PROPERTIES OF AN ABSTRACT PSEUDORESOLVENT
AND WELL-POSEDNESS OF THE DEGENERATE
CAUCHY PROBLEM

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Abstract. The degenerate Cauchy problem in a Banach space is studied on the basis of
properties of an abstract analytical function, satisfying the Hilbert identity, and a related pair
of operators $A, B$.

1. We consider in a Banach space $X$ an operator-valued function of complex variable
$R(\lambda) \in \mathcal{B}(X)$, satisfying the Hilbert identity:
\begin{equation}
\forall x \in X \quad R(\lambda)R(\mu)x = \frac{R(\mu) - R(\lambda)}{\lambda - \mu} x, \quad \lambda, \mu \in \Omega \subset \mathbb{C}.
\end{equation}
For such function $\ker R(\lambda) =: \mathcal{K}$ and range $R(\lambda) =: \mathcal{R}$ do not depend on $\lambda \ [1]$. If $\mathcal{K} = \{0\}$,
then the function $R(\lambda)$ is called by resolvent, if $\mathcal{K} \neq \{0\}$ - pseudoresolvent. For the case
when $R(\lambda)$ is resolvent, from (1) the equality follows:
\begin{equation}
\lambda - R^{-1}(\lambda) = \mu - R^{-1}(\mu) =: A,
\end{equation}
$D(A) = \mathcal{R}$ and $R(\lambda) = R_A(\lambda)$.

Let $V(t)$ be an exponentially bounded operator-function ($\|V(t)\| \leq Le^{\omega t}$) and
\begin{equation}
r_n(\lambda) := \int_0^\infty \lambda^n e^{-\lambda t} V(t) dt,
\end{equation}
(the integral exists in the Bochner sense). In [2] it was proved, that $r_0(\lambda)$ satisfies (1) for
$\Re \lambda > \omega$, iff $V(t)$ satisfies the semigroup relation:
\begin{equation}
V(t + s) = V(t)V(s), \quad t, s \geq 0,
\end{equation}
r_n(\lambda), n \in \mathbb{N}, satisfies (1); iff
\begin{equation}
(V1) \quad \frac{1}{(n - 1)!} \int_0^s [(s - r)^{n-1}V(t + r) - (t + s - r)^{n-1}V(r)] dr =
\end{equation}
Relation (4) is taken as the basis in the definition of continuous semigroups (see, for example, [1], [3], [4]), relation (V1) – in the definition of \(n\)-times integrated semigroups [2].

**Definition 1.** A one-parameter family of bounded operators \(\{U(t), t \geq 0\}\) is called **strongly continuous semigroup** (or \(C_0\)-semigroup) if the following conditions hold:

1. \(U(t + h) = U(t)U(h), \ t, h \geq 0;\)
2. \(U(0) = f;\)
3. \(U(t)\) is strongly continuous with respect to \(t \geq 0\).

**Definition 2.** Let \(n \in \mathbb{N}\). A one-parameter family of bounded linear operators \(\{V(t), t \geq 0\}\) is called **\(n\)-times integrated exponentially bounded semigroup** if (V1) and the following conditions hold

1. \(V(t)\) is strongly continuous with respect to \(t \geq 0;\)
2. \(\exists K > 0, \omega \in \mathbb{R}: ||V(t)|| \leq \lambda \exp(\omega t), t \geq 0.\)

**Definition 3.** Semigroup \(\{V(t), t \geq 0\}\) is called **nondegenerate** if

\(\forall t \geq 0 V(t)x = 0 \implies x = 0.\)

A \(C_0\)-semigroup is called **0-times integrated semigroup**. Operator \(A = \lambda - R^{-1}(\lambda), D(A) = \mathbb{R}\) is called the **generator** of semigroup.

So, an exponentially bounded semigroup by the Laplace transform defines the function \(R(\lambda)\), satisfying to (1) and conditions:

\[\exists l > 0, \omega \in \mathbb{R}: \left| \frac{d^k}{d\lambda^k} \left( \frac{R(\lambda)}{\lambda^n} \right) \right| \leq \frac{Lk!}{(\lambda - \omega)^{k+1}}, k = 0, 1, \ldots\]

(For case \(n = 0\)(5) are the conditions of Miyadera-Feller-Phillips-Hille-Yosida, or MFPHY-conditions).


**Theorem (Arendt-Widder).** Let \(n \in \{0\} \cup \mathbb{N}\), \(R(\lambda) : (\omega, \infty) \rightarrow X\). The condition (5) is equivalent to existence of a function \(V(t) : [0, \infty) \rightarrow X\), satisfying \(V(0) = 0\), and

\[\lim_{\delta \rightarrow 0} \sup_{h \leq \delta} h^{-1}||V(t + h) - V(t)|| \leq Le^{-\omega t}, \ t \geq 0,\]

such that

\[R(\lambda) = \int_0^\infty \lambda^{n+1}e^{-\lambda t}V(t)dt.\]

Moreover, \(R(\lambda)\) has an analytic extension to \(\{\lambda \in \mathbb{C} : \text{Re}\lambda > \omega\}\), which is given by (7).

By virtue of this theorem and results on the connection between well-posedness of the Cauchy problem

\[\frac{du(t)}{dt} = Au(t), \ t \geq 0, \ u(0) = x,\]

and existence of semigroup (see, for example [1-4]) we have
Theorem 1. Let \( n \in \{0\} \cup \mathbb{N}, A \in \mathcal{L}(X < X) \). The following statements are equivalent:
(I) For \( R(\lambda) = R_A(\lambda) \) condition (5) is fulfilled;
(II) \( A \) is the generator of \((n + 1)\)-times integrated semigroup with property (6);
(III) \((CP)\) is \((n + 1)\)-well-posed, that is: for any \( x \in D(A^{n+2}) \) unique solution, such that
\[
||u(t)|| \leq Le^{\omega t}||x||_{n+1}, \quad ||x||_{n+1} = \sum_{i=0}^{n+1} ||A^i x||
\]
events. If \( \overline{D(A)} = X \), then \((CP)\) \( n \)–well-posed.

Existence of a resolvent in \( \Omega = \{ \lambda \in \mathbb{C} : \text{Re} \lambda > \omega, |\text{Im} \lambda| < L \exp(\eta/n \text{Re} \lambda) \} \) and fulfilment in \( \Omega \) of the estimates (MFPHY)-type are equivalent to existence of a local \( n \)-times integrated semigroup \( \{V(t), 0 \leq t < T \}, T > \eta \) [5, 6]. If for \( R(\lambda) > \omega \) only operator \((A - \lambda)^{-1} C\), called by \( C \)-resolvent \((C \in \mathcal{B}(X)\) is an invertible operator) satisfies to conditions:
\[
||(A - \lambda)^{-k} C|| \leq L(R(\lambda) - \omega)^{-k}, \quad k = 0, 1, \ldots,
\]
then \( C \)-semigroup with operator \( A \) exists [7], the Cauchy problem \((CP)\) is only \( C \)-well-posed and for such \((CP)\) regularizator, connected with \( C \)-semigroups, may be constructed [8].

3. Let now \( R(\lambda) \) be a pseudoresolvent, such that \( ||\lambda R(\lambda)|| \) is bounded. Then like [1, section VIII], the following proposition on construction of \( \ker R \) and \( \text{range} \ R \) may be proved.

Proposition 1. For any \( x \in X_1 := \overline{\Pi}, \lambda R(\lambda) x \to_{\lambda \to \infty} x, K \cap X_1 = \{0\}, X_1 \oplus K = X_1 \oplus \mathcal{K} \) is the subspace in \( X \). For a reflexive space \( X, \) \( X = X_1 \oplus \mathcal{K}. \)

If \( R(\lambda) \) is a pseudoresolvent with MFPHY-conditions, then by Arendt-Widder theorem it generates 1-time integrated (degenerate) semigroup \( V(t) \) with property (6), such that
\[
R(\lambda) = \int_0^\infty e^{-\lambda t} V(t) dt.
\]
For \( x \in F := \{ x \in X : V(t)x \in C^1([0, \infty)), X \} \) we have
\[
R(\lambda) x = \int_0^\infty e^{-\lambda t} V'(t) x dt,
\]
hence \( U(t)x := V'(t)x, x \in F, \) satisfies (U1). Due to (6) set \( F \) is closed, by definition \( F \) \( U(t) \) is strongly continuous on \( F, \) we call it by degenerate \( C_0 \)-semigroup. From semigroup property (U1) we have the projector
\[
P := U^2(0) = U(0) : F \to Q := \text{range} \ U(0)
\]
and the decomposition of \( F \) into the direct sum:
\[
F = Q \oplus \ker(U(0)) = Q \oplus \mathcal{K}.
\]
By Definition 1 \( \tilde{U}(t) = U(t)|_Q \) is \( C_0 \)-semigroup with the generator \( G : \)
\[
Gx = \lim_{t \to 0} \frac{\tilde{U}(t)x - x}{t}, \quad x \in D(G), \quad \overline{D(G)} = Q.
\]
4. For the case of a pseudoresolvent, as distinct from the case when $R^{-1}(\lambda)$ exists, we can’t connect $G$ with operator, defined by (2) and the Cauchy problem (CP). We show, that $R(\lambda)$ is a pseudoresolvent with MFPHY-conditions, then for any pair of operators $B, A : X \to E$, $E$ – a Banach space, such that $B \in \mathcal{B}(X, E)$, $B$ is invertible on $X_1$ and $Ax = BGx$, $x \in D(G)$, the degenerate Cauchy problem

\[(DCP) \quad B \frac{du(t)}{dt} = Au(t), \quad t \geq 0, \quad u(0) = x, \quad \ker B \neq \{0\},\]

is well-posed on $\mathcal{R}$.

**Definition 4.** Let $A, B \in \mathcal{L}(X, E)$. $(DCP)$ is called uniformly well-posed on $D \subseteq X$, if for any $x \in D$ a solution exists, is unique and

$$\forall \, T > 0 \, \exists L > 0 : \sup_{t \in [0, T]} ||u(t)|| \leq L||x||.$$ 

It is easily seen that $D \subseteq \mathcal{M} = \{x \in D(A) : Ax \in \text{range}B\}$.

**Proposition 2.** Let $A, B \in \mathcal{L}(X, E)$ such, that operator $(\lambda B - A)^{-1}B$ is bounded, $\lambda \in \Omega \subset C$. Then $(\lambda B - A)^{-1}B$ satisfies the Hilbert identity and

$$\mathcal{R} = \text{range}(\lambda B - A)^{-1}B = \mathcal{M}.$$ 

**Proof.** The proof of the resolvent identity is routine. We show $\mathcal{R} = \mathcal{M}$. Let $x \in \mathcal{R}$, then $x = (\lambda B - A)^{-1}By$, $y \in X$, hence $Ax = B(\lambda x - y)$, and $x \in \mathcal{M}$. Conversely, if $Ax = By$ for some $y \in X$, then $(\lambda B - A)x = B(\lambda x - y)$ and $x \in \mathcal{R}$.  

**Definition 5.** Let $X, E$ are Banach spaces. $A, B \in \mathcal{L}(X, E)$ are called generators of degenerate $n$-times integrated semigroup $\{V(t), t \geq 0\} \in \mathcal{B}(X, X)$ if $A$ is closed, $B$ – bounded and

$$R_{A, B}(\lambda) := (\lambda B - A)^{-1}B = \int_0^\infty \lambda^n e^{-\lambda t}V(t)dt, \quad \text{Re} \lambda > \omega.$$ 

**Theorem 2.** Let $A, B$ be generators of degenerate (1-time) integrated semigroup $V(t)$, satisfying (6), then

(11) \quad $$R_{A, B}(\lambda)V(t) = V(t)R_{A, B}(\lambda), \quad t \geq 0, \quad \text{Re} \lambda > \omega;$$

(12) \quad $$tx = BV(t)x - A \int_0^t V(s)xdx, \quad x \in X_1 \oplus \mathcal{K}, \quad \mathcal{K} = \ker B;$$

(13) \quad $$B \frac{d}{dt}V'(t)x = AV'(t)x, \quad x \in R(\lambda)(X_1).$$

$V'(t)$ is degenerate $C_0$-semigroup on $F = X_1 \oplus \mathcal{K}$.

**Proof.** Let $\lambda, \mu > \omega$, as $R(\lambda) = R_{A, B}(\lambda)$ is a pseudoresolvent, for any $x \in X$

$$\int_0^\infty e^{-\mu t}V(t)R(\lambda)xdt = R(\mu)R(\lambda)x =$$

$$= R(\lambda)R(\mu)x = \int_0^\infty e^{-\mu t}R(\lambda)V(t)xdt$$

and hence by uniqueness of the Laplace transform we have (11).
Let \( x \in X \), \( \text{Re}\lambda > \omega \), then
\[
\int_0^\infty \lambda^2 e^{-\lambda t} tB x dt = Bx = (\lambda B - A)R(\lambda)x = \int_0^\infty \lambda^2 e^{-\lambda t} BV(t)x dt - A \int_0^\infty \lambda e^{-\lambda t} V(t)x dt.
\]

Let now \( x \in \mathcal{R} \), \( x = R(\lambda_0) y \), \( y \in X \), then
\[
||AV(t)x|| = ||AV(t)R(\lambda_0)y|| = ||AR(\lambda_0)V(t)y|| = \lambda_0||BR(\lambda)V(t)y - BV(t)y|| \leq L||e^{\omega t}||\lambda_0||x|| + ||y||,
\]
that means the Laplace transform of \( AV(t)x \) is defined as \( A \) is closed we have
\[
\int_0^\infty \lambda^2 e^{-\lambda t} tB x dt = \int_0^\infty \lambda^2 e^{-\lambda t}[BV(t)x - \int_0^t AV(s)x ds]dt
\]
and
\[
tBx = BV(t)x - A \int_0^t AV(s)x ds, \quad x \in \mathcal{R}.
\]
This equality is true for \( x \in \overline{\mathcal{R}} \) and \( x \in \ker B \) too, hence we have (12).

It was shown above, that \( R(\lambda) \) in (8) satisfies MPHY-conditions then \( U(t) = V'(t) \)
is degenerate \( C_0 \)-semigroup on \( F = Q \oplus K \) and \( C_0 \)-semigroup on \( Q \). For the generator of
such semigroup \( D(G) \) is dense in \( Q \), hence \( F_1 := \{x \in X : \forall t \geq 0 \exists U'(t)x\} \) is dense in
\( Q \) and since \( \ker B = K \subset F_1 \) we have \( \overline{F_1} = F \). For \( x \in F_1 \)
\[
\lambda R(\lambda)x = \int_0^\infty \lambda e^{-\lambda t} U(t)x dt = U(0)x + \int_0^\infty \lambda e^{-\lambda t} U'(t)x dt,
\]
and
\[
\lim_{\lambda \to \infty} \lambda R(\lambda)x = U(0)x.
\]
Since operators \( \lambda R(\lambda) \) are bounded, (14) is true for \( x \in F \). Hence by Proposition 1
\( Q \subset X_1 \). Let \( x \in R(\lambda)(X_1) := \mathcal{R}_1 \), \( x = R(\lambda)y \), \( y \in X_1 \), we show (13) and \( X_1 \subset Q \), hence
\( X_1 = Q \). For \( y \in X_1 \) (12) is true, we apply operator \( (\lambda B - A)^{-1} \) to (12), having used
(11) and the equality
\[
(\lambda B - A)^{-1}Ay = \lambda(\lambda B - A)^{-1}By - y, \quad y \in D(A),
\]
we obtain
\[
\int \lambda^2 e^{-\lambda t} BV(t)x dt - A \int_0^t AV(s)x ds,
\]
and
\[
\lim_{\lambda \to \infty} \lambda R(\lambda)x = U(0)x.
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\]
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\int \lambda^2 e^{-\lambda t} BV(t)x dt - A \int_0^t AV(s)x ds,
\]
and
\[
\lim_{\lambda \to \infty} \lambda R(\lambda)x = U(0)x.
\]
5. In view of Proposition 2, set $\mathcal{R}$ coincides with maximal well-posedness class for (DCP): $\mathcal{R} = \mathcal{M}$. We establish conditions on the pseudoresolvent, connected with operators $A$, $B$, which assure (DCP) well-posedness on $\mathcal{R}$.

**Theorem 4.** Let $A, B \in \mathcal{L}(X, E)$, $A$ is closed, $B$ and $R(\lambda) = (\lambda B - A)^{-1}B$ for some $\lambda$ are bounded. Then the following statements are equivalent.

(I) (DCP) is uniformly well-posed on $\mathcal{R}$.

(II) $A$, $B$ are the generators of a degenerate $C_0$-semigroup.

(III) For $R(\lambda)$ MFPHY-conditions are fulfilled and $X = K \oplus R$.

**Proof.** (I)$\Rightarrow$(II). Define on $\mathcal{R}$ operators $\tilde{U}(t)$ as solution operators: for $x \in \mathcal{R}$ $\tilde{U}(t)x := u(t)$. Similarly to the nondegenerate case the operators $\tilde{U}(t)$ form a semigroup on $X_1 = \mathcal{R}$ and satisfy the equality

$$
(\lambda B - A)^{-1} \int_0^\infty \lambda^2 e^{-\lambda t} \tilde{U}(t)x dt = Bx, \quad x \in X_1.
$$

Operator $(\lambda B - A)$ is invertible for $\text{Re} \lambda > \omega$: really, let $x \in \ker(\lambda B - A)$, then for $v(t) := \exp(\lambda t)x$ we have

$$
Bv'(t) = \lambda Bv(t) = Av(t), \quad v(0) = x,
$$

and $||v(t)|| = ||\tilde{U}(t)|| \leq L e^{\omega t}||x||$, $t \geq 0$. On the one hand $\ln ||v(t)||/t = \text{Re} \lambda + \ln ||x||/t$, on the other $\ln ||v(t)||/t \leq \omega + \ln L ||x||/t$, and $\text{Re} \lambda \leq \omega$. For generator of $\tilde{U}$ from (16) follows:

$$
(\lambda B - A)^{-1}Bx = (\lambda - G)^{-1}x, \quad x \in X_1, \quad \text{Re} \lambda > \omega,
$$

operator $P : X \to X_1$ is projector in $X$ and $U(t) := \tilde{U}(t)P$ is degenerate $C_0$-semigroup.

(II)$\Rightarrow$(III). It is not difficult to verify, that $A$, $B$ are the generators of integrated semigroup $V(t) := \int_0^t \tilde{U}(t)dt$, satisfying (6). Hence by (8) MFPHY-conditions for $R(\lambda)$ are fulfilled and by Theorem 2 $X = F = K \oplus X_1$.

(III)$\Rightarrow$(I). If (III), then $R(\lambda)(X_1) = R(\lambda)(X) = \mathcal{R}$ and by Theorem 3 (I) follows.

In [6] some results on connection between properties of a pseudoresolvent and well-posedness of the differential inclusion: $u'(t) \in Ju(t), \quad T \geq 0, \quad u(0) = x$, where $J := B^{-1}A$ with $D(J) = \mathcal{M}$, are obtained by the technique of multivalued operators.

**References**


