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A VARIATIONAL METHOD FOR UNIVALENT FUNCTIONS CONNECTED WITH ANTIGRAPHY

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Abstract. The paper is devoted to a class of functions analytic and univalent in the unit disk that are connected with an antigraphy $e^{i\varphi}\overline{w} + i\rho e^{i\frac{\varphi}{2}}$. Variational formulas and Grunsky inequalities are derived. As an application there are given some estimations in the considered class of functions.

1. Introduction. $H(U)$ denotes, as usual, the space of all functions analytic in the unit disk $U = \{z : |z| < 1\}$. Let $\rho \in \mathbb{R}$, $\varphi \in [0, 2\pi]$, $a \in \mathbb{C}$, and $\rho \neq 2\mathrm{Im}\{e^{-i\frac{\varphi}{2}}a\}$. $S_{a\rho\varphi}$ denotes the class of all functions that are analytic, univalent in the unit disk U and satisfy the conditions

$$(1) \qquad f(0) = a \qquad \text{and} \qquad f(z_1) \neq e^{i\varphi}\overline{f(z_2)} + i\rho e^{i\frac{\varphi}{2}}, \quad z_1, z_2 \in U.$$

The class $S_{a\rho\varphi}$ is, in some sense, similar to the classes of Gel'fer, Bieberbach-Eilenberg, Grunsky-Shah and bounded functions. We can write the definitions of these classes in a common form as follows:

Let J be a class of all functions that are analytic and univalent in U and satisfy the conditions

$$f(0) = a \qquad \text{and} \qquad w \in f(U) \Longrightarrow \omega(w) \notin f(U).$$

For $a = 1$ and $\omega(w) = -w$ J is the class of Gel'fer functions, for $a = 0$ and $\omega(w) = \frac{1}{w}$ – the class of Bieberbach-Eilenberg functions, for $a = 0$ and $\omega(w) = -\frac{1}{w}$ – the class of Grunsky-Shah functions, for $a = 0$ and $\omega(w) = \frac{1}{\overline{w}}$ – the class of bounded functions, and finally for $\omega(w) = e^{i\varphi}\overline{w} + i\rho e^{i\frac{\varphi}{2}}$ – the class $S_{a\rho\varphi}$. Each of these homographies and antigraphies has the property that the inverse function is the same.

The class $S_{10\pi}$ coincides with the class of univalent functions with positive real part.

2. Variational formulas. Let $f \in S_{a\rho\varphi}$ and $D = f(U)$. It is clear that the domain

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D has the property

$$w \in D \Longrightarrow e^{i\varphi}\overline{w} + i\rho e^{i\frac{\varphi}{2}} \notin D. \tag{2}$$

Using the Golusin's method we can derive the variational formula for the function f.

THEOREM 1. *Let $f \in S_{a\rho\varphi}$, $z_0 \in U$. Then for sufficiently small ε there exists a function $f^* \in S_{a\rho\varphi}$ such that*

$$f^*(z) = f(z) + \varepsilon\Bigg\{ e^{i\alpha}\left[\frac{(f(z)-a)(f(z)-b)}{f(z)-f(z_0)} - \frac{(f(z_0)-a)(f(z_0)-b)}{z_0 f'^2(z_0)}\frac{zf'(z)}{z-z_0}\right] + \tag{3}$$
$$+ e^{-i\alpha}\left[\frac{(f(z)-a)(f(z)-b)}{f(z)-e^{i\varphi}\overline{f(z_0)} - i\rho e^{i\frac{\varphi}{2}}} + \frac{\overline{(f(z_0)-a)(f(z_0)-b)}}{\overline{z_0 f'^2(z_0)}}\frac{z^2 f'(z)}{1-\overline{z_0}z}\right]\Bigg\} + o(\varepsilon)$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$, while $\varepsilon \to 0$, uniformly on compact subsets of U.

Proof. In order to find the variation of the function f we shall at first define such a variation w^* of the boundary ∂D that does not violate the property (2) for the domain D^* bounded by $w^*(\partial D)$. Define the function

$$w^*(w) = w + \varepsilon v(w), \tag{4}$$

where $\varepsilon > 0$, $v(w)$ is a function analytic in the closure of such a domain Δ that contains ∂D and satisfies the condition

$$w \in \Delta \Longrightarrow e^{i\varphi}\overline{w} + i\rho e^{i\frac{\varphi}{2}} \in \Delta$$

and having the property

$$v(e^{i\varphi}\overline{w} + i\rho e^{i\frac{\varphi}{2}}) = e^{i\varphi}\overline{v(w)}. \tag{5}$$

Following [3] one can prove that the function (4) is univalent for sufficiently small ε and $w^*(\partial D)$ is a boundary of a domain D^* having the property (2).

Let $P = \{z : r \le |z| < 1\}$, $r \in (0,1)$ be such a ring that $f(P) \subset \Delta$. The function

$$F(z,\varepsilon) = w^*(f(z)) - a, \quad z \in P$$

satisfies the assumptions of Golusin theorem [2] for the function $f(z)-a$. So the function f^* such that $f^*(U) = D^*$ and $f^*(0) = a$ has the form

$$f^*(z) = f(z) + \varepsilon\left\{ v(f(z)) - zf'(z)S(z) + zf'(z)\overline{S\left(\frac{1}{\overline{z}}\right)}\right\} + o(\varepsilon), \tag{6}$$

where $S(z)$ is a principal part of the development into a Laurent series of the function $\frac{v(f(z))}{zf'(z)}$ and $\frac{o(\varepsilon)}{\varepsilon} \to 0$, while $\varepsilon \to 0$, uniformly on compact subsets of U. The function f^* belongs to the class $S_{a\rho\varphi}$ and is a variation of the function f.

Now, we define the function $v(w)$ as follows

$$v(w) = (w-a)(w-b)\left(\frac{e^{i\alpha}}{w-w_0} + \frac{e^{-i\alpha}}{w - e^{i\varphi}\overline{w_0} - i\rho e^{i\frac{\varphi}{2}}}\right),$$

where $w_0 = f(z_0)$, $z_0 \in U$, $b = e^{i\varphi}\overline{a} + i\rho e^{i\frac{\varphi}{2}}$, α is an arbitrary real number. It is clear that $v(w)$ satisfies the condition (5). The variation (6) in this case takes the form (3). ∎

We can also obtain other variational formulas. If $w_0 \notin \overline{D}$ and $e^{i\varphi}\overline{w_0}+i\rho e^{i\frac{\varphi}{2}} \notin \overline{D}$ then we have

$$f^*(z) = f(z) + \varepsilon\Big\{e^{i\alpha}\frac{(f(z)-a)(f(z)-b)}{f(z)-w_0} + \\ + e^{-i\alpha}\frac{(f(z)-a)(f(z)-b)}{f(z)-e^{i\varphi}\overline{w_0}-i\rho e^{i\frac{\varphi}{2}}}\Big\} + o(\varepsilon), \tag{7}$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$, while $\varepsilon \to 0$, uniformly on compact subsets of U.

Compositions of a function $f \in S_{a\rho\varphi}$ with univalent functions g such that $g(0) = 0$ and $g(U) \subset U$ give other variations of f, for example:

$$f^*(z) = f(e^{i\varepsilon}z) = f(z) + i\varepsilon z f'(z) + o(\varepsilon), \quad \varepsilon \in \mathbb{R} \tag{8}$$

and

$$f^*(z) = f(k_\alpha^{-1}((1-\varepsilon)k_\alpha(z))) = f(z) - \varepsilon z f'(z)\frac{e^{i\alpha}+z}{e^{i\alpha}-z} + o(\varepsilon), \tag{9}$$

where $k_\alpha(z) = \frac{z}{(1+e^{-i\alpha}z)^2}$, $\alpha \in \mathbb{R}$, $\varepsilon > 0$, and where $\frac{o(\varepsilon)}{\varepsilon} \to 0$, while $\varepsilon \to 0$, uniformly on compact subsets of U.

3. Schiffer equation. $S_{a\rho\varphi}$ is a normal family of functions. It becomes compact if we add the constant function $g = a$. The family of functions close to the function $f \in S_{a\rho\varphi}$ that we have just constructed is rich enough to consider the maximal problem in the class $S_{a\rho\varphi}$. Let ψ be a complex, continuous functional defined over $S_{a\rho\varphi}$. Suppose that $\mathrm{Re}\{\psi\}$ has a Fréchet derivative at the point $f \in S_{a\rho\varphi}$. Then there exists a functional $L_f \in H'(U)$ such that

$$\mathrm{Re}\{\psi(f^*)\} = \mathrm{Re}\{\psi(f)\} + \varepsilon\mathrm{Re}\{L_f(h)\} + o(\varepsilon), \tag{10}$$

for every function

$$f^*(z) = f(z) + \varepsilon h(z) + o(\varepsilon),$$

such that $h \in H(U)$, $\frac{o(\varepsilon)}{\varepsilon} \to 0$, while $\varepsilon \to 0$, uniformly on compact subsets of U.

THEOREM 2. *Let ψ be a complex functional defined and continuous over the class $S_{a\rho\varphi}$ and let $\mathrm{Re}\{\psi\}$ have a Fréchet derivative L_f at the point $f \in S_{a\rho\varphi}$. If $\mathrm{Re}\{\psi\}$ attains its maximal value in the class $S_{a\rho\varphi}$ at f then f satisfies the equation*

$$\frac{(\zeta f'(\zeta))^2}{(f(\zeta)-a)(f(\zeta)-b)}A(f(\zeta)) = B(\zeta), \tag{11}$$

where $A(w)$ and $B(z)$ are given by the formulas:

$$A(w) = L_f\left(\frac{(f(z)-a)(f(z)-b)}{f(z)-w}\right) + \overline{L_f\left(\frac{(f(z)-a)(f(z)-b)}{f(z)-e^{i\varphi}\overline{w}-i\rho e^{i\frac{\varphi}{2}}}\right)}, \tag{12}$$

$$B(\zeta) = L_f\left(\frac{\zeta z f'(z)}{z-\zeta}\right) + \overline{L_f(zf'(z))} - \overline{L_f\left(\frac{zf'(z)}{1-\overline{\zeta}z}\right)},$$

$r < |\zeta| < 1$, $r \in (0,1)$. *The function $B(\zeta)$ is an analytic function in the ring $P_r = \{\zeta : r < |\zeta| < \frac{1}{r}\}$, is real and non-positive on ∂U.*

P r o o f. If the functional $\mathrm{Re}\{\psi\}$ attains at $f \in S_{a\rho\varphi}$ its maximal value and f^* has the form (3) then (10) leads to

$$\frac{(z_0 f'(z_0))^2}{(f(z_0)-a)(f(z_0)-b)} A(f(z_0)) = B(z_0),$$

where $A(w)$ and $B(\zeta)$ are given by the formulas (12). Combining (8) with (10) and (9) with (10) and using the fact that f is maximal we conclude that $B(\zeta)$ is real and non-positive on ∂U, which completes the proof. ∎

As a consequence of applying the variational formula (7) to (10) we have the following theorem:

THEOREM 3. *Let ψ and f satisfy the assumptions of the previous theorem, A be such a function meromorphic in $\mathbb{C}$ that $A \neq 0$. If w_0 and $e^{i\varphi}\overline{w_0} + i\rho e^{i\frac{\varphi}{2}}$ are not in $f(U)$ then at least one of these points is on the boundary $\partial f(U)$. Particularly the set $\mathbb{C}-(f(U)\cup h(U))$, where $h(z) = e^{i\varphi}\overline{f(z)} + i\rho e^{i\frac{\varphi}{2}}$ has no interior points.*

4. Grunsky inequalities. Defining the functional ψ in a special way we can obtain the complete square on the left-hand side of (11) and then find a solution of this equation in an implicit form. Such a functional leads also to Grunsky inequalities and then to some simple estimations in the class $S_{a\rho\varphi}$. Let

$$\psi(f) = \lambda^2 \log \frac{f'(0)}{a-b} + 2\lambda L\left(\log \frac{f(z)-a}{z(f(z)-b)}\right) + \tag{13}$$
$$+L^2\left(\log \frac{f(z)-f(\zeta)}{z-\zeta}\right) - |L|^2\left(\log(f(z) - e^{i\varphi}\overline{f(\zeta)} - i\rho e^{i\frac{\varphi}{2}})\right),$$

where L is a functional from $H'(U)$ such that
$L(1) = 0$, $L^2(\varphi(z,\zeta)) = L(L(\varphi(z,\zeta)))$, $|L|^2(\varphi(z,\overline{\zeta})) = L(\overline{L(\varphi(z,\overline{\zeta}))})$ for $\varphi(z,\zeta)$ analytic in $U\times U$, λ is an arbitrary real number.

The Fréchet derivative of $\mathrm{Re}\{\psi\}$ exists for every $f \in S_{a\rho\varphi}$ and has the form

$$\mathrm{Re}\{L_f(h)\} = \mathrm{Re}\left\{\lambda^2\frac{h'(0)}{f'(0)} + 2\lambda L\left(\frac{(a-b)h(z)}{(f(z)-a)(f(z)-b)}\right) + L^2\left(\frac{h(z)-h(\zeta)}{f(z)-f(\zeta)}\right) - \right. \tag{14}$$
$$\left. -\overline{|L|^2\left(\frac{h(z)}{f(z)-e^{i\varphi}\overline{f(\zeta)}-i\rho e^{i\frac{\varphi}{2}}}\right)} + |L|^2\left(\frac{e^{i\varphi}\overline{h(\zeta)}}{f(z)-e^{i\varphi}\overline{f(\zeta)}-i\rho e^{i\frac{\varphi}{2}}}\right)\right\}.$$

THEOREM 4. *If the functional (13) attains its maximal value at the point $f \in S_{a\rho\varphi}$ then f satisfies the equation*

$$\lambda \log \frac{f(\zeta)-a}{\zeta(f(z)-a)} + L\left(\log\frac{f(z)-f(\zeta)}{z-\zeta}\right) - \overline{L\left(\log(f(z)-e^{i\varphi}\overline{f(\zeta)}-i\rho e^{i\frac{\varphi}{2}})\right)} + \tag{15}$$
$$+\overline{L(\log(1-\overline{\zeta}z))} = \lambda\log\frac{f'(0)}{a-b} + L\left(\log\frac{f(z)-a}{z}\right) - \overline{L(\log(f(z)-b))}.$$

The maximal value $\mathrm{Re}\{\psi(f)\} = -|L|^2(\log(1-\overline{\zeta}z))$.

P r o o f. Let $f \in S_{a\rho\varphi}$ be a maximal function for the functional $\mathrm{Re}\{\psi\}$. According to the theorem 2 the function f satisfies the equation (11). In our case this equation has the form

$$(\zeta f'(\zeta))^2\left(\lambda\frac{a-b}{(f(\zeta)-a)(f(\zeta)-b)}-L\left(\frac{1}{f(z)-f(\zeta)}\right)+\right.$$
$$\left.+e^{-i\varphi}\overline{L\left(\frac{1}{f(z)-e^{i\varphi}\overline{f(\zeta)}-i\rho e^{i\frac{\varphi}{2}}}\right)}\right)^2=-B(\zeta). \tag{16}$$

From the Caccioppoli-Kőthe integral representation of the functional from $H'(U)$ [1] and from the fact that $B(\zeta)$ is non-positive on ∂U and from (16), following [4], we conclude that the function

$$C(\zeta)=\lambda\frac{(a-b)\zeta f'(\zeta)}{(f(\zeta)-a)(f(\zeta)-b)}-L\left(\frac{\zeta f'(\zeta)}{f(z)-f(\zeta)}-\frac{\zeta}{z-\zeta}\right)+$$
$$+\overline{L\left(\frac{e^{i\varphi}\overline{\zeta f'(\zeta)}}{f(z)-e^{i\varphi}\overline{f(\zeta)}-i\rho e^{i\frac{\varphi}{2}}}-\frac{1}{1-\overline{\zeta}z}\right)}$$

is analytic in U and has such a continuous continuation tu $\overline{U}$ that is real on ∂U. Furthermore, we notice that it is constant and this constant is equal to λ and we have

$$\zeta f'(\zeta)\left(\lambda\frac{a-b}{(f(\zeta)-a)(f(\zeta)-b)}-L\left(\frac{1}{f(z)-f(\zeta)}\right)+\right.$$
$$\left.+e^{-i\varphi}\overline{L\left(\frac{1}{f(z)-e^{i\varphi}\overline{f(\zeta)}-i\rho e^{i\frac{\varphi}{2}}}\right)}\right)=\lambda-L\left(\frac{\zeta}{z-\zeta}\right)+\overline{L\left(\frac{1}{1-\overline{\zeta}z}\right)}. \tag{17}$$

Now it is easy to verify that

$$\frac{(a-b)\zeta f'(\zeta)}{(f(\zeta)-a)(f(\zeta)-b)}=\zeta\frac{\partial}{\partial\zeta}\log\frac{f(\zeta)-a}{\zeta(f(\zeta)-b)}, \tag{18}$$
$$\frac{\zeta f'(\zeta)}{f(z)-f(\zeta)}-\frac{\zeta}{z-\zeta}=-\zeta\frac{\partial}{\partial\zeta}\log\frac{f(z)-f(\zeta)}{z-\zeta},$$
$$\frac{e^{i\varphi}\overline{\zeta f'(\zeta)}}{f(z)-e^{i\varphi}\overline{f(\zeta)}-i\rho e^{i\frac{\varphi}{2}}}=-\overline{\zeta}\frac{\partial}{\partial\overline{\zeta}}\log(f(z)-e^{i\varphi}\overline{f(\zeta)}-i\rho e^{i\frac{\varphi}{2}}),$$
$$\frac{1}{1-\overline{\zeta}z}=1-\overline{\zeta}\frac{\partial}{\partial\overline{\zeta}}\log(1-\overline{\zeta}z).$$

Applying (18) to (17) we get

$$\lambda\log\frac{f(\zeta)-a}{\zeta(f(\zeta)-b)}+L\left(\log\frac{f(z)-f(\zeta)}{z-\zeta}\right)-\overline{L\left(\log(f(z)-e^{i\varphi}\overline{f(\zeta)}-i\rho e^{i\frac{\varphi}{2}})\right)}+$$
$$+\overline{L(\log(1-\overline{\zeta}z))}=c, \tag{19}$$

where

$$c=\lambda\log\frac{f'(0)}{a-b}+L\left(\log\frac{f(z)-a}{z}\right)-\overline{L(\log(f(z)-b))}.$$

We shall prove that $\mathrm{Re}\{c\}=0$. Notice at first that it follows from the theorem 3 that the boundaries $\partial f(U)$ and $\partial h(U)$ have a common point ω. Then there exist two sequences (ζ_n^1) and (ζ_n^2) of points from U such that $f(\zeta_n^1)\to\omega$ and $h(\zeta_n^2)\to\omega$. Putting correspondingly

ζ_n^1 and ζ_n^2 into (19) and passing to the limit we conclude that $\mathrm{Re}\{c\} = 0$ that is

$$\mathrm{Re}\left\{\lambda \log \frac{f'(0)}{a-b} + L\left(\log \frac{f(z)-a}{z}\right) - \overline{L(\log(f(z)-b))}\right\} = 0. \tag{20}$$

(19) leads also to another equation

$$\lambda L\left(\log \frac{f(\zeta)-a}{\zeta(f(\zeta)-b)}\right) + L^2\left(\log \frac{f(z)-f(\zeta)}{z-\zeta}\right) - \tag{21}$$
$$-|L|^2\left(\log(f(z) - e^{i\varphi}\overline{f(\zeta)} - i\rho e^{i\frac{\varphi}{2}})\right) + |L|^2(\log(1-\overline{\zeta}z)) = 0.$$

Finally adding (21) and (20) multiplied by λ, we obtain

$$\mathrm{Re}\left\{\lambda^2 \log \frac{f'(0)}{a-b} + 2\lambda L\left(\log \frac{f(z)-a}{z(f(z)-b)}\right) + L^2\left(\log \frac{f(z)-f(\zeta)}{z-\zeta}\right) - \right.$$
$$\left. -|L|^2\left(\log(f(z) - e^{i\varphi}\overline{f(\zeta)} - i\rho e^{i\frac{\varphi}{2}})\right)\right\} = -|L|^2(\log(1-\overline{\zeta}z)),$$

which completes the proof. ∎

The next theorem is not a simple consequence of the previous one because the class $S_{a\rho\varphi}$ is not compact.

THEOREM 5. *If $\lambda \in \mathbb{R} - \{0\}$ then every $f \in S_{a\rho\varphi}$ satisfies the inequality*

$$\mathrm{Re}\left\{\lambda^2 \log \frac{f'(0)}{a-b} + 2\lambda L\left(\log \frac{f(z)-a}{z(f(z)-b)}\right) + L^2\left(\log \frac{f(z)-f(\zeta)}{z-\zeta}\right) - \right. \tag{22}$$
$$\left. -|L|^2\left(\log(f(z) - e^{i\varphi}\overline{f(\zeta)} - i\rho e^{i\frac{\varphi}{2}})\right)\right\} \le -|L|^2(\log(1-\overline{\zeta}z)).$$

The equality occurs for some function $g \in S_{a\rho\varphi}$.

Proof. We shall prove that there exists a maximal function $f \in S_{a\rho\varphi}$ for the functional ψ given by the formula (13). This functional is continuous. It is also bounded from above. It follows from the fact that $|f'(0)|$ is bounded, and $\frac{f-a}{f'(0)} \in S$ if $f \in S_{a\rho\varphi}$ (S - the class of all functions analytic and univalent in U with normalisation $f(0) = f'(0)-1 = 0$), from Growth theorem, from the estimation

$$\mathrm{Re}\left\{L^2\left(\log \frac{g(z)-g(\zeta)}{z-\zeta}\right)\right\} \le -|L|^2(\log(1-\overline{\zeta}z)), \text{ for } g \in S \text{ [2, p. 116]}, \tag{23}$$

and from the integral representation of the functional from $H'(U)$. Suppose that $\lambda \neq 0$. The class $S_{a\rho\varphi}$ is a normal family. Using the fact that $\frac{f-a}{f'(0)} \in S$ if $f \in S_{a\rho\varphi}$ we can in a similar manner as in [4] prove that the functional (13) attains its maximal value at some $f \in S_{a\rho\varphi}$. ∎

In the case $\lambda = 0$ the inequality (22) also holds but we do not know if there exists in $S_{a\rho\varphi}$ a function for which occurs the equality. However we can prove that this result cannot be improved.

THEOREM 6. *Each function $f \in S_{a\rho\varphi}$ satisfies the inequality*

$$\operatorname{Re}\left\{L^2\left(\log\frac{f(z)-f(\zeta)}{z-\zeta}\right)-|L|^2\left(\log(f(z)-e^{i\varphi}\overline{f(\zeta)}-i\rho e^{i\frac{\varphi}{2}})\right)\right\}\le \tag{24}$$
$$\le -|L|^2(\log(1-\overline{\zeta}z)).$$

This inequality cannot be improved.

P r o o f. Applying to (24) the following facts:
(i) there exists a function $\hat{f} \in S$ for which in (23) occurs equality,
(ii) each function from the class S can be approximated by bounded functions from S,
(iii) if $g \in S$ is a bounded function then for sufficiently small $r > 0$ the function $a + rg \in S_{a\rho\varphi}$,
it is easy to see that the left-hand side of (24) can be arbitrarily near the right-hand side, so this result is best possible. ∎

5. Examples. To illustrate the theorems given above, consider two special functionals from $H'(U)$. At first let the functional L have the form

$$L(g)=\sum_{m=1}^{N}\lambda_m\left[g(z_m)-g(0)\right],\ \text{where } g\in H(U),\ z_1,\dots,z_N\in U,\lambda_1,\dots,\lambda_N\in\mathbb{C}.$$

Then (22) leads to the following inequality :

$$\operatorname{Re}\left\{\left(\lambda-\sum_{m=1}^{N}\lambda_m\right)^2\log\frac{f'(0)}{a-b}+2\lambda\sum_{m=1}^{N}\lambda_m\log\frac{f(z_m)-a}{z_m(f(z_m)-b)}+\right.$$
$$+\sum_{n,m=1}^{N}\lambda_n\lambda_m\log\frac{f(z_m)-f(z_n)}{z_m-z_n}\frac{z_nz_m(a-b)}{(f(z_n)-a)(f(z_m)-a)}-$$
$$\left.-\sum_{n,m=1}^{N}\lambda_n\overline{\lambda}_m\log\frac{f(z_n)-e^{i\varphi}\overline{f(z_m)}-i\rho e^{i\frac{\varphi}{2}}}{a-e^{i\varphi}\overline{f(z_m)}-i\rho e^{i\frac{\varphi}{2}}}\cdot\frac{a-b}{f(z_n)-b}\right\}\le$$
$$\le-\sum_{n,m=1}^{N}\lambda_n\overline{\lambda}_m\log(1-z_n\overline{z}_m),$$

where for $\frac{f(z_m)-f(z_n)}{z_m-z_n}$ we take $f'(z_m)$ in the case $n = m$.

Putting $N = 1$, $\lambda = \lambda_1 = 1$, $z_1 = z$ in the above inequality we obtain the following estimation:

$$\frac{|f'(z)|}{\left|f(z)-e^{i\varphi}\overline{f(z)}-i\rho e^{i\frac{\varphi}{2}}\right|}\le\frac{1}{1-|z|^2}$$

and for $z = 0$ we have

$$|f'(0)|\le|a-b|.$$

Considering the functional

$$L(g)=\sum_{m=1}^{N}\lambda_m g'(z_m),\ \text{where } g\in H(U),\quad z_1,\dots,z_N\in U,\lambda_1,\dots,\lambda_N\in\mathbb{C}.$$

and applying it to the inequality (22) we get

$$\mathrm{Re}\Bigg\{\lambda^2\log\frac{f'(0)}{a-b}+2\lambda\sum_{m=1}^{N}\lambda_m\left(\frac{(a-b)f'(z_m)}{(f(z_m)-a)(f(z_m)-b)}-\frac{1}{z_m}\right)+$$

$$+\sum_{n,m=1}^{N}\lambda_n\lambda_m\left(\frac{f'(z_m)f'(z_n)}{(f(z_m)-f(z_n))^2}-\frac{1}{(z_m-z_n)^2}\right)-$$

$$-\sum_{n,m=1}^{N}\lambda_n\overline{\lambda}_m\frac{e^{i\varphi}\overline{f'(z_m)}f'(z_n)}{(f(z_n)-e^{i\varphi}\overline{f(z_m)}-i\rho e^{i\frac{\varphi}{2}})^2}\Bigg\}\le$$

$$\le\sum_{n,m=1}^{N}\lambda_n\overline{\lambda}_m\frac{1}{(1-z_n\overline{z_m})^2}.$$

Because $\lim\limits_{n\to m}\dfrac{f'(z_m)f'(z_n)}{(f(z_m)-f(z_n))^2}=\dfrac{1}{6}\{f(z_m),z_m\}$, where $\{f(z_m),z_m\}$ denotes the Schwarzian derivative, then in the case $N=1$, $z_1=z$ we have

$$\mathrm{Re}\Bigg\{\lambda^2\log\frac{f'(0)}{a-b}+2\lambda\lambda_1\left(\frac{(a-b)f'(z)}{(f(z)-a)(f(z)-b)}-\frac{1}{z}\right)+\frac{1}{6}\lambda_1^2\{f(z),z\}-$$

$$-|\lambda_1|^2\frac{e^{i\varphi}|f'(z)|^2}{(f(z)-e^{i\varphi}\overline{f(z)}-i\rho e^{i\frac{\varphi}{2}})^2}\Bigg\}\le|\lambda_1|^2\frac{1}{(1-|z|^2)^2}.$$

For $\lambda=0$ we get the following estimation:

$$|\{f(z),z\}|\le\frac{6}{(1-|z|^2)^2}-\frac{6|f'(z)|^2}{\left|f(z)-e^{i\varphi}\overline{f(z)}-i\rho e^{i\frac{\varphi}{2}}\right|^2}.$$

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