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BI-AXIAL GEGENBAUER FUNCTIONS OF THE FIRST AND SECOND KIND

ALAN COMMON

Institute of Mathematics and Statistics
University of Kent
Canterbury, Kent, CT2 7NF, England
E-mail: A.K.Common@ukc.ac.uk

Abstract. The classical orthogonal polynomials defined on intervals of the real line are related to many important branches of analysis and applied mathematics. Here a method is described to generalise this concept to polynomials defined on higher dimensional spaces using Bi-Axial Monogenic functions.

The particular examples considered are Gegenbauer polynomials defined on the interval $[-1,1]$ and the Gegenbauer functions of the second kind which are weighted Cauchy integral transforms over this interval of these polynomials. Related polynomials are defined which are orthogonal on the unit ball $\mathbb{B}^p \equiv \{\vec{x} \in \mathbb{R}^p;\ |\vec{x}| \leq 1\}$ using Bi-Axial Monogenic generating functions on $\mathbb{R}^m$. Then corresponding generalised Gegenbauer functions of the *second kind* are defined using generalised weighted Bi-Axial Monogenic Cauchy transforms of these polynomials over $\mathbb{B}^p$.

These generalised Gegenbauer functions of first and second kind reduce to the standard case when $p = 1$ and are solutions of related second order differential equations which become identical in the one dimensional case.

1. Gegenbauer functions. The Gegenbauer polynomials

$$C_n^{(\alpha)}(x)\ ;\ \ n = 0, 1, 2, \ldots\ \ ;\ \ \alpha > -\frac{1}{2}$$

have the orthogonality property

$$\int_{-1}^{1} C_n^{(\alpha)}(x) C_m^{(\alpha)}(x)(1 - x^2)^{\alpha - \frac{1}{2}} dx = 0\ ;\ \ m \neq n\ . \tag{1.1}$$

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They satisfy the second order differential equation

$$(1.2)\qquad (1-x^2)^{-\alpha+\frac{1}{2}}\frac{d}{dx}\left\{(1-x^2)^{\alpha+\frac{1}{2}}\frac{d}{dx}C_n^{(\alpha)}(x)\right\}+n(n+2\alpha)C_n^{(\alpha)}(x)=0\ .$$

They may be defined from a Rodrigues formula

$$(1.3)\qquad C_n^{(\alpha)}(x)=\frac{(-)^n\Gamma(\alpha+\frac{1}{2})\Gamma(n+2\alpha)}{2^n n!\Gamma(2\alpha)\Gamma(\alpha+n+\frac{1}{2})(1-x^2)^{\alpha-\frac{1}{2}}}\ \frac{d^n}{dx^n}\ [(1-x^2)^{\alpha-\frac{1}{2}+n}]$$

and have the generating function corresponding to the relation

$$(1.4)\qquad (1+z^2)^{-\alpha}=(1+x^2)^{-\alpha}\sum_{n=0}^{\infty}\left[\frac{-iy}{(1+x^2)^{\frac{1}{2}}}\right]^n C_n^{(\alpha)}\left[\frac{x}{(1+x^2)^{\frac{1}{2}}}\right]$$

where $z=x+iy$. We see that this generating function, written in a non-standard form, is the analytic continuation of $[1+x^2]^{-\alpha}$ from the real axis to general $z=x+iy$.

Finally Gegenbauer functions of the second kind are given by

$$(1.5)\quad Q_n^{(\alpha)}(z)=\frac{1}{2}(z^2-1)^{-\alpha+\frac{1}{2}}\int_{-1}^{1}\frac{(1-t^2)^{\alpha-\frac{1}{2}}C_n^{(\alpha)}(t)dt}{(z-t)}\qquad n=0,1,2,\dots\ ;\quad \alpha>-\frac{1}{2}$$

and satisfy the same second differential equation (1.2) as the $C_n^{(\alpha)}(x)$.

2. Bi-axial monogenic functions. An important class of monogenic functions in $\mathbb{R}^m$ is that defined over bi-axially symmetric domains [8].

The approach is to consider functions on $\mathbb{R}^m$ taking values in a complex Clifford algebra $\mathcal{A}$. The generating vectors of the Clifford algebra $\mathcal{A}$ are $\{e_\ell;\ \ell=1,\dots,m\}$ satisfying the defining relations,

$$(2.1)\qquad e_\ell e_j+e_je_\ell=-2\delta_{j\ell}e_0;\ \ j,\ell=1,\dots,m\ ,$$

where e_0 is the unit element in $\mathcal{A}$.

To every point in $\mathbb{R}^m$ there corresponds a vector in the algebra

$$(2.2)\qquad \vec{x}=\sum_{\ell=1}^{m}x_\ell e_\ell\ .$$

The function f on an open set Ω of $\mathbb{R}^m$ taking values in $\mathcal{A}$ is said to be left monogenic when

$$(2.3)\qquad \partial_{\vec{x}}f\equiv\sum_{\ell=1}^{m}e_\ell\frac{\partial f}{\partial x_\ell}=0,\ \ \forall x\in\Omega\ .$$

We consider the splitting $\mathbb{R}^m=\mathbb{R}^p+\mathbb{R}^q$ and denote a general element $\vec{x}$ of $\mathbb{R}^m$ by

$$(2.4)\qquad \vec{x}=\vec{x}_1+\vec{x}_2=\rho_1\vec{\omega}_1+\rho_2\vec{\omega}_2$$

where $\rho_1=|\vec{x}_1|$, $\rho_2=|\vec{x}_2|$ and $\vec{x}_1\in\mathbb{R}^p$, $\vec{x}_2\in\mathbb{R}^q$.

Then any suitable Clifford valued function $f(\vec{x}_1,\vec{x}_2)$ on $\mathbb{R}^m$ will have the expansion

$$(2.5)\qquad f(\vec{x}_1,\vec{x}_2)=\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}f_{k,\ell}(\rho_1,\rho_2)P_{k,\ell}(\vec{x}_1,\vec{x}_2)$$

where $P_{k,\ell}(\vec{x}_1,\vec{x}_2)$ are bi-axial spherical monogenics defined by Jank and Sommen [7] They have the property of being homogeneous of degree k in $\vec{x}_1$, ℓ in $\vec{x}_2$ and

$$\partial_{\vec{x}_1} P_{k,\ell}(\vec{x}_1,\vec{x}_2) = \partial_{\vec{x}_2} P_{k,\ell}(\vec{x}_1,\vec{x}_2) = 0 \ . \tag{2.6}$$

Bi-axial monogenic functions are monogenic functions of the form

$$f(\vec{x}_1,\vec{x}_2) \equiv f_{k,\ell}(\rho_1,\rho_2) P_{k,\ell}(\vec{x}_1,\vec{x}_2) \ . \tag{2.7}$$

An example of this class are the *Generalised Cauchy transforms* of scalar function $g(\lambda)$,

$$\Lambda^{(1)}_{k,\ell}(g)(\vec{x}) = \frac{1}{\omega_p}\int_{\mathbb{R}^p} \frac{[\vec{x}_1+\vec{x}_2-\vec{u}]g(\lambda)P_{k,\ell}(\vec{\eta},\vec{x}_2)d^p\vec{u}}{|\vec{x}_1+\vec{x}_2-\vec{u}|^{m+2\ell}} \tag{2.8}$$

$$\Lambda^{(2)}_{k,\ell}(g)(\vec{x}) = \frac{1}{\omega_p}\int_{\mathbb{R}^p} \frac{[\vec{x}_1+\vec{x}_2-\vec{u}]\vec{\eta}g(\lambda)P_{k,\ell}(\vec{\eta},\vec{x}_2)d^p\vec{u}}{|\vec{x}_1+\vec{x}_2-\vec{u}|^{m+2\ell}} \tag{2.9}$$

where $\vec{u}=\lambda\vec{\eta}$, $|\vec{\eta}|=1$ and ω_p is the area of the unit sphere in p-dimensions. It may be shown that these transforms are Bi-axial monogenic functions of the form (2.7) [3].

This paper is a review of recent studies of Bi-axial monogenic functions jointly with F. Sommen and details of proofs may be found in the quoted references.

3. Bi-axial Gegenbauer functions. One way to generalise the polynomials $C_n^{(\alpha)}(x)$ is to start from Rodrigues formula (1.3) as was done by Cnops, [5].

On the other hand, one can start from the generating function definition of these polynomials [1,2]. We see that the generating function given by (1.4) is analytic except for z^2 real and less than or equal to minus one and tends to $(1+x^2)^{-\alpha}$ as $y\to 0$. In the bi-axial case we define generalised Gegenbauer polynomials through a generating function of the form (2.7),

$$f(\vec{x}) = f_{k,\ell}(\rho_1,\rho_2)P_{k,\ell}(\vec{x}_1,\vec{x}_2)$$

such that as $\rho_2\to 0$,

$$f(\vec{x}) \approx (1+\rho_1^2)^{-[\alpha+k+\ell+(p+q)/2-1]}\rho_1^k\rho_2^\ell P_{k,\ell}(\vec{\omega}_1,\vec{\omega}_2) \ . \tag{3.1}$$

Now the R.H.S. has a unique monogenic extension to domains with $\rho_2\neq 0$ [F. Sommen: Lecture Notes, Ghent] and we have shown [2] that it may be written in the form

$$f(\vec{x}) = (1+\rho_1^2)^{-[\alpha+k+\ell+(p+q)/2-1]} \times \sum_{j=0}^{\infty}\left[\frac{1}{1+\rho_1^2}\right]^{j/2} C^{(\alpha)}_{j;p,q;k,\ell}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{\frac{1}{2}}}\right]\vec{x}_2^{\,j}P_{k,\ell}(\vec{x}_1,\vec{x}_2) \tag{3.2}$$

where setting

$$\vec{u} = \vec{x}_1/(1+\rho_1^2)^{\frac{1}{2}} \tag{3.3}$$

then for $j=0,1,2,\ldots$

$$C^{(\alpha)}_{2j;p,q;k,\ell}(\vec{u}) = \frac{(-)^j(\alpha+k+\ell+p/2+q/2-1)_j(k+p/2)_j}{(\ell+q/2)_j\Gamma(j+1)} \times {}_2F_1(\alpha+k+\ell+p/2+q/2+j-1,-j;\ k+p/2;\ -\vec{u}^2\ ) \tag{3.4}$$

$$(3.5)\qquad C^{(\alpha)}_{2j+1;p,q;k,\ell}(\vec{u}) = \frac{(-)^j(\alpha+k+\ell+p/2+q/2-1)_{j+1}(k+1+p/2)_j}{(\ell+q/2)_{j+1}\Gamma(j+1)} \times \vec{u}_2F_1(\alpha+k+\ell+p/2+q/2+j,-j;\ k+p/2+1;\ -\vec{u}^2\ )\ .$$

The $C^{(\alpha)}_{n;p,q;k,\ell}(\vec{u})$ are polynomials of degree n in $\vec{u}$ and reduce to standard Gegenbauer polynomials when $p=q=1,\quad k=\ell=0$ as does the generating function $f(\vec{x})$.

It may be shown from the monogenicity of the generating function (3.2) that there is a corresponding Rodrigues formulae,

$$(3.6)\qquad C^{(\alpha)}_{2j;p,q;k,\ell}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{\frac{1}{2}}}\right]P_{k,\ell}(\vec{x}_1,\vec{x}_2) = \frac{(-)^j(1/2)_j}{(2j)!(\ell+q/2)_j}(1+\rho_1^2)^{\alpha+q/2+p/2+k+\ell-1+j} \times(\partial_{\vec{x}_1})^{2j}[(1+\rho_1^2)^{-(\alpha+q/2+p/2+k+\ell-1)}P_{k,\ell}(\vec{x}_1,\vec{x}_2)]\ .$$

$$(3.7)\qquad C^{(\alpha)}_{2j+1;p,q;k,\ell}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{\frac{1}{2}}}\right]P_{k,\ell}(\vec{x}_1,\vec{x}_2) = \frac{(-)^{j+1}(1/2)_{j+1}}{(2j+1)!(\ell+q/2)_{j+1}}(1+\rho_1^2)^{\alpha+q/2+p/2+k+\ell-1/2+j} \times(\partial_{\vec{x}_1})^{2j+1}[(1+\rho_1^2)^{-(\alpha+q/2+p/2+k+\ell-1)}P_{k,\ell}(\vec{x}_1,\vec{x}_2)]\ .$$

for $j=0,1,2\ldots$.
These polynomials then have the corresponding orthogonality property

$$(3.8)\qquad \int_{\mathbb{R}^p}\overline{P_{k_1,\ell_1}}(\vec{x}_1,\vec{x}_2)\overline{C^{(\alpha)}_{n;p,q;k_1,\ell_1}}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{\frac{1}{2}}}\right]C^{(\alpha)}_{j;p,q;k_2,\ell_2}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{\frac{1}{2}}}\right] \times P_{k_2,\ell_2}(\vec{x}_1,\vec{x}_2)(1+\rho_1^2)^{-[\alpha+p/2+q/2+(k_1+k_2+\ell_1+\ell_2)/2]}d^p\vec{x}_1 = 0$$

when $k_1\neq k_2$ and or $\ell_1\neq\ell_2$ or $n\neq j$.

These orthogonality properties reduce to the standard ones for Gegenbauer polynomials when $k_1=k_2=\ell_1=\ell_2=0$ and $p=q=1$.

These generalised Gegenbauer polynomials also have the Rodrigues representation

$$(3.9)\qquad C^{(\alpha)}_{2j;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{x}_2) = \frac{(-)^j(\frac{1}{2})_j(\alpha+q/2+p/2+k+\ell-1)_j}{(\ell+q/2)_j(\alpha+q/2+\ell+j)_j\Gamma(2j+1)} \times\ (1+\vec{x^2}_1)^{-[\alpha+q/2+\ell-1]}(\partial_{\vec{x}_1})^{2j}[(1+\vec{x}_1)^{\alpha+q/2+\ell+2j-1}P_{k,\ell}(\vec{x}_1,\vec{x}_2\ )]$$

and

$$(3.10)\quad C^{(\alpha)}_{2j+1;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{x}_2) = \frac{(-)^j(\frac{1}{2})_{j+1}(\alpha+q/2+p/2+k+\ell-1)_{j+1}}{(\ell+q/2)_{j+1}(\alpha+q/2+\ell+j)_{j+1}\Gamma(2j+2)} \times\ (1+\vec{x^2}_1\ \ )^{-[\alpha+q/2+\ell-1]}(\partial_{\vec{x}_1})^{2j+1}[(1+\vec{x}_1)^{\alpha+q/2+\ell+2j}P_{k,\ell}(\vec{x}_1,\vec{x}_2\ )]$$

for $j=0,1,2,\ldots$.

This follows from the fact that these polynomials are related to the axial Gegenbauer polynomials $\mathbb{C}^{(\alpha)}_{n;p,k}(\vec{x}_1)$ defined in [5]. In fact the $C^{(\alpha)}_{n;p,q,k,\ell}(\vec{x}_1)$ are just constant scalar multiples of $\mathbb{C}^{(\alpha+q/2+\ell-1)}_{n;p,k}(\vec{x}_1)$.

4. Bi-Axial Gegenbauer functions of the second kind. The bi-axial Gegenbauer polynomials satisfy the second order differential equation [4],

$$(4.1)\quad (1+\vec{x^2}_1\,)^{-(\alpha+q/2+\ell-1)}\partial_{\vec{x}_1}\left\{(1+\vec{x^2}_1\,)^{\alpha+q/2+\ell}\partial_{\vec{x}_1}\left[C^{(\alpha)}_{n;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_2)\right]\right\}$$
$$= \beta(n,\alpha+q/2+\ell-1,k)C^{(\alpha)}_{n;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_2)$$

where

$$(4.2)\qquad \beta(n,\alpha,k) = n(2\alpha+n+p+2k)\ ;\quad n=0,2,4,\dots$$

$$(4.3)\qquad = (2\alpha+n+1)(n+p+2k-1)\ ;\quad n=1,3,5,\dots\ .$$

In analogy to the standard case, we will construct bi-axial Gegenbauer functions of the second kind which give the second independent solution of these equations. The approach is to consider generalised Cauchy transforms of the bi-axial Gegenbauer polynomials defined in the previous section (as in the standard case).

It may be noted from (3.4), (3.5) that $C^{(\alpha)}_{2j;p,q;k,\ell}(\vec{u})$ is a scalar valued function whilst $C^{(\alpha)}_{2j+1;p,q;k,\ell}(\vec{u})$ is vector valued. We may then define the transforms

$$(4.4)\qquad \Lambda^{(\alpha)}_{n;p,q;k,\ell}(\vec{x}) = \frac{1}{\omega_p}\int_{\mathbb{B}^p}\frac{[\vec{x}_1+\vec{x}_2-\vec{u}](1+\vec{u}^2)^{\alpha+q/2+\ell-1}C^{(\alpha)}_{n;p,q;k,\ell}(\vec{u})P_{k,\ell}(\vec{u},\vec{x}_2)}{|\vec{x}_1+\vec{x}_2-\vec{u}|^{m+2\ell}}d^p\vec{u}$$
$$n=0,1,2,\dots$$

which are of the form (2.8) for n even and of the form (2.9) for n odd and are monogenic for $\mathbb{R}^m\backslash\mathbb{R}^p$. We use these transforms in the following:

Definition. The bi-axial Gegenbauer functions of the second kind are given by

$$(4.5)\qquad Q^{(\alpha)}_{n;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_2)$$
$$\equiv \operatorname*{Lt}_{|\vec{x}_2|\to 0}\left\{\frac{[-(\vec{x}_1+\vec{x}_2)^2-1]^{-\alpha+q/2+\ell}}{|\vec{x}_2|^{\ell}\omega_p}\int_{\mathbb{B}^p}\frac{(\vec{x}_1+\vec{x}_2-\vec{u})(1+\vec{u}^2)^{\alpha+q/2+\ell-1}}{|\vec{x}_1+\vec{x}_2-\vec{u}|^{m+2\ell}}\right.$$
$$\left.\times\ C^{(\alpha)}_{n;p,q;k,\ell}(\vec{u})P_{k,\ell}(\vec{u},\vec{x}_2)\right\}d^p\vec{u}$$

where $n,p,q,k,\ell\in\mathcal{N}\ :\ \alpha>0$.

Taking the limit $|\vec{x}_2|\to 0$ and using the Rodrigues formula (3.9), (3.10)

$$(4.6)\qquad Q^{(\alpha)}_{2j;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_1) = \frac{(-)^j(1/2)_j(\alpha+q/2+p/2+k+\ell-1)_j}{(\ell+q/2)_j(\alpha+q/2+\ell+j)_j\Gamma(2j+1)}$$
$$\times\Theta^{(\alpha)}_{2j;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_2)$$

$$(4.7)\quad Q^{(\alpha)}_{2j+1;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_1) = \frac{(-)^j(1/2)_{j+1}(\alpha+q/2+p/2+k+\ell-1)_{j+1}}{(\ell+q/2)_{j+1}(\alpha+q/2+\ell+j)_{j+1}\Gamma(2j+2)}$$
$$\times\Theta^{(\alpha)}_{2j+1;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_2)$$

where

$$\Theta^{(\alpha)}_{n;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_2) = \frac{[-\vec{x}_1^2-1]^{-\alpha+q/2+\ell}}{\omega_P} \times \int_{\mathbb{B}^p} \frac{(\vec{x}_1-\vec{u})(\partial_{\vec{u}})^n[(1+\vec{u}^2)^{\alpha+q/2+\ell-1+n}P_{k,\ell}(\vec{u},\vec{\omega}_2)]}{|\vec{x}_1-\vec{u}|^{m+2\ell}} d^p\vec{u}; \quad n=0,1,2,\ldots. \tag{4.8}$$

It may be demonstrated using the Funk-Hecke theorem [6] that the R.H.S. of (4.8) is indeed proportional to $P_{k,\ell}(\vec{x}_1,\vec{\omega}_2)$. Uing (4.7), (4.8), the following result may be proved [4]:

Theorem. *The bi-axial Gegenbauer functions* $Q^{(\alpha)}_{n;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_2)$ *defined in (3.4) for* $\alpha > 0$; $p,q,k,\ell \in \mathcal{N}$ *satisfy the differential equation*

$$(-\vec{x}_1^2-1)^{\ell+q/2-\alpha}\partial_{\vec{x}_1}\left\{(-\vec{x}_1^2-1)^{\alpha-\ell-q/2+1}\partial_{\vec{x}_1}\left[Q^{(\alpha)}_{n,p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_2)\right]\right\} = \gamma(p,q,k,\ell,n)Q^{(\alpha)}_{n;p,q;k,\ell}(\vec{x}_1)P_{k,\ell}(\vec{x}_1,\vec{\omega}_2)\,; \quad n=0,1,2,\ldots \tag{4.9}$$

where

$$\gamma(p,q,k,\ell,2j) = -(2\alpha+2j)(m+2\ell+2k+2j-2) \tag{4.10}$$

and

$$\gamma(p,q,k,\ell,2j+1) = -(2\alpha++p+2k+2j)(q+2\ell+2j) \tag{4.11}$$

for $j=0,1,2,\ldots$ *and* $\vec{x}_1 \in \mathbb{R}^p \backslash \mathbb{B}^p$.

In the complex scalar case the Gegenbauer functions of the second kind satisfy the same differential equation as the corresponding Gegenbauer polynomials. In the bi-axial case the two sets of differential equations (4.1) and (4.9) are related but NOT identical. However from their definitions (4.2), (4.3) and (4.10), (4.11)

$$\beta(n,\alpha+q/2+\ell-1,k) = -\gamma(p,q,k,\ell,n) \tag{4.12}$$

for $p=q=1$, $k=\ell=0$ so that in this case the equations satisfied by the bi-axial Gegenbauer polynomials are the same as those for the bi-axial Gegenbauer functions of the second kind. Therefore our results do agree with the standard complex variable case since the latter corresponds to taking $p=q=1$.

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