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GENERALIZED HURWITZ MAPS OF THE TYPE $S \times V \to W$, ANTI-INVOLUTIONS, AND QUANTUM BRAIDED CLIFFORD ALGEBRAS

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Abstract. The notion of a J^3-triple is studied in connection with a geometrical approach to the generalized Hurwitz problem for quadratic or bilinear forms. Some properties are obtained, generalizing those derived earlier by the present authors for the Hurwitz maps $S \times V \to V$. In particular, the dependence of each scalar product involved on the symmetry or antisymmetry is discussed as well as the configurations depending on various choices of the metric tensors of scalar products of the basis elements.

Then the interrelation with quantum groups and related Clifford-type structures is indicated via anti-involutions which also play a central role in the theory of symmetric complex manifolds.

Finally, the theory is linked with a natural generalization of general linear inhomogeneous groups as quantum braided groups. This generalization is in the spirit of the theory initiated and developed by S. Majid, however, our construction differs in the interrelation between the homogeneous and inhomogeneous parts of the group. In order to study the quantum braided or-

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thogonal groups, we consider a kind of quantum geometry in the covector space. This enables us to investigate a quantum braided Clifford algebra structure related to the spinor representation of that group.

Introduction. The importance of the normed maps $S \times V \to W$ is, to some extent, caused by the superstring model of the fermionic and bosonic states [8, 21, 24], and the applications to stochastic mechanics of particle systems [40]. In the case of $W = V$ and the coincidence of the corresponding metric tensors of scalar products of the basic elements defined below in (7), we consider again the Hurwitz problem [22, 23, 45-47, 49, 52, 1-5], and the concept of a Hurwitz pair [34-39]. Our present approach is, in some sense, related to the original approach of the famous Hurwitz's papers [15, 16]. For (hyper)complex-analytical aspects, we refer to [13].

We begin by introducing a concept of a J^3-triple [6, 24]. Then we discuss various variants of a generalized Hurwitz conditions like

$$(1)\qquad (a,b)_S(x,y)_V = \frac{1}{2}[(a\cdot x, b\cdot y)_W + \epsilon_1(b\cdot x, a\cdot y)_W]$$
$$\text{for } x,y\in V \text{ and } a,b\in S,\ \ \epsilon_1 = 1 \ \text{ or } \ -1 \ \ (\textit{the } J^3\textit{-condition}).$$

In particular we are interested in admissible triples of numbers $\epsilon_j = 1$ or -1, $j = 1,2,3$, such that

$$(2)\qquad (b,a)_S = \epsilon_1(a,b)_S,\quad (y,x)_V = \epsilon_2(x,y)_V,\quad (Y,X)_W = \epsilon_3(X,Y)_W,$$

and prove a simple but important lemma stating that $\epsilon_1\epsilon_2 = \epsilon_3$. The above considerations make it necessary to distinguish the reducible and irreducible J^3-triples. This distinction enables us to formulate and prove a reduction theorem which gives a method of constructing J^3-triples and determines a relationship with Hurwitz pairs.

1. The J^3-triples. Consider three finite-dimensional real vector spaces S, V, W of dimension p, n, and N, respectively, equipped with non-degenerate real scalar products $(\ ,\ )_S$, $(\ ,\ )_V$, $(\ ,\ )_W$, which are pseudo-euclidean or symplectic according to (2). Precisely,

$$(3)\qquad (a,b)_S\in\mathbb{R},\quad (b,a)_S = \varepsilon_1(a,b)_S \quad\text{with}\quad \varepsilon_1 = 1 \ \text{ or } \ -1,\ (\gamma a,b)_S = \gamma(a,b)_S$$
$$\text{and}\quad (a,b+c)_S = (a,b)_S + (a,c)_S \quad\text{for}\quad a,b,c\in S;\quad \gamma\in\mathbb{R};$$

$$(4)\qquad (x,y)_V\in\mathbb{R},\quad (y,x)_V = \varepsilon_2(a,b)_V \quad\text{with}\quad \varepsilon_2 = 1 \ \text{ or } \ -1,\ (\gamma x,y)_V = \gamma(x,y)_V$$
$$\text{and}\quad (x,y+z)_V = (x,y)_V + (x,z)_V \quad\text{for}\quad x,y,z\in V;\quad \gamma\in\mathbb{R};$$

$$(5)\qquad (X,Y)_W\in\mathbb{R},\ (Y,X)_W = \varepsilon_3(X,Y)_W \quad\text{with}\quad \varepsilon_3 = 1 \ \text{ or } \ -1,\ (\gamma X,Y)_W = \gamma(X,Y)_W$$
$$\text{and}\quad (X,Y+Z)_W = (X,Y)_W + (X,Z)_W \quad\text{for}\quad X,Y,Z\in W;\quad \gamma\in\mathbb{R};$$

In S, V and W we choose the bases (ϵ_α), (e_j) and (E_A), respectively, with

$$(6)\qquad \alpha = 1,\dots,p;\quad p = \dim S;\quad \begin{array}{ll} j = 1,\dots,n; & n = \dim V;\\ A = 1,\dots,N; & N = \dim W.\end{array}$$

Hence

$$a = a^\alpha\epsilon_\alpha := \sum_\alpha a^\alpha\epsilon_\alpha,\quad x = x^j e_j,\quad X = X^A E_A,\ \text{etc.}$$

The metric tensors read:

$$\eta \equiv [\eta_{\alpha\beta}] := [(\epsilon_\alpha, \epsilon_\beta)_S], \qquad \begin{aligned} \kappa &\equiv [\kappa_{jk}] := [(e_j, e_k)_V], \\ K &\equiv [K_{AB}] := [(E_A, E_B)_W], \end{aligned} \tag{7}$$

respectively. By the postulates (3)–(5) there exist the tensors

$$\eta^{-1} \equiv [\eta^{\alpha\beta}], \quad \kappa^{-1} \equiv [\kappa^{jk}], \quad K^{-1} \equiv [K^{AB}],$$

and, if η^t denotes the transpose of η etc., we have

$$\eta^t = \epsilon_1 \eta, \quad \det \eta \neq 0; \quad \kappa^t = \epsilon_2 \kappa, \quad \det \kappa \neq 0; \quad K^t = \epsilon_3 K, \quad \det K \neq 0.$$

Remark 1. If, in particular, $\epsilon_1 = 1$, then we can choose the basis (ϵ_α) so that

$$\eta = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p \text{ times}}, -1, \ldots, -1), \quad \text{and hence} \quad \eta^{-1} = \eta.$$

In terms of the metric tensors, the scalar products read:

$$(a, b)_S = \eta_{\alpha\beta} a^\alpha b^\beta, \quad (x, y)_V = \kappa_{jk} x^j y^k, \quad (X, Y)_W = K_{AB} X^A Y^B.$$

Under a J^3-*mapping* $\cdot$ corresponding to the triple (W, V, S) we mean any bilinear mapping $S \times V \to W$ for which the condition (1) holds. This means that, besides (1), we require the properties

$$(a + b) \cdot x = a \cdot x + b \cdot x, \quad a \cdot (x + y) = a \cdot x + b \cdot y$$
$$\text{and} \quad \alpha a \cdot x = a \cdot \alpha x = \alpha(a \cdot x) \quad \text{for } x, y \in V;\ a, b \in S, \text{ and } \gamma \in \mathbb{R}. \tag{8}$$

Because of (8), the J^3-mapping is uniquely determined by the "multiplication" scheme for base vectors:

$$\varepsilon_\alpha \cdot e_j = c^A_{j\alpha} E_A \quad \text{with } \alpha, j, A \text{ as in (6)}. \tag{9}$$

The scheme (9), together with the postulates (3)–(5), yields, in particular, the following formulae for the real structure constant $c^A_{j\alpha}$:

$$c^A_{j\alpha} = (E^A, \varepsilon_\alpha e_j)_W, \quad \text{with} \quad E^A := K^{BA} E_B. \tag{10}$$

With the use of the $N \times n$-rectangular structure

$$C_\alpha := [c^A_{j\alpha}], \quad \overline{C_\alpha} := K C^t_\alpha \kappa^{-1} \equiv [c^B_{j\alpha} K_{AB} \kappa^{kj}], \quad \alpha = 1, \ldots, p, \tag{11}$$

we get

LEMMA 1. *The matrices C_α of a J^3-triple satisfy the relations*

$$C_\alpha \overline{C_\beta} + \epsilon_1 C_\beta \overline{C_\alpha} = 2\eta_{\alpha\beta} I_n, \tag{12}$$

where I_n stands for the identity $n \times n$-matrix.

Proof. We rewrite the J^3-condition (1) in the co-ordinate form, and we have

$$(a, b)_S (x, y)_V = a^\alpha b^\beta x^j y^k \eta_{\alpha\beta} \kappa_{jk}, \qquad \begin{aligned} (a \cdot x, b \cdot y)_W &= a^\alpha b^\beta x^j y^k c^A_{j\alpha} K_{AB} c^B_{k\beta}, \\ (b \cdot x, a \cdot y)_W &= b^\beta a^\alpha x^j y^k c^A_{j\beta} K_{AB} c^B_{j\alpha}, \end{aligned}$$

so

$$a^\alpha b^\beta x^j y^k (c^A_{j\alpha} K_{AB} c^B_{k\beta} + \epsilon_1 c^A_{j\beta} K_{AB} c^B_{j\alpha}) \kappa^{-1}_{jk} = a^\alpha b^\beta x^j y^k \eta_{\alpha\beta},$$

i.e.,

$$C_\alpha K C_\beta^t \kappa^{-1} + \epsilon_1 C_\beta K C_\alpha^t \kappa^{-1} = 2\eta_{\alpha\beta} I.$$

In consequence, by (11), we get (12) as desired.

Remark 2. Formula (12) is equivalent to

$$(13)\quad (a\eta b^t)(x\kappa y^t) = \frac{1}{2}[(a\cdot x)K(b\cdot y)^t + \varepsilon_1 (b\cdot x)K(a\cdot y)^t] \quad \text{for } x,y\in V \quad \text{and} \quad a,b\in S,$$

which itself is equivalent to the J^3-condition (1).

Any triple (W,V,S) equipped with a J^3-mapping will be called a J^3-*triple*. It is clear that the existence of a J^3-mapping imposes conditions excluding several cases and that is the problem we are going to deal with.

LEMMA 2. *If* $\epsilon_1\epsilon_2 = \epsilon_3 = 1$ *in* (3)-(5), *then* (1) *is equivalent to*

$$(14)\qquad (a,a)_S(x,x)_V = (a\cdot x, a\cdot x)_W \quad \text{for } x\in V \text{ and } a\in S.$$

Proof. The implication (1)$\Longrightarrow$(14) is immediate. To prove the converse we polarize (14) with respect to x getting

$$(a,a)_S(x,y)_V + (a,a)_S(y,x)_V = (a\cdot x, a\cdot y)_W + (a\cdot y, a\cdot x)_W$$

that is (cf. [35]), since $\epsilon_1 = \epsilon_3 = 1$, also

$$(15)\qquad (a,a)_S(x,y)_V = (a\cdot x, a\cdot y)_W, \quad \text{for } x,y\in V \text{ and } a\in S \ \ (\textit{the } J^2\textit{-condition}).$$

Analogously, by polarizing (14) with respect to a we get

$$(16)\qquad (a,b)_S(x,x)_V = (a\cdot x, b\cdot x)_W \quad \text{for } x,y\in V \text{ and } a,b\in S \ \ (\textit{the } J^1\textit{-condition}).$$

Indeed,

$$(17)\qquad \begin{aligned}(a+b,a+b)_S(x,y)_V &= (a\cdot x, a\cdot y)_W + (a\cdot x, b\cdot y)_W + (b\cdot x, a\cdot y)_W \\ &+ (b\cdot x, b\cdot y)_W = (a,a)_S(x,y)_V + (b,b)_S(x,y)_V + 2(a,b))_S(x,y)_V,\end{aligned}$$

and hence

$$(18)\qquad (a,b)_S(x,y)_V = \frac{1}{2}[(a\cdot x, b\cdot y)_W + (b\cdot x, a\cdot y)_W].$$

LEMMA 3. *If* $\epsilon_1 = 1$ *and* $\epsilon_2 = \epsilon_3 = -1$, *then* (1) *is equivalent to the* J^2*-condition* (15).

Proof. The implication (1)$\Longrightarrow$(15) is immediate. To prove the converse we just follow (17)-(18).

LEMMA 4. *If* $\epsilon_2 = 1$ *and* $\epsilon_3 = \epsilon_1 = -1$ *in* (3)–(5), *then* (1) *is equivalent to the* J^1*-condition* (16).

Proof. The conclusion follows from Lemma 3 by the formal interchange of the roles of V and S.

Remark 3. If the metric tensor η of S is euclidean, the J^3-mapping $S\times V\to W$ is injective with respect to V for any fixed nonzero vector a of S. In the general case the assertion holds for any fixed anisotropic $a\in S$.

For the proof, it is sufficient to observe that, by (1) applied to $x_1,x_2\in V$, we get $(a,b)_S(x_1-x_2,y)_V = (ax_1-ax_2,by)_W$.

Remark 4. The restriction of the Hurwitz mapping $S \times V \to W$ to any isotropic vector in S sends V into an isotropic subspace of W.

The remark follows directly from (1).

LEMMA 5. *If* $\epsilon_3 = 1$ *and* $\epsilon_1 = \epsilon_2 = -1$ *in* (3)–(5), *then* (1) *implies*

$$(a,b)_S(x,y)_V = \frac{1}{2}[a\cdot x, b\cdot y)_W - b\cdot x, a\cdot y)_W] \quad \textit{for } x,y\in V \quad \textit{and} \quad a,b\in S.$$

Proof. Trivial.

LEMMA 6. $\epsilon_1\epsilon_2 = \epsilon_3$.

Proof. By (1)-(2), we have

$$(a,b)_S(x,y)_V = \epsilon_1(b,a))_S\epsilon_2(y,x)_V = \frac{1}{2}\epsilon_1\epsilon_2[(b\cdot y, a\cdot x)_W + \epsilon_1(a\cdot y, b\cdot x)_W].$$

On the other hand, by (1)-(2) again,

$$\begin{aligned}(a,b)_S(x,y)_V &= \frac{1}{2}[(a\cdot x, b\cdot y)_W + \epsilon_1(b\cdot x, a\cdot y)_W] \\ &= \frac{1}{2}\epsilon_3[(b\cdot y, a\cdot x)_W + \epsilon_1(a\cdot y, b\cdot x)_W].\end{aligned}$$

Comparing the relations obtained we arrive at $\epsilon_1\epsilon_2 = \epsilon_3$.

Let us replace (1) by the condition

$$\begin{aligned}(a,b)_S(x,y)_V =& \alpha_1(a\cdot x, b\cdot y)_W + \alpha_2(a\cdot y, b\cdot x))_W + \alpha_3(b\cdot x, a\cdot y)_W \\ &+\alpha_4(b\cdot y, a\cdot x)_W \quad \text{for } x,y\in V \text{ and } a,b\in S\end{aligned}$$

with some fixed $\alpha_1,\ldots,\alpha_4 \in \mathbb{R}$. We are going to show that this condition reduces to the condition (1).

Indeed, since the postulates (3)–(5) contain (2), we obtain, equivalently,

$$(a,b)_S(x,y)_V = \beta_1(a\cdot x, b\cdot y)_W + \beta_2(b\cdot x, a\cdot y)_W \qquad \text{with } \beta_1 = \alpha_1 + \epsilon_3\alpha_4, \quad \beta_2 = \alpha_3 + \epsilon_3\alpha_2. \tag{19}$$

LEMMA 7. *We have* $\beta_2 = \epsilon_1\beta_1$ *so, up to an unessential constant* β_1, (19) *reduces to* (1).

Proof. We apply (19) and (1):

$$\begin{aligned}\beta_1(a\cdot x, b\cdot y)_W + \beta_2(b\cdot x, a\cdot y)_W &= (a,b)_S(x,y)_V = \epsilon_2(a,b)_S(y,x)_V = \epsilon_2[\beta_1(a\cdot y, b\cdot x)_W \\ &+\beta_2(b\cdot y, a\cdot x)_W] = \epsilon_2\epsilon_3[\beta_1(b\cdot x, a\cdot y)_W + \beta_2(a\cdot x, b\cdot y)_W.\end{aligned}$$

By Lemma 6, $\epsilon_2\epsilon_3 = \epsilon_1$ and hence $(\epsilon_1\beta_1 - \beta_2)[\epsilon_1(a\cdot x, b\cdot y)_W - (b\cdot x, a\cdot y)_W] = 0$. If we set $y = x$, we can see that $\beta_2 = \epsilon_1\beta_1$, as desired.

2. Reducibility. Hereafter we suppose that $\epsilon_1 = 1$ which, to some extent, is motivated by Lemmas 3-4, and by Corollaries 1-2 below. In order to have a better possibility of studying the "multiplication" $\cdot$ as a bilinear mapping, we denote it by ϕ. A particularly important case appears when the J^3-triple (W,V,S) is irreducible, i.e. when the J^3-mapping ϕ does not leave invariant proper subspaces of (W,V) and their complements. This means that (cf. [10], p. 91):

(i) A J^3-triple (W, V, S) is *reducible* whenever there are

(a) real vector subspaces W_1 and W_2 of W with $W_1 \cap W_2$ different from $\{0\}$, and

(b) real vector subspaces V_1 and V_2 of $V, \{0\} \neq V_1 \neq V$, $V_1 \oplus V_2 = V$,

such that

$$\phi[S \times V_1] \subset W_1, \qquad \phi[S \times V_2] \subset W_2.$$

(ii) A J^3-triple (W, V, S) is *irreducible* whenever it is not reducible.

If a J^3-triple (W, V, S) is irreducible, $W = V$, and if there is a unit element ϵ_0 in S with respect to the J^3-mapping ϕ: $S \times V \to V$, i.e., ϵ_0 is the identity mapping in the space of endomorphisms of V, the J^3-triple reduces to a pseudo-euclidean Hurwitz pair; those pairs were investigated in [36-39, 17-20, 25, 27, 28, 50, 14, 31, 33, 44, 12].

Example 1. In order to discuss the triple (V, V, S) with $n = 2$, $p = 2$, and scalar products $(a, a)_S = (a^1)^2 - (a^2)^2$, $(f, g)_V = f^1 g^2 - g^1 f^2$, where $a = a^1\epsilon_1 + a^2\epsilon_2$, $f = f^1 e_1 + f^2 e_2$, $g = g^1 e_1 + g^2 e_2$, which is reducible, we have to define the J^3-mapping ϕ. In terms of the original J^3-mapping ϕ, the restricted J^3-mappings read as follows:

$$\phi(a, f^1 e_1) = \phi(a^1\epsilon_1 + a^2\epsilon_2, f^1 e_1) = a^1 f^1 e_1 + a^2 f^1 e_1 = (a^1 + a^2) f^1 e_1,$$
$$\phi(a, f^2 e_2) = \phi(a^1\epsilon_1 + a^2\epsilon_2, f^2 e_2) = a^1 f^2 e_2 - a^2 f^2 e_2 = (a^1 - a^2) f^2 e_2.$$

We have, e.g.,

$$\kappa = \begin{bmatrix} 0 & | & 1 \\ - & | & - \\ 1 & | & 0 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 1 & | & 0 \\ - & | & - \\ 0 & | & 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 & | & 0 \\ - & | & - \\ 0 & | & -1 \end{bmatrix}$$

and

$$\epsilon_1 e_1 = e_1, \quad \epsilon_2 e_1 = e_1; \qquad \epsilon_1 e_2 = e_2, \quad \epsilon_2 e_2 = -e_2.$$

Hence V splits into the direct sum $V_1 \oplus V_2$ of one-dimensional invariant spaces; the scalar products $(f^1, g^1)_1$ in V_1 and $(f^2, g^2)_2$ in V_2 can be defined as usual products of real numbers. Obviously, the restricted J^3-mappings send $S \times V_1$ into V_1 and $S \times V_2$ into V_2, respectively, so (V, V, S) is indeed reducible.

Example 2. In order to discuss the triple (W, V, S) with $N = 8$, $n = 4$, $p = 3$, and scalar products

$$(a, a)_S = (a^1)^2 + (a^2)^2 + (a^3)^2, \qquad (f, g)_V = f^1 g^1 + \ldots + f^4 g^4,$$
$$(F, G)_W = F^1 G^1 + \ldots + F^8 G^8,$$

where

$$a = a^1\epsilon_1 + a^2\epsilon_2 + a^3\epsilon_3, \quad f = f^1 e_1 + \ldots + f^4 e_4, \quad g = g^1 e_1 + \ldots + g^4 e_4,$$
$$F = F^1 E_1 + \ldots + F^8 E_8, \quad G = G^1 E_1 + \ldots + G^8 E_8,$$

which is reducible, we have to define the J^3-mapping ϕ. In terms of the original J^3-mapping ϕ, the restricted J^3-mappings read as follows:

$$\phi(a, f^1 e_1 + f^2 e_2) = (a^1 f^1 - a^3 f^2) E_1 + (a^3 f^1 + a^1 f^2) E_2 - a^2 f^1 E_3 - a^2 f^2 E_4,$$
$$\phi(a, f^3 e_3 + f^4 e_4) = (a^1 f^3 - a^2 f^) E_5 + (a^2 f^3 - a^1 f^4) E_6 - a^3 f^4 E_7 + a^3 f^3 E_8;$$

obviously, they map $S \times V_1$ into W_1 and $S \times V_2$ into W_2, respectively. We have, e.g.,

$$\kappa = I_4, C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ - & - & - & - & | & - & - & - & - \\ 0 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & | & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & | & 0 & 0 & 0 & 0 \\ - & - & - & - & | & - & - & - & - \\ 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\ - & - & - & - & | & - & - & - & - \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & -1 & 0 \end{bmatrix}$$

and

$$\begin{array}{llllll} \epsilon_1 e_1 = E_1, & \epsilon_2 e_1 = -E_3, & \epsilon_3 e_1 = E_2, & \epsilon_1 e_3 = E_5, & \epsilon_2 e_3 = E_6, & \epsilon_3 e_3 = E_8, \\ \epsilon_1 e_2 = E_2, & \epsilon_2 e_2 = -E, & \epsilon_3 e_2 = -E_1, & \epsilon_1 e_4 = E_6, & \epsilon_2 e_4 = -E_5, & \epsilon_3 e_4 = -E_7. \end{array}$$

Hence V splits into the direct sum $V_1 \oplus V_2$ of real vector subspaces of V with vector functions of the form $f^1 e_1 + f^2 e_2$ and $f^3 e_3 + f^4 e_4$, respectively, while W splits into the direct sum $W_1 \oplus W_2$ of real vector subspaces of W with vector functions of the form $F^1 E_1 + \ldots + F^4 E_4$ and $F^5 E_5 + \ldots F^8 E_8$, respectively. The scalar products $(\ ,\ )_{V_1}$ etc. are given by the formulae

$$\begin{aligned} &(f^1 e_1 + f^2 e_2, g^1 e_1 + g^2 e_2)_{V_1} = f^1 g^1 + f^2 g^2, \\ &(f^3 e_3 + f_4^e, g^3 e_3 + g^4 e_4)_{V_2} = f^3 g^3 + f^4 g^4, \\ &(F^1 E_1 + \ldots + F^4 E_4, G^1 E_1 + \ldots + G^4 E_4)_{W_1} = F^1 G^1 + \ldots + F^4 G^4, \\ &(F^5 E_5 + \ldots + F^8 E_5, G^5 E_5 + \ldots + G^8 E_5)_{W_2} = F^5 G^5 + \ldots + F^8 G^8. \end{aligned}$$

Suppose now that $n \geq N$ and consider arbitrary $(N-n)\times N$-matrices B''_α, in particular the zero $(N-n)\times N$-matrix $0_{n-n,N}$, and the analogous zero-matrices $0_{r,s}$ and $0_r := 0_{r,r}$. Define the $N \times N$-matrices

$$B_\alpha := \begin{bmatrix} C_\alpha \\ B''_\alpha \end{bmatrix}, \quad \Pi_n := \begin{bmatrix} I_n & 0_{n,N-n} \\ 0_{n-n,n} & 0_{n-n} \end{bmatrix}, \quad \mathcal{C}_\alpha := \begin{bmatrix} C_\alpha \\ 0_{N-n,n} \end{bmatrix}.$$

Evidently, we have $\mathcal{C}_\alpha = \Pi_n B_\alpha$. Moreover, by a straightforward verification, we get

Lemma 8. *The modified structure matrices $\mathcal{C}_\alpha$ and B_α of a J^3-triple satisfy the relations*

$$\mathcal{C}_\alpha K \mathcal{C}^t_\beta + \epsilon_1 \mathcal{C}_\beta K \mathcal{C}^t_\alpha = 2\eta_{\alpha\beta} \begin{bmatrix} \kappa & 0_{n,N-n} \\ 0_{n-n,n} & 0_{N-n} \end{bmatrix}, \tag{20}$$

$$\Pi_n(B_\alpha K B^t_\beta + \epsilon_1 B_\beta K B^t_\alpha)\Pi_n = 2\Pi_n X_{\alpha\beta}\Pi_n, \tag{21}$$

where

$$X_{\alpha\beta} := \begin{bmatrix} \eta_{\alpha\beta}\kappa & \mu_{\alpha\beta} \\ \mu'_{\alpha\beta} & \nu_{\alpha\beta} \end{bmatrix}$$

for arbitrary real $(N-n)\times(N-n)$*-matrices* $\nu_{\alpha\beta}$ *and* $\mu^t_{\alpha\beta}$, $\mu'_{\alpha\beta}$ *being arbitrary real* $(N-n)\times n$*-matrices*, $\alpha,\beta = 1,\dots,p$.

In terms of the J^3-triples Theorem 1 in [31] can be formulated as follows:

THEOREM 1. *Suppose that the matrices* $X_{\alpha\beta}$ *are of the form*

$$X_{\alpha\beta} = \eta_{\alpha\beta}X, \quad \textit{where} \quad X = \begin{bmatrix} \kappa & \mu \\ \mu' & \nu \end{bmatrix}, \quad \det X \neq 0 \tag{22}$$

and (*as before*) K, κ *and* η *are the metric tensors of* W, V, *and* S, *respectively*, (W,V,S) *being an arbitrary* J^3*-triple with* $\epsilon_1 = 1$, *while* ν *is an arbitrary real* $(N-n)\times(N-n)$*-matrix and* μ^t, μ' *are arbitrary real* $(N-n)\times$*-matrices. Then the modified structure matrices* B_α, $\alpha = 1,\dots,p$, *corresponding to* (W,V,S), *satisfy the relations*

$$B_\alpha K B^t_\beta + B_\beta K B^t_\alpha = 2\eta\alpha\beta X, \qquad \alpha,\beta = 1,\dots,p. \tag{23}$$

The (*purely imaginary*) $N\times N$*-matrices* γ_α, *defined by*

$$B_\alpha = i\gamma_\alpha B_p, \qquad \alpha = 1,\dots,p-1, \quad i = +\sqrt{-1}, \tag{24}$$

are the familiar generators of a real Clifford algebra $C^(r,s)$ *in the imaginary Maiorana representation. The integers* r, s *are defined by the diagonal form of the metric tensor* η:

$$\eta := \mathrm{diag}(\underbrace{1,\dots,1}_{r+1\ times}, \underbrace{-1,\dots,-1}_{s\ times}) \quad \eta_{pp} = 1, \tag{25}$$

or

$$\eta := \mathrm{diag}(\underbrace{1,\dots,1}_{s\ times}, \underbrace{-1,\dots,-1}_{r+1\ times}) \quad \eta_{pp} = -1. \tag{26}$$

Thus two isomorphism classes of Clifford algebras are obtained unless η *is positive definite*, *in which case only one Clifford algebra is obtained. The generators* γ_α *satisfy the conditions*

$$\overline{\gamma_\alpha} = -\gamma_\alpha, \quad \mathrm{re}\,\gamma_\alpha = 0, \quad \text{where} \quad \overline{\gamma_\alpha} = X^{-1}\gamma^t_\alpha X, \quad \alpha = 1,\dots,p-1; \tag{27}$$

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\widehat{\eta}_{\alpha\beta}I_N, \quad \alpha,\beta = 1,\dots,p-1; \tag{28}$$

$$\widehat{\eta}_{\alpha\beta} := \eta_{\alpha\beta}/\eta_{pp}, \qquad \alpha,\beta = 1,\dots,p-1, \tag{29}$$

and are determined up to the conjugation induced by (25) *and* (27), *i.e.*,

$$\gamma'_\alpha = \begin{bmatrix} \lambda & 0_{n,N-n} \\ \Gamma & \Omega \end{bmatrix} \gamma_\alpha \begin{bmatrix} \lambda & 0_{n,N-n} \\ \Gamma & \Omega \end{bmatrix}^{-1}, \quad \alpha = 1,\dots,p-1. \tag{30}$$

Furthermore, *the metric* K *is determined by*

$$K = \eta_{pp}B^{-1}_p X B^{-1t}_p. \tag{31}$$

The matrices η, X, K *and* B_α, $\alpha = 1,\dots,p$ *or*, *equivalently*, η, X, K *and* γ_α, $\alpha = 1,\dots,p-1$, *and* B_p *satisfying the relations* (12) *with* $\epsilon_1 = 1$, (20), (21), (23)–(29) *and*

(31), *determine the* J^3*-triple* (W, W, S) *with metric tensors* K *of the first copy of* W, X *of the second copy of* W, η, *of* S, *and*

$$\dim W = N, \qquad K^t = \epsilon_3 K, \qquad X^t = \epsilon_3 X. \tag{32}$$

Proof. Note that $(\varepsilon_0, \varepsilon_0)_S = 1$, where ε_0 is the identity mapping in the space of endomorphisms of $W : S \times W \to W$. Therefore, by a pseudo-orthogonal change of basis we may assume that the multiplicative unit is ε_τ for any fixed index τ. This gives $B_\tau = I_N$ by the "multiplication" scheme (9) and $\eta_{\tau\tau} = 1$ by the definition of η. If we construct the Clifford algebra by taking $t = \tau$, then, using the notation of (26), we have $\eta_{\tau\tau} = 1$ and $B_t = I_N$. If, on the other hand, we choose $\tau \neq t$ for which $\eta_{tt} = -1$, then by (24) we have $i\gamma_\tau B_\tau = I_N$ and, consequently, by (27)–(29), we arrive at $B_t = i\gamma_\tau$. The other statements follow by a direct calculation; cf. [31].

The above described J^3-triple is called *generalized* (*pseudo-euclidean*) *Hurwitz pair* [36-39]: instead of two identical vector spaces W equipped with different metric tensors K and X, and a third vector space S equipped with a metric tensor η, we consider one vector space W equipped with K and X, and the vector space S equipped with $\eta : (W, K, X; S, \eta) := (W, K; W, X; S, \eta)$. In such a way, under the hypotheses of Theorem 1, the J^3-triple $(W, K, X; S, \eta)$ is equivalent to the Hurwitz pair (W, S) mentioned in that theorem.

It is easily seen that the notion of generalized (pseudo-euclidean) Hurwitz pair can also be characterized in an intrinsic way, independently from the choice of bases, in terms of scalar products and the J^3-condition (1) only, just as a particular case of the J^3-triple; cf. Sect. 1 and [39], Introduction and Sect. 1, where a more exhaustive explanation of links with known results on Clifford algebras is given.

The following result had been proved by us in 1986 [37-39]:

THEOREM 2. *All generalized* (*pseudo-euclidean*) *Hurwitz pairs are given by the following table*

$r\backslash s$	0	1	2	3	4	5	6	7
0	S	-	A	$\underline{\text{A}}$	AS	-	AS	$\underline{\text{S}}$
1	AS	A	$\underline{\text{A}}$	A	AS	AS	$\underline{\text{A}}$	AS
2	AS	$\underline{\text{A}}$	A	-	S	$\underline{\text{S}}$	AS	-
3	$\underline{\text{S}}$	AS	AS	S	$\underline{\text{S}}$	S	AS	AS
4	AS	-	AS	$\underline{\text{S}}$	S	-	A	$\underline{\text{A}}$
5	AS	AS	$\underline{\text{A}}$	AS	AS	A	$\underline{\text{A}}$	A
6	S	$\underline{\text{S}}$	AS	-	AS	$\underline{\text{A}}$	A	-
7	$\underline{\text{S}}$	S	AS	AS	$\underline{\text{S}}$	AS	AS	S

It has to be read as follows: *All* r *and* s *are given modulo* 8. *An* S *in the appropriate box means there is a symmetric inner product* $(\ ,\ )_V$, *an* A *that there is an antisymmetric one. The cases where there are two inequivalent representations are underlined.*

The result has been reformulated by Randriamihamison [44] and Cnops [12] (Corollary 2), however, because of minor differences in the definition of irreducibility and equivalence of representations there are differences in 12 boxes (in the remaining 52 boxes with the same result we just put +):

$r\backslash s$	0	1	2	3	4	5	6	7	diff.
0	+	+	+	A	+	+	+	S	-2
1	+	+	+	+	+	+	<u>AS</u>	+	+2
2	+	A	+	+	+	S	+	+	-2
3	<u>AS</u>	+	+	+	+	+	+	+	+2
4	+	+	+	S	+	+	+	A	-2
5	+	+	<u>AS</u>	+	+	+	+	+	+2
6	+	S	+	+	+	A	+	+	-2
7	+	+	+	+	<u>AS</u>	+	+	+	+2
diff.	+2	-2	+2	-2	+2	-2	+2	-2	

The last row (resp. column) "diff." shows the difference between the total number of inequivalent solutions of the generalized Hurwitz problem in the Randriamihamison-Cnops and Ławrynowicz-Rembieliński approaches indicated in the corresponding columns (resp. rows). In consequence, in the both approaches the total number of inequivalent solutions indicated in all 64 boxes coincides: it amounts to 96.

3. Anti-involutions. One of special Clifford structures is the *quaternion structure* connected with the metric $ds^2 := (da^0)^2 + \ldots + (da^3)^2$, $a^\alpha \in \mathbb{R}^4$, and the solvability of the corresponding (Hurwitz [15, 16]) J^3-condition

$$[(a^0)^2+\ldots+(a^3)^2][(x^0)^2+\ldots+(x^3)^2] = [(a\cdot x)^0]^2+\ldots+[(a\cdot x)^3]^2, \quad (a\cdot x)^A = c^A_{j\alpha}a^\alpha x^j,$$

with respect to $c^A_{j\alpha}$, where a^α, x^j and $c^A_{j\alpha}$ are real. Consider the familiar quaternion algebra $\mathbb{H}$. Following the programme of Bingener and Lehmkuhl [9] we are interested in having elements x, p satisfying a q-commutator relation

$$[x,p]_q := xp - qpx = 0, \qquad [x,p]_1 \equiv [x,p] := xp - px \tag{33}$$

with an arbitrary unit q of degree zero, in particular a fixed complex number. Then, we define the *q-quantum quaternion algebra* $\mathbb{H}_q$ [43] as the following quadruple $(\mathbf{A}, *, \mathbf{Q}, c^*)$.

1^0 $\mathbf{A}$ is an involutive algebra with the unit, generated by $a^0, \ldots, a^3$ satisfying

$$[a^1,a^2]=0, \quad [a^0,a^3] = -\frac{1}{2}i(1+q^{-1})([a^1,a^1]_q + [a^2,a^2]_q),$$
$$[a^0,a^1]_q = i[a^3,a^1]_q, \quad [a^0,a^1]_{q^{-1}} = -i[a^3,a^1]_{q^{-1}},$$
$$[a^0,a^2]_q = i[a^3,a^2]_q, \quad [a^0,a^2]_{q^{-1}} = -i[a^3,a^2]_{q^{-1}}.$$

2^0 The *anti-involution* $*$ is defined in A by

$$a^0_* = a^0, \quad a^1_* = \frac{1}{2}[(q+q^{-1})a^1 + i(q-q^{-1})a^2],$$
$$a^2_* = \frac{1}{2}[-i(q-q^{-1})a^1 + (q+q^{-1})a^2], \quad a^3_* = a^3.$$

3^0 $\mathbf{Q}$ is an $\mathbf{A}$-module generated by the quaternionic units ϵ_α satisfying

$$\epsilon_0^2 = \epsilon_0,\ \epsilon_0\epsilon_k = \epsilon_k\epsilon_0 = \epsilon_k,\ \epsilon_j\epsilon_k = -\delta_{jk}\epsilon_0 + \sum_{l=1}^{3} \epsilon_{jk}^l \epsilon_l,\ j,k = 1,2,3;\ a^\alpha \epsilon_\beta = \epsilon_\beta a^\alpha.$$

4^0 The *anti-involution* c^* is the natural extension of $*$ to Q by its identification with the quaternionic conjugation of ϵ_α, namely $c^*\epsilon_0 = \epsilon_0$, $c^*(\epsilon_1, \epsilon_2, \epsilon_3) = -(\epsilon_1, \epsilon_2, \epsilon_3)$.

5^0 The specific linear combinations

$$\begin{aligned} a =& a^\alpha \epsilon_\alpha = a^0\epsilon_0 + (a^1, a^2, a^4)(\epsilon_1, \epsilon_2, \epsilon_3), \\ \bar{a} =& (c^*a^\alpha)(c^*\epsilon_\alpha) = (c^*c^0)\epsilon_0 - (c^*a^1, c^*a^2, c^*a_3)(\epsilon_1, \epsilon_2, \epsilon_3) \end{aligned}$$

are called the *q-quaternion* and the *conjugate q-quaternion*, respectively.

6^0 The *co-multiplication* $\widehat{\otimes}$ is introduced in $\mathbf{Q}$ by

$$\begin{aligned} a\widehat{\otimes}a =& (a^\alpha \otimes a^\beta)\epsilon_\alpha\epsilon_\beta = [a^0 \otimes a^0 - (a^1, a^2, a^3) \otimes (a^1, a^2, a^3)]\epsilon_0 \\ &+ [a^0 \otimes (a^1, a^2, a^3) + (a^1, a^2, a^3) \otimes a^0 + (a^1, a^2, a^3) \times (a^1, a^2, a^3)](\epsilon_1, \epsilon_2, \epsilon_3), \end{aligned}$$

and the *co-unity* is defined by $\epsilon(a) = e^0$, where $\otimes$ denotes the tensor product while x – the usual three-vector product.

The q-quaternion norm can be introduced by

$$\begin{aligned} \|a\|_q^2 =& \bar{a}a = a\bar{a} = a^0a^0 + (c^*a^1, c^*a^2, c^*a^3)(a^1, a^2, a^3) \\ =& (a^0)^2 + (a^3)^2 + \frac{1}{2}(q + q^{-1})[(a^1)^2 + (a^2)^2]. \end{aligned}$$

The idea of q-deformation can naturally be extended to a *quantum space* V which is the quotient algebra $\mathcal{F}/J$, where $\mathcal{F}$ is an associative algebra with the unit element over $\mathbb{R}$ or $\mathbb{C}$, freely generated by the elements $x^1, \ldots, x^n$, while J is a two-sided ideal in $\mathcal{F}$. In the case of the Yang-Baxter category of such spaces [30, 26, 51], J is defined by a collection of bilinear reordering rules for generators x^k : $\mathbf{x} \otimes \mathbf{x} = B\mathbf{x} \otimes \mathbf{x}$, where $\mathbf{x}$ denotes the column $(x^k)^T$, $\otimes$ denotes the usual direct product, and $B \in \mathrm{End}(\mathbb{C}^n \times \mathbb{C}^n)$. The matrix B is assumed to satisfy the *Yang-Baxter equation* guaranteeing the associativity of V :

$$\begin{aligned} &B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23}, \quad \text{where } B_{12} = B \otimes I_n, \\ &B_{23} = I_n \otimes B, \text{ and } (I_n)_j^k = \delta_j^k,\ j = 1, \ldots, n. \end{aligned}$$

The *twisted external algebra associated with* V is generated by x^j and dx^k, where d is the exterior differential operator obeying the standard conditions of linearity, nilpotency and the Leibniz rule with gradation. We consider an endomorphism C such that the following reordering rules and consistency conditions are satisfied:

$$\begin{aligned} &\mathbf{x} \otimes d\mathbf{x} = Cd\mathbf{x} \otimes \mathbf{x}, \quad C \in \mathrm{End}(\mathbb{C}^n \times \mathbb{C}^n), \quad &&\text{and} \quad (B_{12} - I_n)(C_{12} + I_n) = 0, \\ &B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23}, &&\text{where} \quad C_{12} = C \otimes I_n, \\ &C_{12}C_{13}C_{12} = C_{23}C_{12}C_{23}, && \qquad\qquad C_{23} = I_n \otimes C. \end{aligned} \tag{34}$$

Hereafter we assume that

$$x^jx^k = (q_j/q_k)x^kx^j \text{ (not summed)},\ j,k = 1, \ldots, n, \tag{35}$$

and introduce in V the antilinear anti-involution $*$, defined on the generators by

$$x^{j*} = x^j, \quad j = 1, \ldots, n. \tag{36}$$

The parameters q_j are supposed to lie on the unit circle $\{|q| = 1,\ q \in \mathbb{C}\}$, so in fact we are dealing with a kind of deformation being a straightforward extension of the *parametrical method*, well known from the theory of univalent functions and quasiconformal mappings [29]. Since the corresponding rules for the exterior algebra read:

$$x^j dx^k = (q_j/q_k)dx^k x^j, \quad \text{and hence } dx^j dx^k = -(q_j/q_k)dx^k dx^j,$$

we finally arrive at a *deformed Grassmann algebra.*

A special choice

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-1/r & q/r & 0 \\ 0 & 1/q & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ p \end{bmatrix}, \ d\mathbf{x} = \begin{bmatrix} dx \\ dp \end{bmatrix}, \quad \begin{matrix} (dx)^2 = 0,\ c^*(x) = p, \\ (dp)^2 = 0,\ c^*(p) = x \end{matrix} \tag{37}$$

gives the already mentioned case (33). In the case where $x, p \in \mathbb{C}$ and $q \in \mathbb{R}$ we may consider the antilinear involution $\bar{c}(x) = \bar{x}$, $\bar{c}(p) = \bar{p}$ (complex conjugation). Let $\{x, p\}_q := xp + qpx$, $\{x, p\}_1 \equiv \{x, p\} := xp + px$. We have [11, 32]:

THEOREM 3. *Suppose* (34), (35), *and* $\{dx, dp\}_q = 0$. *Then*

$$C = \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 1/q & 0 & 0 \\ 0 & 0 & 0 & t \end{bmatrix}, \quad \begin{matrix} [x, dx]_s = 0,\ |s| = 1,\ s \in \mathbb{C}, \\ [p, dx]_{1/q} = 0 \\ [x, dp]_q = 0 \\ [p, dp]_t = 0,\ |t| = 1,\ t \in \mathbb{C}. \end{matrix} \tag{38}$$

If the derivatives f_x, f_p *of a differentiable function* f *of* x, p *are defined by*

$$f_x dx + f_p dp = df, \tag{39}$$

then

(40)

$$f_x(x,p) = \lim_{x' \to x} \frac{f(sx', p) - f(x', p)}{(s-1)x'}, \qquad f_p(x,p) = \lim_{p' \to p} \frac{f(x, tp') - f(x, p)}{(t-1)p'} \quad \textit{for } s, t \neq 1.$$

In the case of the complex conjugation $\bar{c}$ *and* $q, r \in \mathbb{R} \setminus \{0, 1\}$, *for* z, $\bar{z}$ *and* dz, $d\bar{z}$ *defined by*

$$z = x + ip, \quad \bar{z} = x - ip \quad \textit{and} \quad dz = dx + idy,\ d\bar{z} = dx - idy, \tag{41}$$

we get

$$(dz)^2 = \frac{1-q}{1+q} dz d\bar{z} = \frac{q-1}{q+1} d\bar{z} dz = -(d\bar{z})^2 \tag{42}$$

and the Cauchy-Riemann equations $f_{\bar{z}}(z, \bar{z}) = 0$ *for a mapping* f *holomorphic in a neighbourhood in* $\mathbb{C}$, *in the form* $q f_x = -i f_p$.

R e m a r k 5. Symbols of the type

$$\lim_{x \to x'} \frac{F(x', p)}{x'} \quad \text{resp.} \quad \lim_{p \to p'} \frac{G(x, p')}{p'}$$

mean decreasing by one the powers of x resp. p in the series defining the function $F(x, p)$

resp. $G(x,p)$ without assuming the existence of $1/x$ resp. $1/p$. As usual,

$$f_z := \frac{1}{2}(f_x - if_y), \qquad f_{\bar{z}} := \frac{1}{2}(f_x + if_y).$$

P r o o f. The system of equations (34) for C involves four independent real parameters which can be chosen as

$$\arg q,\ \arg r,\ \arg s,\ \arg t \quad \text{if} \quad |q| = |r| = |s| = |t| = 1,$$

or just q, r, s, t if they are real, thus yielding (36), (38), (40), and the desired form of the Cauchy-Riemann equations.

R e m a r k 6. Theorem 2 illustrates a duality between phase spaces related to Manin's quantum plane [42] and one-dimensional symmetric complex manifolds in the case of the only classical Klein projective line $\mathbb{C}P^1$ [7, 48]. Let us recall that a *symmetric* complex manifold is a complex manifold $\mathbb{M}$ together with an *anti-involution* c on $\mathbb{M}$, i.e. an antiholomorphic mapping c on $\mathbb{M}$ such that $c^2 = -\mathrm{id}$. A *morphism* of symmetric complex manifolds from $(\mathbb{M}, c)$ to $(\mathbb{M}', c')$ is a holomorphic mapping $\phi : \mathbb{M} \to \mathbb{M}'$ such that $c' \circ \phi = \phi \circ c$. It is natural to ask about the duality mentioned in the cases of the only possible two non-classical Klein projective lines $\Delta = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathbb{R}P^2$. The positive answer is given in Theorem 3A and 3B below, respectively.

An analogous reasoning leads to the following counterparts of Theorem 3:

THEOREM 3A. *Suppose* (34), (37), *and* $\{dx, dp\}_{q/r} = 0$ *and* $[p, dp]_r = 0$. *Then*

$$C = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & 1/r & q & 0 \\ 0 & r/q & 0 & 0 \\ 0 & 0 & 0 & r \end{bmatrix}, \quad \begin{array}{l} [x, dx]_r = 0, \\ [p, dx]_{r/q} = 0 \\ [x, dp]_q = (t-1)dx\ p, \\ |r| = 1,\ r \in \mathbb{C}. \end{array} \tag{38}$$

If the derivatives f_x, f_p of a differentiable function f of x, p are defined by (39), *then*

$$f_x(x,p) = \lim_{x' \to x} \frac{f(rx', rp) - f(x, rp)}{(r-1)x'}, \qquad f_p(x,p) = \lim_{p' \to p} \frac{f(x, rp') - f(x,p)}{(r-1)p'} \quad \text{for } r \neq 1.$$

In the case of the complex conjugation $\bar{c}$ and $q \in \mathbb{R} \setminus \{0\}$, $r \in \mathbb{R} \setminus \{0, 1\}$, for z, $\bar{z}$ and dz, $d\bar{z}$ defined by (41), *we get*

$$(dz)^2 = \frac{r-q}{r+q} dz d\bar{z} = \frac{q-r}{q+r} d\bar{z} dz = -(d\bar{z})^2$$

and the Cauchy-Riemann equations $f_{\bar{z}}(z, \bar{z}) = 0$ for a mapping f holomorphic in a neighbourhood in $\mathbb{C}$, in the form $qf_x = -if_p$.

THEOREM 3B. *Suppose* (34), (37), *and* $[dx, dp]_q = 0$. *Then*

$$C = \begin{bmatrix} 1/r & 0 & 0 & 0 \\ 0 & 0 & q/r & 0 \\ 0 & 1/q & 1/r - 1 & 0 \\ 0 & 0 & 0 & 1/r \end{bmatrix}, \quad \begin{array}{l} [x, dx]_s = 0,\ |r| = 1,\ r \in \mathbb{C}, \\ \{p, dx\}_{1/q} = 0 \\ [x, dp]_q = -2dx\ p, \\ \{p, dp\} = 0. \end{array}$$

If the derivatives f_x, f_p *of a differentiable function* f *of* x, p *are defined by* (39), *then*

$$f_x(x,p)=\lim_{x'\to x}\frac{f(x/r',p)-f(x,p)}{(1/r-1)x'},\qquad f_p(x,p)=\lim_{p'\to p}\frac{f(x/r,p'/r)-f(x/r,p)}{(1/r-1)p'}.$$

In the case of the complex conjugation $\bar{c}$ *and* $q\in\mathbb{R}\setminus\{0\}$, $r\in\mathbb{R}\setminus\{0,1\}$, *for* z, $\bar{z}$ *and* dz, $d\bar{z}$ *defined by* (41), *we get*

$$(dz)^2=\frac{q-r}{q+r}dzd\bar{z}=\frac{r-q}{r+q}d\bar{z}dz=-(d\bar{z})^2$$

and the Cauchy-Riemann equations $f_{\bar{z}}(z,\bar{z})=0$ *for a mapping* f *holomorphic in a neighbourhood in* $\mathbb{C}$, *in the form* $qf_x=-if_p$.

4. Quantum braided Clifford algebras. Matrix groups like $GL(n)$, $SO(n)$, etc. were generalized in two ways recently. Both are based on deformation of the algebra of functions on the groups generated by co-ordinate function T_j^k that commute:

$$T_j^kT_r^s=T_r^sT_j^k,\quad \text{i.e., } T_1T_2=T_2T_1. \tag{43}$$

In a *quantum deformation* of the initial group these commutation relations are modified by a matrix $R=[R_{jk}^{rs}]$ so that the functions do not commute but satisfy the relations

$$R_{12}T_1T_2=T_2T_1R_{12}.$$

In this relation the elements of R are real or complex numbers, but T is formed by generally noncommuting elements of an algebra. In a *braided deformation* of the initial group the relations (43) are modified by a matrix $Z=[Z_{jk}^{rs}]$ with elements being real or complex numbers so that

$$T_1Z_{12}T_2Z_{12}^{-1}=Z_{21}^{-1}T_2Z_{21}T_1.$$

We are going to generalize general linear inhomogeneous groups as quantum braided groups in the spirit of Majid's theory [41], but our construction [30] differs in the interrelations between the homogeneous and inhomogeneous parts of the group. Namely let us consider the co-module action in the form

$$\begin{bmatrix} x^1\\ \dots\\ x^n\\ \mathbf{1}\end{bmatrix}=\begin{bmatrix} & | & a^1\\ \Lambda & | & \dots\\ & | & a^n\\ - & - & -\\ 0 & | & \mathbf{1}\end{bmatrix}\begin{bmatrix} x^1\\ \dots\\ x^n\\ \mathbf{1}\end{bmatrix}, \tag{44}$$

where the matrix elements Λ_k^j and a^k have to be the *quantum braided* generators, while $\mathbf{1}$ is the unit element of the algebra. We assume the isotropy of V which means that the generators form a commutative subalgebra of V, i.e.

$$\Lambda_k^j\Lambda_s^r=\Lambda_s^r\Lambda_k^j. \tag{45}$$

We arrive [30] at

THEOREM 4. *Under the assumptions* (35), (36) *and* (45), *the co-module action* (44) *preserves the reordering rules* (35) *for the generators and the "reality" conditions* (36) *for the antilinear anti-involution* $*$. *Moreover*,

$$\Lambda_k^jx^r=(q_j/q_k)x^r\Lambda_k^j,\quad \Lambda_k^{j*}=(q_j/q_k)\Lambda_k^j$$

so Λ^j_k *can be represented as* $\Lambda^j_k = (q_j/q_k)^{1/2}\lambda^j_k$, *where* λ^j_k *are "real", i.e.* $\lambda^{j*}_k = \lambda^j_k$. *Furthermore*,

$$\Lambda^j_k a^r = (q_j/q_k)a^r\Lambda^j_k, \quad a^j a^k = (q_j/q_k)a^k a^j \quad \text{and} \quad a^j x^k = (q_j/q_k)x^k a^j.$$

P r o o f. It can be done by a straightforward verification.

Now, following the standard procedure, we can define the *line element*

$$ds^2 = \dot{x}^j g_{jk}\dot{x}^k d\tau \quad \text{with} \quad dx^j(\tau) = \dot{x}^j(\tau)d\tau.$$

The reality od ds^2 implies that $g^*_{jk} = g_{*j}$. Moreover, we have to assume that ds^2 belongs to the centre of the whole algebra. Hence

$$\Lambda^j_k g_{rs} = (q_k/q_j)^2 g_{rs}\Lambda^j_k, \qquad g_{jk}g_{rs} = (q_j q_k/q_r q_s)^2 g_{rs}g_{jk},$$
$$g_{jk}a^r = (q^2_r/q_j q_k)a^r q_{jk}, \qquad g_{jk}x^r = (q^2_r/q_j q_k)x^r g_{jk}.$$

Therefore $g_{jk} = (q_j/q_k)g_{kj}$. Hereafter elements of the matrix g^{-1} will be denoted by g^{jk}. Selecting a subgroup of the group in question, under the condition of invariance of ds^2, we get $\Lambda^\dagger a\Lambda = g$, where $\dagger$ denotes the composition of matrix transposition and star-conjugation, i.e. $(q_j/q_k)\Lambda^j_k g_{jr}\Lambda^r_s = g_{ks}$.

Finally, we can extend the deformed Grassmann algebra in question to the corresponding *deformed Clifford algebra*. As basic rules for its generators we accept the relations

$$\lambda^j\lambda^k + (q_j/q_k)\lambda^k\lambda^j = 2g^{jk}I, \tag{45}$$

where I is a finite or infinite-dimensional matrix. By Theorem 4, we also have the following braiding (multiplication) rules:

$$\wedge^j_k\lambda^r = (q_j/q_k)\lambda^r\wedge^j_k, \quad g^{jk}\lambda^r = (q_j q_k/q^2_r)\lambda r g^{jk}, \quad \text{and} \quad g^{jk}g^{rs} = (q_j q_k/q_r q_s)g^{rs}g^{jk}.$$

Summing up, we may conclude the paper by noticing a succesful composition of at least three different geometrical concepts added to the notions of Clifford algebra and deformed Clifford algebra. Their importance in mathematical analysis and physical applications is undoubtful, but of course requires further studies.

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