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PROPERTIES OF AN ABSTRACT PSEUDORESOLVENT AND WELL-POSEDNESS OF THE DEGENERATE CAUCHY PROBLEM

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Abstract. The degenerate Cauchy problem in a Banach space is studied on the basis of properties of an abstract analytical function, satisfying the Hilbert identity, and a related pair of operators A, B.

1. We consider in a Banach space X an operator-valued function of complex variable $R(\lambda) \in \mathcal{B}(X)$, satisfying the Hilbert identity:

$$\forall\, x \in X \quad R(\lambda)R(\mu)x = \frac{R(\mu) - R(\lambda)}{\lambda - \mu}x, \quad \lambda, \mu \in \Omega \subset \mathbb{C}. \tag{1}$$

For such function $\ker R(\lambda) =: \mathcal{K}$ and range $R(\lambda) =: \mathcal{R}$ do not depend on λ [1]. If $\mathcal{K} = \{0\}$, then the function $R(\lambda)$ is called by resolvent, if $\mathcal{K} \neq \{0\}$ - pseudoresolvent. For the case when $R(\lambda)$ is resolvent, from (1) the equality follows:

$$\lambda - R^{-1}(\lambda) = \mu - R^{-1}(\mu) =: A, \tag{2}$$

$D(A) = \mathcal{R}$ and $R(\lambda) = R_A(\lambda)$.

Let $V(t)$ be an exponentially bounded operator-function ($||V(t)|| \le Le^{\omega t}$) and

$$r_n(\lambda) := \int_0^\infty \lambda^n e^{-\lambda t} V(t)dt, \tag{3}$$

(the integral exists in the Bochner sense). In [2] it was proved, that $r_0(\lambda)$ satisfies (1) for $\mathrm{Re}\lambda > \omega$, iff $V(t)$ satisfies the semigroup relation:

$$V(t+s) = V(t)V(s), \qquad t, s \ge 0, \tag{4}$$

$r_n(\lambda)$, $n \in \mathbb{N}$, satisfies (1); iff

$$\frac{1}{(n-1)!}\int_0^s [(s-r)^{n-1}V(t+r) - (t+s-r)^{n-1}V(r)]dr = \tag{V1}$$

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$$= V(t)v(s), \qquad s,t \geq 0;\ V(0) = 0.$$

Relation (4) is taken as the basis in the definition of continuous semigroups (see, for example, [1], [3], [4]), relation (V1) – in the definition of n-times integrated semigroups [2].

Definition 1. A one-parameter family of bounded operators $\{U(t),\ t \geq 0\}$ is called *strongly continuous semigroup* (or C_0*-semigroup*) if the following conditions hold:
(U1) $U(t+h) = U(t)U(h), \quad t,h \geq 0$;
(U2) $U(0) = f$;
(U3) $U(t)$ is strongly continuous with respect to $t \geq 0$.

Definition 2. Let $n \in \mathbb{N}$. A one-parameter family of bounded linear operators $\{V(t), t \geq 0\}$ is called *n-times integrated exponentially bounded semigroup* if (V1) and the following conditions hold
(V2) $V(t)$ is strongly continuous with respect to $t \geq 0$;
(V3) $\exists\ K > 0, \omega \in \mathbb{R} : ||V(t)|| \leq \lambda \exp(\omega t), t \geq 0$.

Definition 3. Semigroup $\{V(t), t \geq 0\}$ is called *nondegenerate* if
(V4) $\forall\ t \geq 0 V(t)x = 0 \Longrightarrow x = 0$.

A C_0-semigroup is called 0-*times integrated semigroup.* Operator $A = \lambda - R^{-1}(\lambda)$, $D(A) = \mathcal{R}$ is called the *generator* of semigroup.

So, an exponentially bounded semigroup by the Laplace transform definies the function $R(\lambda)$, satisfying to (1) and conditions:

$$\exists\ l > 0,\ \omega \in \mathbb{R} : \Big| |\frac{d^k}{d\lambda^k}\Big(\frac{R(\lambda)}{\lambda^n}\Big)\Big| | \leq \frac{Lk!}{(\lambda-\omega)^{k+1}}, \quad k = 0, 1, \ldots \tag{5}$$

(For case $n = 0$(5) are the conditions of Miyadera-Feller-Phillips-Hille-Yosida, or MFPHY-conditions).

2. Arendt [2] extended on arbitrary Banach space X the abstract criterion for Laplace transform.

Theorem (Arendt-Widder). *Let $n \in \{0\} \cup \mathbb{N}$, $R(\lambda) : (\omega, \infty) \to X$. The condition* (5) *is equivalent to existence of a function $V(t) : [0, \infty) \to X$, satisfying $V(0) = 0$, and*

$$\lim_{\delta \to 0} \sup_{h \leq \delta} h^{-1} ||V(t+h) - V(t)|| \leq Le^{\omega t}, \quad t \geq 0, \tag{6}$$

such, that

$$R(\lambda) = \int_0^\infty \lambda^{n+1} e^{-\lambda t} V(t) dt. \tag{7}$$

Moreover, $R(\lambda)$ has an analytic extention to $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \omega\}$, which is given by (7).

By virtue of this theorem and results on the connection between well-posedness of the Cauchy problem

$$\frac{du(t)}{dt} = Au(t),\ t \geq 0,\ u(0) = x, \tag{CP}$$

and existence of semigroup (see, for example [1-4]) we have

THEOREM 1. *Let* $n\in\{0\}\cup\mathbb{N}$, $A\in\mathcal{L}(X<X)$. *The following statements are equivalent*:

(I) *For* $R(\lambda)=R_A(\lambda)$ *condition* (5) *is fulfilled*;

(II) A *is the generator of* $(n+1)$*-times integrated semigroup with property* (6);

(III) (CP) *is* $(n+1)$*-well-posed*, *that is*: *for any* $x\in D(A^{n+2})$ *unique solution*, *such that*

$$||u(t)||\leq Le^{\omega t}||x||_{n+1},\qquad ||x||_{n+1}=\sum_{i=0}^{n+1}||A^i x||$$

exists. If $\overline{D(A)}=X$, *then* (CP) n*–well-posed.*

Existence of a resolvent in $\Omega=\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda>\omega,\ |\mathrm{Im}\lambda|<L\exp(r/n\mathrm{Re}\lambda)\}$ and fulfilment in Ω of the estimates (MFPHY)-type are equivalent to existence of a local n-times integrated semigroup $\{V(t),0\leq t<T\}$, $T>r$ [5], [6]. If for $R(\lambda)>\omega$ only operator $(A-\lambda)^{-1}C$, called by C-resolvent ($C\in\mathcal{B}(X)$ is an invertible operator) satisfies to conditions:

$$||(\lambda-A)^{-k}C||\leq L(R(\lambda)-\omega)^{-k},\qquad k=0,1,\dots,$$

then C-semigroup with operator A exists [7], the Cauchy problem (CP) is only C-well-posed and for such (CP) regularizator, connected with C-semigroups, may be constructed [8].

3. Let now $R(\lambda)$ be a pseudoresolvent, such that $||\lambda R(\lambda)||$ is bounded. Then like [1, section VIII], the following proposition on construction of $\ker R$ and *range* R may be proved.

PROPOSITION 1. *For any* $x\in X_1:=\overline{R}$, $\lambda R(\lambda)x\to_{\lambda\to\infty}x$, $\mathcal{K}\cap X_1=\{0\}$, $X_1\oplus\mathcal{K}=\overline{X_1\oplus\mathcal{K}}$ *is the subspace in* X. *For a reflexive space* X, $X=X_1\oplus\mathcal{K}$.

If $R(\lambda)$ is a pseudoresolvent with MFPHY-conditions, then by Arendt-Widder theorem it generates 1-time integrated (degenerate) semigroup $V(t)$ with property (6), such that

$$R(\lambda)=\int_0^\infty \lambda e^{-\lambda t}V(t)dt. \tag{8}$$

For $x\in F:=\{x\in X:V(t)x\in C^1([0,\infty),X)\}$ we have

$$R(\lambda)x=\int_0^\infty e^{-\lambda t}V'(t)xdt,$$

hence $U(t)x:=V'(t)x$, $x\in F$, satisfies (U1). Due to (6) set F is closed, by definition F $U(t)$ is strongly continuous on F, we call it by degenerate C_0-semigroup. From semigroup property (U1) we have the projector

$$P:=U^2(0)=U(0):F\to Q:=\mathrm{range}U(0)$$

and the decomposition of F into the direct sum:

$$F=Q\oplus\ker(U(0))=Q\oplus\mathcal{K}. \tag{9}$$

By Definition 1 $\tilde{U}(t)=U(t)|_Q$ is C_0-semigroup with the generator $G:$

$$Gx=\lim_{t\to0}\frac{\tilde{U}(t)x-x}{t},\qquad x\in D(G),\quad \overline{D(G)}=Q. \tag{10}$$

4. For the case of a pseudoresolvent, as distinct from the case when $R^{-1}(\lambda)$ exists, we can't connect G with operator, defined by (2) and the Cauchy problem (CP). We show, that $R(\lambda)$ is a pseudoresolvent with MFPHY-conditions, then for any pair of operators $B, A : X \to E$, E − a Banach space, such that $B \in \mathcal{B}(X, E)$, B is invertible on X_1 and $Ax = BGx$, $x \in D(G)$, the degenerate Cauchy problem

$$(DCP) \qquad B\frac{du(t)}{dt} = Au(t), \quad t \geq 0, \quad u(0) = x, \quad \ker B \neq= \{0\},$$

is well-posed on $\mathcal{R}$.

Definition 4. Let $A, B \in \mathcal{L}(X, E)$. (DCP) is called uniformly well-posed on $D \subseteq X$, if for any $x \in D$ a solution exists, is unique and

$$\forall\, T > 0\ \exists L > 0 : \sup_{t\in[0,T]} ||u(t)|| \leq L||x||.$$

It is easily seen that $D \subseteq \mathcal{M} = \{x \in D(A) : Ax \in \mathit{range}B\}$.

Proposition 2. *Let $A, B \in \mathcal{L}(X, E)$ such, that operator $(\lambda B - A)^{-1}B$ is bounded, $\lambda \in \Omega \subset C$. Then $(\lambda B - A)^{-1}B$ satisfies the Hilbert identity and*

$$\mathcal{R} = \mathit{range}(\lambda B - A)^{-1}B = \mathcal{M}.$$

Proof. The proof of the resolvent identity is routine. We show $\mathcal{R} = \mathcal{M}$. Let $x \in \mathcal{R}$, then $x = (\lambda B - A)^{-1}By$, $y \in X$, hence $Ax = B(\lambda x - y)$, and $x \in \mathcal{M}$. Conversely, if $Ax = By$ for some $y \in X$, then $(\lambda B - A)x = B(\lambda x - y)$ and $x \in \mathcal{R}$. ■

Definition 5. Let X, E are Banach spaces. $A, B \in \mathcal{L}(X, E)$ are called generators of degenerate n-times integrated semigroup $\{V(t), t \geq 0\} \in \mathcal{B}(X, X)$ if A is closed, B − bounded and

$$R_{A,B}(\lambda) := (\lambda B - A)^{-1}B = \int_0^\infty \lambda^n e^{-\lambda t} V(t)dt, \qquad \mathrm{Re}\lambda > \omega.$$

Theorem 2. *Let A, B be generators of degenerate* (1*-time*) *integrated semigroup $V(t)$, satisfying* (6), *then*

$$R_{A,B}(\lambda)V(t) = V(t)R_{A,B}(\lambda), \qquad t \geq 0,\ \mathrm{Re}\lambda > \omega; \tag{11}$$

$$tBx = BV(t)x - A\int_0^t V(s)xds, \qquad x \in X_1 \oplus \mathcal{K},\ \mathcal{K} = \ker B; \tag{12}$$

$$B\frac{d}{dt}V'(t)x = AV'(t)x, \qquad x \in R(\lambda)(X_1). \tag{13}$$

$V'(t)$ is degenerate C_0-semigroup on $F = X_1 \oplus \mathcal{K}$.

Proof. Let $\lambda, \mu > \omega$, as $R(\lambda) = R_{A,B}(\lambda)$ is a pseudoresolvent, for any $x \in X$

$$\int_0^\infty \mu e^{-\mu t} V(t)R(\lambda)xdt = R(\mu)R(\lambda)x =$$
$$= R(\lambda)R(\mu)x = \int_0^\infty \mu e^{-\mu t} R(\lambda)V(t)xdt$$

and hence by uniqueness of the Laplace transform we have (11).

Let $x \in X$, $\mathrm{Re}\lambda > \omega$, then

$$\int_0^\infty \lambda^2 e^{-\lambda t} tBxdt = Bx = (\lambda B - A)R(\lambda)x =$$
$$\int_0^\infty \lambda^2 e^{-\lambda t} BV(t)xdt - A\int_0^\infty \lambda e^{-\lambda t} V(t)xdt.$$

Let now $x \in \mathcal{R}$, $x = R(\lambda_0)y$, $y \in X$, then

$$||AV(t)x|| = ||AV(t)R(\lambda_0)y|| = ||AR(\lambda_0)V(t)y|| =$$
$$= ||\lambda_0 BR(\lambda)V(t)y - BV(t)y|| \le L||B||e^{\omega t}(|\lambda_0|||x|| + ||y||),$$

that means the Laplace transform of $AV(t)x$ is defined as A is closed we have

$$\int_0^\infty \lambda^2 e^{-\lambda t} tBxdt = \int_0^\infty \lambda^2 e^{-\lambda t}[BV(t)x - \int_0^t AV(s)xds]dt$$

and

$$tBx = BV(t)x - A\int_0^t AV(s)xds, \qquad x \in \mathcal{R}.$$

This equality is true for $x \in \overline{\mathcal{R}}$ and $x \in \ker B$ too, hence we have (12).

It was shown above, that $R(\lambda)$ in (8) satisfies MFPHY-conditions then $U(t) = V'(t)$ is degenerate C_0-semigroup on $F = Q \oplus \mathcal{K}$ and C_0-semigroup on Q. For the generator of such semigroup $D(G)$ is dense in Q, hence $F_1 := \{x \in X : \forall\, t \ge 0\ \exists\, U'(t)x\}$ is dense in Q and since $\ker B = \mathcal{K} \subset F_1$ we have $\overline{F_1} = F$. For $x \in F_1$

$$\lambda R(\lambda)x = \int_0^\infty \lambda e^{-\lambda t} U(t)xdt = U(0)x + \int_0^\infty \lambda e^{-\lambda t} U'(t)xdt,$$

and

$$\lim_{\lambda\to\infty} \lambda R(\lambda)x = U(0)x. \tag{14}$$

Since operators $\lambda R(\lambda)$ are bounded, (14) is true for $x \in F$. Hence by Proposition 1 $Q \subset X_1$. Let $x \in R(\lambda)(X_1) =: \mathcal{R}_1$, $x = R(\lambda)y$, $y \in X_1$, we show (13) and $X_1 \subset Q$, hence $X_1 = Q$. For $y \in X_1$ (12) is true, we apply operator $(\lambda B - A)^{-1}$ to (12), having used (11) and the equality

$$(\lambda B - A)^{-1}Ay = \lambda(\lambda B - A)^{-1}By - y, \qquad y \in D(A),$$

we obtain

$$tx = V(t)x - \int_0^t V(s)(\lambda(\lambda B - A)^{-1}B - I)yds, \tag{15}$$

hence $x \in F$ and $\mathcal{R}_1 \subset F$. By Proposition 1 for $x \in \mathcal{R}$ $\lambda R(\lambda)x \to_{\lambda\to\infty} x$, hence $\overline{\mathcal{R}_1} = X_1$ and $X_1 \subset \overline{F} = F$, as $X_1 \cap \mathcal{K} = \{0\}$, $X_1 \subset Q$, $F = X_1 \oplus \mathcal{K}$.

By differentiating (15), applying operator B and differentiating once more we obtain (13). ∎

THEOREM 3. *Let* $A, B \in \mathcal{L}(X, E)$ *is closed*, $B \in \mathcal{B}(X, E)$ *and* $R(\lambda) = (\lambda B - A)^{-1}B$ *satisfies to MFPHY-conditions*, *then* (DCP) *is well-posed on* $\mathcal{R}_1 = R(\lambda)(X_1)$.

P r o o f. Existence of the solution $u(t) = U(t)x$, $x \in \mathcal{R}_1$, follows from Theorem 2, uniqueness may be proved as in the nondegenerate case (see [3], [4]). ∎

5. In view of Proposition 2, set $\mathcal{R}$ coincides with maximal well-posedness class for (DCP): $\mathcal{R} = \mathcal{M}$. We establish conditions on the pseudoresolvent, connected with operators A, B, which assure (DCP) well-posedness on $\mathcal{R}$.

THEOREM 4. *Let $A, B \in \mathcal{L}(X, E)$, A is closed, B and $R(\lambda) = (\lambda B - A)^{-1}B$ for some λ are bounded. Then the following statements are equivalent.*

(I) (DCP) *is uniformly well-posed on $\mathcal{R}$.*

(II) *A, B are the generators of a degenerate C_0-semigroup.*

(III) *For $R(\lambda)$ MFPHY-conditions are fulfilled and $X = \mathcal{K} \oplus \mathcal{R}$.*

Proof. (I)$\Longrightarrow$(II). Define on $\mathcal{R}$ operators $\tilde{U}(t)$ as solution operators: for $x \in \mathcal{R}$ $\tilde{U}(t)x := u(t)$. Similarly to the nondegenerate case the operators $\tilde{U}(t)$ form a semigroup on $X_1 = \overline{\mathcal{R}}$ and satisfy the equality

$$(\lambda B - A)^{-1} \int_0^\infty \lambda^2 e^{-\lambda t} \tilde{U}(t) x dt = Bx, \qquad x \in X_1. \tag{16}$$

Operator $(\lambda B - A)$ is invertible for $\mathrm{Re}\lambda > \omega$: really, let $x \in \ker(\lambda B - A)$, then for $v(t) := \exp(\lambda t)x$ we have

$$Bv'(t) = \lambda Bv(t) = Av(t), \qquad v(0) = x,$$

and $||v(t)|| = ||\tilde{U}(t)|| \leq Le^{\omega t}||x||$, $t \geq 0$. On the one hand $\ln ||v(t)||/t = \mathrm{Re}\lambda + \ln ||x||/t$, on the other $\ln ||v(t)/t \leq \omega + \ln L||x||/t$, and $\mathrm{Re}\lambda \leq \omega$. For generator of $\tilde{U}$ from (16) follows:

$$(\lambda B - A)^{-1}Bx = (\lambda - G)^{-1}x, \qquad x \in X_1, \quad \mathrm{Re}\lambda > \omega,$$

operator $Px := (\lambda - G)(\lambda_0 B - A)^{-1}Bx$, $P : X \to X_1$ is projector in X and $U(t) := \tilde{U}(t)P$ is degenerate C_0-semigroup.

(II)$\Longrightarrow$(III). It is not difficult to verify, that A, B are the generators of integrated semigroup $V(t) := \int_0^t U(t)dt$, satisfying (6). Hence by (8) MFPHY-conditions for $R(\lambda)$ are fulfilled and by Theorem 2 $X = F = \mathcal{K} \oplus X_1$.

(III)$\Longrightarrow$(I). If (III), then $R(\lambda)(X_1) = R(\lambda)(X) = \mathcal{R}$ and by Theorem 3 (I) follows. ∎

In [6] some results on connection between properties of a pseudoresolvent and well-posedness of the differential inclusion: $u'(t) \in Ju(t)$, $T \geq 0$, $u(0) = x$, where $J := B^{-1}A$ with $D(J) = \mathcal{M}$, are obtained by the technique of multivalued operators.

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