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SOME PARTIAL DIFFERENTIAL EQUATIONS IN CLIFFORD ANALYSIS

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Abstract. Using Clifford analysis in a multidimensional space some elliptic, hyperbolic and parabolic systems of partial differential equations are constructed, which are related to the well-known classical equations. To obtain parabolic systems Clifford algebra is modified and some corresponding differential operator is constructed. For systems obtained the boundary and initial value problems are solved.

Introduction. The Clifford analysis has suggested us an idea to construct in a multidimensional space some partial differential equations which are related with the well-known classical equations. Using Clifford algebra the operator generalizing the classical Cauchy-Riemann operator is considered in multidimensional space in [3], [7]. Applying this operator to the element of usual Clifford algebra one can get elliptic systems, and by applying to the element of some universal Clifford algebra, hyperbolic systems are obtained. The natural question has arised how to obtain parabolic systems. For this, we need to consider some modification of Clifford algebra. Thus, in a multidimensional space below elliptic, hyperbolic and parabolic systems are obtained, which are related with Laplace, wave and heat equations, respectively. An information about Clifford algebra one can find, for example, in [3], [5], [6], [7].

1. Some basic notions and definitions. Let $e_1, e_2, \dots, e_n$ be an orthonormal base of the n-dimensional real vector space R^n with respect to the usual scalar product. The universal Clifford algebra $R_{(n,s)}$ over R^{n+1} has the basis

$$e_0, e_1, \dots, e_n, e_1e_2, \dots, e_{n-1}e_n, \dots, e_1e_2 \dots e_n,$$

by defining the basic multiplication rules as

$$e_0^2 = 1, \quad e_j^2 = -1, \quad j = 1, 2, \dots, s, \tag{1}$$

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$$e_j^2 = 1, \quad j = s+1, \dots, n; \quad e_j e_k + e_k e_j = 0, \quad 1 \le j < k \le n$$

where e_0 is its identity element. It is a real 2^n-dimensional non-commutative ($n \ge 2$) vector space. $R_{(n,n)} \equiv R_{(n)}$ is usual Clifford algebra. Thus, the basis consists of the elements $e_A = e_{\alpha_1} e_{\alpha_2} \dots e_{\alpha_k}$, where $A : \{\alpha_1, \alpha_2, \dots, \alpha_k\} \in \{1, 2, \dots, n\}$ and $1 \le \alpha_1 < \alpha_2 < \dots < \alpha_k \le n$. An arbitrary element $u \in R_{(n,s)}$ may be written as

$$u = \sum_A u_A e_A, \quad u_A \in R, \quad 0 \le \alpha_1 < \alpha_2 < \dots < \alpha_k \le n. \tag{2}$$

For any $u, v \in R_{(n,s)}$ the product is defined as

$$u \cdot v = \sum_{A,B} u_A v_B e_A e_B. \tag{3}$$

A convolution $u \longrightarrow \bar{u}$ called conjugation is defined by requiring that

$$\bar{u} = \sum_A u_A \bar{e}_A \tag{4}$$

with

$$\bar{e}_0 = e_0, \quad \bar{e}_j = -e_j, \quad j = 1, 2, \dots, n, \quad \bar{e}_A = \bar{e}_{\alpha_k} \dots \bar{e}_{\alpha_1}. \tag{5}$$

Let a domain $\Omega \subset R^{n+1}$ and a function $u(x)$:

$$\Omega \longrightarrow R_{(n,s)}; x(x_0, x_1, \dots, x_n) \in \Omega.$$

Consider the operators

$$\bar{\partial} = \sum_{j=0}^{n} \frac{\partial}{\partial x_j} e_j \quad \partial = \frac{\partial}{\partial x_0} e_0 - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} e_j. \tag{6}$$

Using (1), one can obtain the Coulomb operator

$$\bar{\partial}\partial = \partial\bar{\partial} = [\sum_{j=0}^{s} \frac{\partial^2}{\partial x_j^2} - \sum_{j=s+1}^{n} \frac{\partial^2}{\partial x_j{}^2}] e_0. \tag{7}$$

A function $u(x) \in C^1(\Omega)$ with values in $R_{(n,s)}$ is said to be regular in Ω if

$$\bar{\partial} u = 0, \quad u(x) = \sum_A u_A(x) e_A. \tag{8}$$

For the regular function $u(x)$ with values in $R_{(n)}$ by virtue of (7) we have:

$$\Delta u = 0, \tag{9}$$

where the Laplace operator Δ is taken with respect to all $x_0, x_1, \dots, x_n$. For the regular function $u(x)$ with values in $R_{(n,n-1)}$ by virtue of (7) one can get the wave equation

$$\Delta u - \frac{\partial^2 u}{\partial x_n^2} = 0, \tag{10}$$

where Δ is now taken with respect to variables $x_0, x_1, \dots, x_{n-1}$.

2. Modified Clifford algebra and heat equation. Now in place of $\bar{\partial}$ we need to consider an operator which is connected with the heat equation. For this we consider some modified Clifford algebra (like in Grassman algebra). Let the multiplication rules be defined by

$$e_o^2 = 1, \quad e_j^2 = -1, \quad j = 1, 2, \ldots, n-1; \quad e_n^2 = 0 \tag{11}$$

$$e_j e_k + e_k e_j = 0, \quad j \neq k = 1, 2, \ldots, n.$$

instead of (1) and equalities (4), (5), (6) remained unchanged. This algebra is denoted by $R^0_{(n)}$. Instead of (8) the following equation will be considered

$$[\bar{\partial} - P_{(n)}]u(x) = 0 \tag{12}$$

where a linear operator $P_{(n)}u$ is defined by the condition

$$\partial P_{(n)} u = \frac{\partial u}{\partial x_n}. \tag{13}$$

Using (6), (11) we have:

$$P_{(n)} u = -\sum_A (-1)^k u_{An}(x) e_A \tag{14}$$

$$A_{(\alpha_1, \alpha_2, \ldots, \alpha_k)} \in \{0, 1, \ldots, n-1\}.$$

Now, it is obvious that if u is the solution of (12), then it also is the solution of the heat equation:

$$\Delta u = \frac{\partial u}{\partial x_n}, \tag{15}$$

where Δ is taken with respect to $x_0, x_1, \ldots, x_{n-1}$.

3. Some partial cases.

a) Let $u(x)$ have a vectorial form:

$$u(x) = u_0(x) e_0 - \sum_{j=1}^{n} u_j(x) e_j \tag{16}$$

As is well known, if $u(x) \in R_{(n)}$, then (8) is equivalent to the Riesz system

$$\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \tag{17}$$

$$\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j} = 0, \quad j, k = 0, 1, \ldots, n,$$

which is for $n > 1$ an overdetermined elliptic system.

Now, let $u(x) \in R_{(n,n-1)}$, then (8) is equivalent to the hyperbolic system

$$\sum_{j=0}^{n-1} \frac{\partial u_j}{\partial x_j} - \frac{\partial u_n}{\partial x_n} = 0, \tag{18}$$

$$\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j} = 0, \quad j, k = 0, 1, \ldots, n.$$

But if $u(x) \in R^0_{(n)}$ the solution of (12), where

$$P_{(n)} u(x) = u_n(x) e_o,$$

then (12) is equivalent to the parabolic system

$$\sum_{j=0}^{n-1} \frac{\partial u_j}{\partial x_j} - u_n = 0, \tag{19}$$

$$\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j} = 0, \quad j,k = 0,1,\dots,n.$$

Thus, (18) and (19) can be considered as hyperbolic and parabolic analogues of Riesz system, respectively. If the scalar function $v(x)$ is any solution of (9), (10), (15), then

$$u_j = \frac{\partial v}{\partial x_j}, \quad j = 0,1,\dots,n$$

is the solution of (17), (18), (19), respectively.

b) Let $n = 2$ (the quaternionic case), and

$$u(x) = u_0(x)e_0 - u_1(x)e_1 - u_2(x)e_2 - u_{12}(x)e_1e_2. \tag{20}$$

Let $u(x) \in R_{(2)}$, then, as is known, (8) is equivalent to the Moisil-Theodorescu system [9], which we have written in a vectorial form [10a]:

$$\operatorname{div} U = 0, \tag{21}$$

$$\operatorname{grad} \varphi + \operatorname{rot} U = 0$$

where $U(u_0, u_1, u_2)$ – three-component vector, $\varphi \equiv u_{12}$ – scalar function, operations div, grad, rot are taken with respect to x_0, x_1, x_2. Let $u(x) \in R_{(2,1)}$. Note that in the cases $R_{(2,1)}$ and $R_{(2,0)}$ equation (8) gives us the hyperbolic system of the same form. That is why it is sufficient to consider only one of them. Thus, (8) is equivalent to the system:

$$\begin{aligned} &\frac{\partial u_0}{\partial x_0} + \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} = 0, \qquad \frac{\partial u_{12}}{\partial x_0} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0,\\ &\frac{\partial u_0}{\partial x_1} - \frac{\partial u_1}{\partial x_0} + \frac{\partial u_{12}}{\partial x_2} = 0, \qquad \frac{\partial u_{12}}{\partial x_1} - \frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_2} = 0. \end{aligned} \tag{22}$$

Considering the complex functions

$$\begin{aligned} \varphi(x) &= u_0 + u_{12} - i(u_1 + u_2),\\ \psi(x) &= u_0 - u_{12} - i(u_1 - u_2), \end{aligned} \tag{23}$$

(22) can be written as

$$\begin{gathered} 2\frac{\partial \varphi}{\partial \bar z} + i\frac{\partial \bar\varphi}{\partial x_2} = 0, \quad 2\frac{\partial \psi}{\partial \bar z} - i\frac{\partial \bar\psi}{\partial x_2} = 0, \quad z = x_0 + ix_1\\ 2\frac{\partial}{\partial \bar z} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1}. \end{gathered} \tag{24}$$

These equations are the partial case of the metaparabolic equations whose general form was considered in [1]. Some initial value problems of such equations were considered in [10].

Let now $u(x) \in R^0_{(2)}$, then since $P_{(2)}u = u_2e_0 - u_{12}e_1$, (12) is equivalent to the parabolic system

$$\begin{aligned} &\frac{\partial u_0}{\partial x_0} + \frac{\partial u_1}{\partial x_1} - u_2 = 0 \qquad \frac{\partial u_{12}}{\partial x_0} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0, \\ &\frac{\partial u_0}{\partial x_1} - \frac{\partial u_1}{\partial x_0} + u_{12} = 0 \qquad \frac{\partial u_{12}}{\partial x_1} - \frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_2} = 0. \end{aligned} \tag{25}$$

Considering complex functions

$$w_1 = u_0 - iu_1, \quad w_2 = u_{12} - iu_2,$$

(25) can be written in the complex form

$$2\frac{\partial w_1}{\partial \bar{z}} + i\bar{w}_2 = 0, \quad 2\frac{\partial w_2}{\partial \bar{z}} + i\frac{\partial \bar{w}_1}{\partial x_2} = 0. \tag{26}$$

c) Let now $n = 3$ and

$$u(x) = u_0e_0 - \sum_{j=1}^{3} u_je_j - \sum_{1=j<k}^{3} u_{jk}e_je_k - u_{123}e_1e_2e_3. \tag{27}$$

If $u(x) \in R_{(3,1)}$, the equation (8) is equivalent to the hyperbolic system:

$$\begin{aligned} \operatorname{div} U - \frac{\partial \varphi}{\partial x_3} = 0, \qquad \operatorname{div} V + \frac{\partial \psi}{\partial x_3} = 0, \\ \operatorname{grad} \psi + \operatorname{rot} U + \frac{\partial V}{\partial x_3} = 0, \qquad \operatorname{grad} \varphi + \operatorname{rot} V - \frac{\partial U}{\partial x_3} = 0, \end{aligned} \tag{28}$$

where $U \equiv (u_0, u_1, u_2)$, $V \equiv (u_{123}, u_{23}, -u_{13})$ are three-component vectors, $u_{12} \equiv \psi$, $u_3 \equiv \varphi$ – scalars, the operators grad, div, rot are taken with respect to x_0, x_1, x_2. Note, that if $\varphi = \psi \equiv 0$ and $x_3 \equiv t$ is a time variable, then (28) are Maxwell's equations (in a vacuum). But, if the unknown quantities in (28) do not depend on x_3, this system forms two seperated Moisil-Theodorescu systems (21).

Let $u(x) \in R^{(0)}_{(3)}$, then by using (27), equation (12) is equivalent to the parabolic system having that

$$P_{(3)}u = \varphi e_0 - u_{13}e_1 - u_{23}e_2 + u_{123}e_1e_2,$$

$$\begin{aligned} \operatorname{div} U - \varphi = 0, \qquad \operatorname{div} V + \frac{\partial \psi}{\partial x_3} = 0, \\ \operatorname{grad} \psi + \operatorname{rot} U + V = 0, \qquad \operatorname{grad} \varphi + \operatorname{rot} V - \frac{\partial U}{\partial x_3} = 0. \end{aligned} \tag{29}$$

Let the scalar functions f_1, f_2 be solutions of (10) or (15) for $n = 3$. Then

$$U = \operatorname{grad} f_1, \quad V = \operatorname{grad} f_2, \quad \varphi = \frac{\partial f_1}{\partial x_3}, \quad \psi = -\frac{\partial f_2}{\partial x_3}$$

are the solutions of (28), (29) respectively.

4.Boundary and initial value problems. Let $S_{(n)}$ be the half hyperspace $x_n \geq 0 (n \geq 1)$. The following problems are correctly posed:

a) Find a bounded solution of (17) or (19) in $S_{(n)}$, when only one condition is given on the boundary:

$$u_j = f(x_0, \dots, x_{n-1}), \quad \text{for } x_n = 0,$$

where j is fixed and takes one of the values $0, 1, \dots, n$.

b) Find a bounded solution of (18) in $S_{(n)}$, when only two conditions are given on the boundary: for $x_n = 0$ u_j, u_n are given, where j is a fixed number from $\{0, 1, \dots, n-1\}$; or $u_j, \frac{\partial u_j}{\partial x_n}$ are given where $j \in \{0, 1, \dots, n\}$ is fixed. Note that the number of given conditions on $x_n = 0$ does not depend on n.

Let $\Phi(U, \psi)$ and $\Psi(V, \varphi)$ be four-component vectors, constructed by the solutions of (28) or (29).

c) Find a bounded solution of (28) in $S_{(3)}$, when both vectors: Φ and Ψ for $x_3 = 0$ are given.

d) Find a bounded solution of (29) in $S_{(2)}$, when any four quantities from eight unknowns for $x_3 = 0$ are given.

Analogous problems can be considered in $S_{(12)}$ for equations (21), (22), (25). The unique solutions of all these problems in certain classes can be ropresented in quadratures using, for example, Fourier integral transform.

Now I want to note the following: The generalized Moisil-Theodorescu system

$$\operatorname{div} U + (A \cdot U) = 0 \tag{30}$$

$$\operatorname{grad} \varphi + \operatorname{rot} U + [B \times U] + C\varphi = 0,$$

where A, B, C are given three component vectors; div, grad and rot are taken with respect to x_0, x_1, x_2, was first considered in [10a] in 1975 (Russian), then it was also considered in [10b] (English). To define the solution of (30) in $S_{(2)}$ it is sufficient to give on $x_2 = 0$ two boundary conditions, but in the bounded domain it is not sufficient [2]. Let S be a domain bounded by the closed smooth surface Γ and L be a closed smooth line on Γ, such that its orthogonal projection L_0 on the plane $x_2 = 0$ bounds the domain of variables x_0, x_1 for S. The following problem was posed and solved in [10a]:

Find a regular solution of (30) in S by the conditions:

$$u_0(x) = f_0(x), \quad \varphi(x) = f(x), \quad x \in \Gamma, \tag{31}$$

$$\alpha u_1(x) + \beta u_2(x) = g(x), \quad x \in L,$$

where f_0, f and α, β, g are given functions on Γ and L respectively.

Recently I have seen the article [8], where the system (30) is considered and some of our old results are obtained again. Unfortunately, the author, Huang Liede, perhaps, does not know our papers [10a], [10b]. Moreover, for nonhomogeneous Moisil-Theodorescu system the boundary conditions of type (31) are considered in [4], and there is no reference to my papers.

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