of Theorem* 2, *for each fixed $m \in \mathbb{N}$,any $\varepsilon, 0 < \varepsilon < 1$ and n large enough the Padé approximant $\pi_{n,m}(z)$ has at least one zero in the annulus $\mathcal{A}_n(\varepsilon)$.*

For $n \in \mathbb{N}$, we denote by P_n the set of the zeros of $R_{n,m}$. Set $P_n := \{\xi_{n,k}\}_{k=1}^{n}$ with the normalization $|1-\xi_{n,k}| \leq |1-\xi_{n,k+1}|, k = 1, \ldots, n-1$. From Theorem 2, we have

$$\text{dist}(P_n, 1) \to 0, \text{ as } n \to \infty.$$

For any positive ε, denote by $\iota_n(\varepsilon)$ the number of the zeros of $\xi_{n,k}$ which lie in the disk of radius ε and centered at $u = 1$. In the present paper we prove

THEOREM 3. *In the conditions of Theorem* 2, *for any ε small enough, we have*

$$\liminf_{n\to\infty} \frac{\iota_n(\varepsilon)}{n} > 0. \tag{4}$$

From here, we have

COROLLARY 3. *In the conditions of Theorem* 2, *for each fixed $m \in \mathbb{N}$,any $\varepsilon, 0 < \varepsilon < 1$ and n large enough the Padé approximant $\pi_{n,m}$ has at least ι_n zeros in the annulus $\mathcal{A}_n(\varepsilon)$, where the numbers ι_n fulfill, as $n \to \infty$, condition* (4).

The structure of the paper is as follows. First, for the sake of perfection, we state the general idea of the proof of Theorem 2; then we provide the proof of Theorem 3.

The basis of all the forthcoming considerations is

LEMMA 1 (see [2]). *In the conditions of Theorem* 2, *for any* n, *it is valid*:

$$R_{n,m}(u) = 1 + \sum_{j=1}^{n} b_{n,j} A_{n,m,j} u^{-j} \tag{5}$$

with

$$b_{n,j} := \prod_{l=1}^{j} \eta_{n-j+l}^{l} \tag{6}$$

The asymptotic behaviour of $A_{n,j,m}$ *is as follows*: *for* $j < n/3m$

$$A_{n,m,j} = \prod_{l=1}^{m} \frac{(j+l)}{m!} + \mathcal{N}_m(j,n), \ \text{ as } n \to \infty,$$

and

$$|n \cdot \mathcal{N}_m(j,n)| < C_1(m) \cdot j^{m+1} \ \text{ as } n \to \infty;$$

for $j \geq n/3m$

$$|A_{n,m,j}| \leq c_1(m) j^{m+1}$$

with $C_1(m)$ *a positive constant not depending on* n.

In what follows, we shall denote by $C(...)$ positive constants that do not depend on n.

2. Proofs of the results

Proof of Theorem 2. Recall that m is fixed and $n \to \infty$.

Arguing in the same way as in [2], we shall establish that for every δ small enough there exists a positive integer n_δ such that for any $n > n_\delta$ the inequality

$$\operatorname{Re} R_{n,m}(e^{-2\delta}) \geq C(0) \cdot e^{n \cdot \alpha(\delta)} \tag{7}$$

is valid, where

$$\alpha(\delta) := \delta^2 / 2d_1$$

and $C(0)$ is a positive constant.

For convenience, we shall use the notation $c_1 := -2d_1$. In the conditions of Theorem 2, $d_1 > 0$. In [2], for each n large enough ($n > n_0$) the inequalities

$$|\eta_n| \leq 1 - d_1/n, \tag{8}$$

and

(9). $$|\eta_n| \leq |\eta_{n+1}|$$

were established. Both latter inequalities lead to

$$|b_{n,j}| \leq (|\eta_n|)^{j(j+1)/2} \leq (1 - d_1/n)^{j(j+1)/2}. \tag{10}$$

Let ε be a fixed positive number, $\varepsilon < 1$.

In our further considerations, we assume that for $n > n_0$ the following inequalities are fulfilled:

$$(n/d_1)\,|\log(1-d_1/n)| \geq 1-\varepsilon, \tag{11}$$

and

$$|\operatorname{Im}\eta_n| \leq C\,(1)\cdot \operatorname{Re}\eta_n/n^2 \tag{12}$$

for a suitable positive constant $C\,(1)$. Without loss of generality, we may assume that $C\,(1) > 1$.

In accordance to the lemma, we may also write

$$|\mathcal{N}_{1,m}(j,n)| \leq C\,(1)\cdot j^{m+1}/n \tag{13}$$

for $j < n/3m$ and

$$|A_{n,j,m}| \leq C\,(1)\, j^{m+1}. \tag{14}$$

otherwise. Select a positive number δ_0 such that

$$0 < 6C(1)m!\delta_0/d_1(1-\varepsilon) < 1/3. \tag{15}$$

and set

$$d(\varepsilon) := d_1(1-\varepsilon).$$

In what follows, we shall assume that each $n > n_0$ satisfies the inequality

$$\operatorname{Re}\eta_n \geq 1-(2d_1+\delta_0)/n > 0. \tag{16}$$

Let δ be a positive number such that $\delta < \delta_0$. Set $\mathcal{D}(\varepsilon,\delta) := 1-6\delta/d(\varepsilon)$. Obviously, there is an integer $n_\delta, n_\delta > n_0$, such that for any $n \geq n_\delta$ the inequalities

$$\operatorname{Re}\eta_n \geq 1-(2d_1+\delta)/n \tag{17}$$

and

$$n\cdot\mathcal{D}(\varepsilon,\delta)|\log\left(1-\frac{d_1}{n\mathcal{D}(\varepsilon,\delta)}\right)| \leq 2 \tag{18}$$

are fulfilled. Set $j_1(\delta) := 6\delta/d(\varepsilon)$. In accordance to (10) and (11) we may write for $j > j_1(\delta)\cdot n$ that

$$|b_{n,j}| \leq e^{-3j\delta},$$

which, in view of (14),(15) and of the choice of δ implies the inequality

$$\|\sum_{j=j_1(\delta)n}^{n} b_{n,j}A_{n,j,m}u^{-j}\|_{|u|=e^{2\delta}} \leq C\,(\delta_0)\,e^{-\delta j_1(\delta)n} \tag{19}$$

Consider the product $\prod_{l=1}^{j}\left(\operatorname{Re}\eta_{n-j+l}\right)^l$. Applying (17), for $j+1 \leq j_1(\delta)\cdot n$ we obtain

$$\prod_{l=1}^{j}\left(\operatorname{Re}\eta_{n-j+l}\right)^l \geq \left(1-\frac{d_1}{n\mathcal{D}(\varepsilon,\delta)}\right)^{j(j+1)/2} \tag{20}$$

For the same number j we get, by (8), (9), (12) and the choice of δ the inequalities

$$|\frac{b_{n,j}}{\prod_{l=1}^{j}\left(\operatorname{Re}\eta_{n-j+l}\right)^l}-1| \leq C\,(2)\cdot\delta^2 \tag{21}$$

where $C\,(2) \leq \frac{18C(1)}{d(\varepsilon)^2}$. The choice of δ ensures that $C(2)\cdot\delta_0^2 < 1/2$.

Further, from inequality (21) we obtain

$$\left(1 - C(2)\delta^2\right)\prod_{l=1}^{j}\left(\operatorname{Re}\eta_{n-j+l}\right)^l \leq \operatorname{Re} b_{n,j} \leq \left(1 + C(2)\delta^2\right)\prod_{l=1}^{j}\left(\operatorname{Re}\eta_{n-j+l}\right)^l$$

and

$$|\operatorname{Im} b_{n,j}| \leq C(2)\delta^2\prod_{l=1}^{j}\left(\operatorname{Re}\eta_{n-j+l}\right)^l$$

Using (13), (16), (17) and the last inequalities, we get

$$\operatorname{Re} A_{n,j,m}\cdot\operatorname{Re} b_{n,j} - \operatorname{Im} A_{n,j,m}\cdot\operatorname{Im} b_{n,j} \geq Q_{\delta_0}(j)\cdot\prod_{l=1}^{j}\left(\operatorname{Re}\eta_{n-j+l}\right)^l.$$

with

$$Q_{\delta_0}(j) := (1 - C(2)\delta_0^2)\cdot\left(\prod_{l=1}^{m}(j+l)/m! - C(1)j^m\cdot\frac{n\delta_0}{d(\varepsilon)}\right) - C(2)\delta_0^2 C(1)j^m\cdot\frac{6\delta_0}{d(\varepsilon)}$$

As we see, Q_{δ_0} is a polynomial of degree exactly m and all its coefficients are positive. Further, in view of (18), (20) and of of δ_0 we may write

$$\operatorname{Re} A_{n,j,m}\cdot\operatorname{Re} b_{n,j} - \operatorname{Im} A_{n,j,m}\cdot\operatorname{Im} b_{n,j} > 0. \tag{22}$$

Recall that the last inequality is valid for $n > n_\delta$ and for any j with $j+1 < j_1(\delta)\cdot n$. Set now $j_2(\delta) := \delta\cdot\mathcal{D}(\varepsilon,\delta)$ and consider $R_{n,m,\delta}\left(e^{2\delta}\right) := \sum_{j=0}^{j_2(\delta)(n+1)-1} b_{n,j}A_{n,j,m}\left(e^{2\delta j}\right)$. In view of (20), for $j < j_2(\delta)(n+1)-1$ and for n large enough we may write

$$\prod_{l=1}^{j}\left(\operatorname{Re}\eta_{n-j+l}\right)^l > e^{-\delta j}.$$

Now, combining (19), (21), (22) and the last result, we obtain

$$\operatorname{Re} R_{n,m}\left(e^{-2\delta}\right) > \sum_{j=0}^{j_2(\delta)(n+1)-1} Q_{\delta_0}(j)e^{\delta j} - C(\delta_0)e^{-6(\delta)^2 n/d(\epsilon,\delta)}$$

Inequality (7) results from here.

Now, it easily follows that the point $u = 1$ attracts, as $n \to \infty$, at least one zero of the sequence $R_{n,m}$. Before presenting the proof,we set $w = \frac{1}{u}$ and $R_{n,m}(1/w) := R_{n,m}(w)$. In accordance to (4), it is valid that

$$R_{n,m}(w) = 1 + \sum_{j=1}^{n} b_{n,j}A_{n,m,j}w^j \tag{5'}$$

Also, in view to (7), we have

$$\operatorname{Re} R_{n,m}(w)_{w=(e^{2\delta})} \geq C(0)\cdot e^{n\cdot\alpha(\delta)} \tag{7'}$$

We introduce the notation $U_a(r)$; that is a disk of radius r, centered at the point a; further, we set $\Gamma_a(r) := \partial U_a(r)$.

We prove Theorem 2 on arguing the contrary. Suppose that $w = 1$ is not a limit point of zeros of the sequence $\{R_{n,m}(w)\}$, as $n \to \infty$; then there is a disk $U_1(e^{-\rho})$ such that $R_{n,m}(w) \neq 0$ for some subsequence $M \subset \mathbb{N}$ there. Set $\tau := \log(1 + e^{-\rho})$ and $V := U_0(1) \bigcup U_1(e^{-\rho})$ Let $X_n(w), n \in M$ be the regular branch of $(R_{n,m})^{1/n}$ determined by the condition $X_n(0) = 1$. Select now a positive number r with $r < 1 - e^{-\rho}$. Obviously, the sequence $\{X_n(w)\}$ is uniformly bounded on $U_0(r)$, and hence, by the well known result of Bernstein-Walsh, inside V, as well. On the other hand, Theorem 1 ensures that $X_n(w) \to 1$, as $n \in M$ uniformly inside $U_0(1)$. Thus, by the Theorem of uniqueness for holomorphic functions, $X_n(w) \to 1$ uniformly inside V. Combining this result and (7') we come to a contradiction.

This contradiction proves Theorem 2.

Proof of Theorem 3. Preserving the notations of Theorem 2, denote now by $\xi_{n,k}, k = 1, \ldots, \iota_n$ the zeros of $R_{n,m}(w)$ in $U_1(e^{-\rho})$. By Theorem 2, $\iota_n \geq 1$. We shall show that

$$\liminf_{n\to\infty} \iota_n/n > 0. \tag{4'}$$

Select a positive number θ such that $r < 1 - e^{-\rho} \cdot e^{\theta}$. Set $\tau(\theta) := \log(1 + e^{-\rho+\theta})$. Without loss of generality we may assume that the number $\tau(\theta)/2$ satiesfies inequality (15).

Suppose to the contrary that there is an infinite sequence $\Lambda \subset \mathbb{N}$ such that

$$\lim_{n\to\infty, n\in\Lambda} \iota_n/n = 0. \tag{23}$$

Set

$$q_n(w) := \prod_{k=1}^{\iota_n} \left(1 - \frac{w}{\xi_{n,k}}\right)$$

and

$$\chi_n(w) := \left\{\frac{R_{n,m}}{q_n(w)}\right\}^{1/n}$$

with $\chi_n(0) = 1$.

Consider the sequence $\{\chi_n\}_{n\in\Lambda}$.

For $q_n(w)$ we have

$$\min_{w\in\Gamma_0(e^{\tau(\theta)})} |q_n(w)| \geq \left\{\frac{e^{-\rho}(e^{\theta} - 1)}{(1 + e^{-\rho})}\right\}^{\iota_n}$$

On the other hand, applying the well known Bernstein-Walsh lemma to $R_{n,m}(w)$, we get

$$\|R_{n,m}(w)\|_{\Gamma_0(\tau(\theta)} \leq \|R_{n,m}(w)\|_{U_0(r)} \cdot \frac{(1 + e^{-\rho+\theta})^n}{r^n}$$

Combining both last inequalities and (23), and applying Theorem 1, we easily get that the sequence $\{\chi_n\}_{n\in\Lambda}$ is uniformly bounded inside V (recall that accordingly to the geometric construction and to the choice of θ, we have $V \subset U_0(\tau(\theta))$. Further, for $w \in U_0(r)$, there holds

$$\left\{\frac{(1 + e^{-\rho} + r)}{(1 - e^{\rho})}\right\}^{\iota_n} \geq |q_n(w)| \geq \left\{\frac{(1 - e^{-\rho} - r)}{(1 + e^{\rho})}\right\}^{\iota_n}$$

Therefore, in view to Theorem 1 and to (23), we may write

$$\chi_n \to 1 \text{ as } n \to \infty,\ n \in \Lambda$$

on the disk $U_0(r)$. Then

$$\chi_n \to 1, \text{ as } n \to \infty, n \in \Lambda \tag{24}$$

uniformly inside the domain V.

Select a positive number ε_0 such that

$$\varepsilon_0 < \frac{e^{-\rho}}{4}$$

Set $\Omega(\varepsilon_0) := \bigcup_{n\in\Lambda} \bigcup_{k=1}^{\iota_n} \left\{w, |w - \xi_{n,k}| < \frac{\varepsilon_0}{\iota_n \cdot n^2}\right\}$. Obviously,

$$mes_1(\Omega(\varepsilon_0)) < \varepsilon_0 < \frac{e^{-\rho}}{4}. \tag{25}$$

Further, for $w \in U_1(e^{-\rho}) - \Omega(\varepsilon_0)$ we have

$$\left\{\frac{2e^{-\rho}}{(1-e^{-\rho})}\right\}^{\iota_n} \geq |q_n(w)|. \tag{26}$$

The choice of ε_0 and (25) imply the existence of a positive number $\delta, \delta < \tau$ such that $e^\delta \in U_1(e^{-\rho}) - \Omega(\varepsilon_0)$. Applying (7) to those numbers δ, using (26) and (23), we conclude that $\chi(e^\delta) > e^{\delta^2/8d_1}$. This inequality contradicts (24).

Consequently, (4') holds and Theorem 3 is valid.

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