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SUBORDINATION THEORY FOR HOLOMORPHIC MAPPINGS OF SEVERAL COMPLEX VARIABLES

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Abstract. The authors obtain a generalization of Jack–Miller–Mocanu's lemma and, using the technique of subordinations, deduce some properties of holomorphic mappings from the unit polydisc in $\mathbb{C}^n$ into $\mathbb{C}^n$.

1. Introduction. Let n be a positive integer and $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean product $\langle z, w\rangle = \sum_{k=1}^n z_k \overline{w}_k$ and the Euclidean norm $|z| = \langle z, z\rangle^{1/2}$. Let U_1^n denote the unit polydisc in $\mathbb{C}^n$, i.e. the set $\{z \in \mathbb{C}^n : \|z\| < 1\}$, where $\|z\| = \max_{1\le j\le n} |z_j|$, and let B_1^n stand for the open unit Euclidean ball in $\mathbb{C}^n$. For $n = 1$, $B_1^n = U_1^n = U = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc in $\mathbb{C}$.

Recently the present authors [3] have obtained a new generalization of the Jack-Miller-Mocanu lemma and, using the technique of subordinations, arrived at some properties of holomorphic mappings defined on the unit polydisc U_1^n. In this paper one deduces other results concerning partial differential subordinations and some inequalities for holomorphic mappings on U_1^n.

Let Ω be a domain in $\mathbb{C}^n$ and let $H(\Omega)$ be the set of holomorphic mappings on Ω. If $f \in H(\Omega)$, denote by $[Df(z)]$, $z \in \Omega$, its Fréchet matrix $[(\partial/\partial z_j) f_k(z)]_{j,k=1,\ldots,n}$ and by $[Df(z)]'$ its transpose. Also, if F is a holomorphic function defined on a domain $D \subseteq \mathbb{C}^n$, then by $(\partial/\partial z)F$ we denote the complex vector $((\partial/\partial z_1)F, \ldots, (\partial/\partial z_n)F)$ and by $[(\partial/\partial z)F]'$ its transpose. (If $z \in \mathbb{C}^n$, then $[z]'$ means the transpose of z.) Since $(\mathbb{C}^n, |\cdot|)$ is a normed space with respect to the Euclidean norm, if $A : \mathbb{C}^n \to \mathbb{C}^n$ is a continuous and linear operator, then by $|A|$ we denote the norm of A, i.e., $|A| = \sup_{|u|=1} |Au|$. For our purpose we shall use the following result.

LEMMA 1.1 [2]. *Let* $0 < r_0 < 1$ *and* $h : r_o\overline{U}_1^n \to \mathbb{C}$ *be a holomorphic function on* $r_o\overline{U}_1^n$ *with* $h(0) = 0$. *If* $z_0 \in r_o\overline{U}_1^n$ *and* $|h(z_0)| = \max\{|h(z)| : z \in r_o\overline{U}_1^n\}$, *then at* $z = z_0$ *we*

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have

$$z_0[\frac{\partial h(z_0)}{\partial z}]' = mh(z_0) \quad \text{and} \quad \mathrm{Re}\{\frac{z_0}{h(z_0)}\frac{\partial^2 h(z_0)}{\partial z^2}[z_0]'\} \geq m(m-1), \quad \text{where} \quad m \geq 1.$$

2. Main results. We start with

THEOREM 2.1. *Let $f : U_1^n \to \mathbb{C}^n$ be a holomorphic mapping on U_1^n such that $f(0) = 0$ and $f(z) \not\equiv 0$, $z \in U_1^n$. Let $0 < r_0 < 1$ and $z_0 \in r_o\overline{U}_1^n$ be such that*

$$|f(z_0)| = \max\{|f(z)| : z \in r_o\overline{U}_1^n\}. \tag{1}$$

Then there exist real numbers m and s such that $s \geq m \geq 1$, and at $z = z_0$ we have

(i) $\langle [Df(z_0)][z_0]', f(z_0)\rangle = m|f(z_0)|^2$ *and*

(ii) $\langle |[Df(z_0)][z_0]'| = s|f(z_0)|$.

Proof. Using the hypothesis we can assume $z_0 \neq 0$ and $f(z_0) \neq 0$. Let $g : U_1^n \to \mathbb{C}$, be defined by $g(z) = \langle f(z), f(z_0)\rangle$, $z \in U_1^n$; then g is holomorphic on U_1^n, $g(0) = 0$ and g satisfies $|g(z_0)| = \max\{|g(z)| : ||z|| \leq r_0\}$. From Lemma 1.1 we deduce that there exists $m \in \mathbb{R}$, $m \geq 1$ such that $z_0[(\partial/\partial z)g(z_0)]' = mg(z_0)$. Yet,

$$z_0[\frac{\partial g(z_0)}{\partial z}]' = \sum_{k=1}^{n} z_0^k \frac{\partial g(z_0)}{\partial z_k} = \sum_{k=1}^{n} z_0^k [\sum_{j=1}^{n} \overline{f_j(z_0)} \frac{\partial \dot{f_j}(z_0)}{\partial z_k}] = \langle [Df(z_0)][z_0]', f(z_0)\rangle.$$

Hence we obtain (i). On the other hand, $|\langle [Df(z_0)][z_0]', f(z_0)\rangle| \leq |f(z_0)||[Df(z_0)][z_0]'|$, so from (i) we have $|[Df(z_0)][z_0]'| \geq m|f(z_0)|$, which implies that there exists $s \in \mathbb{R}$, $s \geq m$, such that $|[Df(z_0)][z_0]'| = s|f(z_0)|$.

Remark 2.1. For $n = 1$ we obtain the result of Jack-Miller-Mocanu's lemma [5], [6].

Let M and s be real numbers such that $M > 0$ and $s \geq 1$. Let further $D \subseteq \mathbb{C}^{2n}$ be a domain such that $(0, 0) \in D$.

DEFINITION 2.1. Let $K_n = \cup_{s \geq 1} K_n^s(M)$, where $K_n^s(M) = \{(u, v) \in \mathbb{C}^{2n} : |u| = M, |v| = sM\}$. Suppose that $K_n \subset D$ and let $V_n(D, M) = \{g : D \to \mathbb{C}^n : g$ is continuous on D, $|g(0, 0)| < M$, $|g(u, v)| \geq M$, for all $(u, v) \in K_n\}$.

By using this definition and the result of Theorem 2.1, we deduce

THEOREM 2.2. *Let $D \subseteq \mathbb{C}^{2n}$ be a domain and f be a holomorphic mapping from U_1^n into $\mathbb{C}^n$ such that $f(0) = 0$ and $f(z) \not\equiv 0$, $z \in U_1^n$. Suppose there exists $g \in V_n(D, M)$ such that*

$$(f(z), [Df(z)][z]') \in D \quad \textit{and} \quad |g(f(z), [Df(z)][z]')| < M \qquad \textit{for all} \quad z \in U_1^n.$$

Then $|f(z)| < M$, $z \in U_1^n$.

Proof. If the relation $|f(z)| < M$ does not hold everywhere in U_1^n, then, using the continuity of the norm and $f(0) = 0$, we deduce that there exists $z_0 \in r_0\overline{U}_1^n$, $0 < r_0 < 1$, and $M = |f(z_0)| = \max\{|f(z)| : ||z|| \leq r_0\}$. Then by Theorem 2.1 there exists $s \in \mathbb{R}$, $s \geq 1$, such that at $z = z_0$ we have $|[Df(z_0)][z_0]'| = s|f(z_0)|$. If we set $u = f(z_0)$ and

$v = [Df(z_0)][z_0]'$, then $(u, v) \in K_n^s(M)$. Hence, by $g \in V_n(D, M)$, we have $|g(u, v)| \geq M$, so we obtain a contradiction with the hypothesis. Therefore $|f(z)| < M$ for all $z \in U_1^n$.

Remark 2.2. It is interesting that this result can be applied for proving that some partial differential equations in $\mathbb{C}^n$ have bounded solutions.

COROLLARY 2.1. *Let $F : U_1^n \to \mathbb{C}^n$ be a holomorphic mapping on U_1^n, which satisfies $F(0) = 0$ and $|F(z)| < M$ for all $z \in U_1^n$. Let $g \in V_n(D, M)$ be a holomorphic mapping and suppose that the differential equation $g(f(z),\ [Df(z)][z]') = F(z),\ f(0) = 0$, has on U_1^n a holomorphic solution f. Then $|f(z)| < M$ for all $z \in U_1^n$.*

DEFINITION 2.2. Let $\omega : U_1^n \to \mathbb{C}^n$ be a holomorphic mapping on U_1^n. We say that ω is a *Schwarz mapping* if $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U_1^n$.

DEFINITION 2.3. Let f be a holomorphic mapping from U_1^n into $\mathbb{C}^n$ and g be a holomorphic mapping from B_1^n into $\mathbb{C}^n$. We say that f is *subordinate* to g ($f \prec g$) if there exists a Schwarz mapping ω (in the sense of Definition 2.2) such that $f = g \circ \omega$ in U_1^n.

Remark 2.3. If f is subordinate to g, then $f(0) = g(0)$ and $f(U_1^n) \subseteq g(B_1^n)$. Yet, if g is biholomorphic on B_1^n, then we can easily show that $f \prec g$ if and only if $f(0) = g(0)$ and $f(U_1^n) \subseteq g(B_1^n)$. Also, if $f \prec g$, then $f(r\overline{U}_1^n) \subseteq g(r\overline{B}_1^n)$ for all $0 < r < 1$.

Let Q_n be denote the family of all biholomorphic mappings g on $\overline{B}_1^n \setminus E(g)$, where

$$E(g) = \{\zeta \in \partial B_1^n : \text{there exists } k,\ 1 \leq k \leq n, \text{ with } \lim_{\substack{|z|<1 \\ z \to \zeta}} g_k(z) = \infty\}.$$

In the next (except for some examples) we shall suppose that $E(g) = \emptyset$; in the other case we can use in the proofs the class Q_n.

Now we can give the following result:

THEOREM 2.3. *Let f be a holomorphic mapping on U_1^n and let g be a biholomorphic mapping on $\overline{B}_1^n$ such that $f(0) = g(0)$. If f is not subordinate to g, then there exist real numbers m and s, $s \geq m \geq 1$, and points $z_0 \in U_1^n$, $0 < ||z_0|| < 1$, $\zeta_0 \in \partial B_1^n$, such that*

(i) $f(z_0) = g(\zeta_0)$, $f(\{z : ||z|| < ||z_0||\}) \subset g(B_1^n)$

and at $z = z_0$ we have

(ii) $\sum_{k=1}^n \overline{\zeta}_0^k \cdot z_0[Df(z_0)]'[(\partial/\partial w)\tilde{g}_k(w_0)]' = m$,

(iii) $s|[Dg(\zeta_0)]^{-1}|^{-1} \leq |[Df(z_0)][z_0]'| \leq s|[Dg(\zeta_0)]|$,

where $\zeta_0 = (\zeta_0^1, \ldots, \zeta_0^n)$, $w_0 = g(\zeta_0)$, $g^{-1}(w_0) = (\tilde{g}_1(w_0), \ldots, \tilde{g}_n(w_0))$.

Proof. Since f is not subordinate to g and $f(0) = g(0)$, then $f(U_1^n) \not\subseteq g(B_1^n)$. Hence there exist $z_0 \in U_1^n$, $||z_0|| = r_0$, $0 < r_0 < 1$ and $\zeta_0 \in \partial B_1^n$ such that $f(z_0) = g(\zeta_0)$ and $f(\{z : ||z|| < ||z_0||\}) \subset g(B_1^n)$. Let $h : r_0\overline{U}_1^n \to \mathbb{C}^n$ be the mapping given by $h(z) = (g^{-1} \circ f)(z)$, $z \in r_0U_1^n$. Then h is holomorphic on $r_0U_1^n$, $h(0) = 0$ and $1 = |h(z_0)| = \max\{|h(z)| : ||z|| \leq r_0\}$. By applying the result of Theorem 2.1 and the continuity and linearity of the operators $Dg(\zeta_0)$ and $[Dg(\zeta_0)]^{-1}$ on $(\mathbb{C}^n, |\cdot|)$, we obtain (ii) and (iii), as desired.

For $n = 1$ we deduce

COROLLARY 2.2 [5, 6]. *Let f and g be holomorphic functions on U and g be univalent on $\overline{U}$, such that $f(0) = g(0)$. If f is not subordinate to g, then there exist $z_0 \in U$, $\zeta_0 \in \partial U$ and $m \geq 1$ such that $f(z_0) = g(\zeta_0)$ and $z_0 f'(z_0) = m\zeta_0 g'(\zeta_0)$.*

Now, using the above results, we are able to introduce the concept of "admissible class" for mappings of several variables.

DEFINITION 2.4. Let $D \subseteq \mathbb{C}^n$, $\Omega \subseteq \mathbb{C}^{2n}$ be domains, $n \geq 1$, let g be a biholomorphic mapping on $\overline{B}_1^n$, and $\zeta_0 \in \partial B_1^n$. Suppose that $H_n^s(\zeta_0, g) = \{(u, v) \in \mathbb{C}^{2n} : u = g(\zeta_0), s|[Dg(\zeta_o)]^{-1}|^{-1} \leq |v| \leq s|[Dg(\zeta_0)]|\}$, where s, $s \geq 1$, is a real number. Let further

$$H_n(g) = \bigcup_{\substack{|\zeta_0|=1 \\ s\geq 1}} H_n^s(\zeta_0, g)$$

and suppose $H_n(g) \subset \Omega$ and $(g(0), 0) \in \Omega$. The *admissible class* $\psi_n^n(\Omega, D, g)$ consists of those mappings $\psi_n : \Omega \times U_1^n \to \mathbb{C}^n$ which are continuous and satisfy $\psi_n(g(0), 0; 0) \in D$ and $\psi_n(v, v; z) \notin D$ for all $(u, v) \in H_n(g)$ and $z \in U_1^n$.

Using the conclusion of Theorem 2.3 and the above definition we obtain:

THEOREM 2.4. *Let f be a holomorphic mapping on U_1^n and let g be a biholomorphic mapping on $\overline{B}_1^n$, such that $f(0) = g(0)$. Suppose that there exists $\psi_n^n(\Omega, D; g)$ such that*

$$(f(z), [Df(z)][z]') \in \Omega \quad \text{and} \quad \psi_n(f(z), [Df(z)][z]') \in D \quad \text{for all} \quad z \in U_1^n.$$

Then f is subordinate to g.

Proof. If the subordination $f \prec g$ does not hold, then, by Theorem 2.3, there exist $z_0 \in U_1^n$, $\zeta_0 \in \partial B_1^n$ and $s \in \mathbb{R}$, $s \geq 1$, such that $f(z_0) = g(\zeta_0)$ and the relations (ii) and (iii) hold. Yet, if we define $u = f(z_0)$ and $v = [Df(z_0)][z_0]'$, then $(u, v) \in H_n^s(\zeta_0, g) \subseteq H_n(g)$. Hence, from Definition 2.4, we deduce $\psi_n(u, v; z_0) \notin D$ which is a contradiction with the hypothesis.

3. Examples. In this section we point out the usefulness of the above results.

Let $z \in \mathbb{C}^n$, $z = (z_1, \ldots, z_n)$; then we say that Re $z \geq 0$ (resp. Re $z > 0$) if and only if Re $z_k \geq 0$ (resp. Re $z_k > 0$) for all $k \in \{1, \ldots, n\}$. Let $\tilde{1} = (1, \ldots, 1) \in \mathbb{C}^n$. Consider the mapping $g : U_1^n \to \mathbb{C}^n$, given by

$$g(z) = (\frac{1 + z_1}{1 - z_1}, \ldots, \frac{1 + z_n}{1 - z_n}) \quad \text{for all} \quad z \in U_1^n. \tag{2}$$

Then it is clear that g is univalent on U_1^n and $g(U_1^n) = E_n$, where $E_n = \{w \in \mathbb{C}^n : \text{Re } w > 0\}$. Now, let $A = \{z \in \partial B_1^n : \text{there exists } k, 1 \leq k \leq n \text{ such that} z_k = 1\}$. In this case $E(g) = A$.

Moreover, we denote by $G_n^s(\zeta_0, \tilde{1})$ the class

$$G_n^s(\zeta_0, \tilde{1}) = \{(u, v) \in \mathbb{C}^{2n} : u = (\frac{1 + \zeta_0^1}{1 - \zeta_0^1}, \ldots, \frac{1 + \zeta_0^n}{1 - \zeta_0^n}), \quad |v| \geq \frac{1}{2}s\},$$

where ζ_0 $(\zeta_0^1, \ldots, \zeta_0^n) \in \partial B_1^n \setminus A$ and $s \geq 1$. Let $G_n(\tilde{1}) = \bigcup_{s\geq 1}\{G_n^s(\zeta_0, \tilde{1}) : \zeta_0 \in \partial B_1^n \setminus A\}$. Let further $\psi_n^n(\tilde{1})$ be the class of those continuous mappings $\psi_n : \Omega \times U_1^n \to \mathbb{C}^n$, which satisfy $\psi_n(\tilde{1}, 0; 0) \in D$ and $\psi_n(u, v; z) \notin D$, for all $z \in U_1^n$ and $(u, v) \in G_n(\tilde{1})$, where $\Omega \subseteq \mathbb{C}^{2n}$ with $G_n(\tilde{1}) \subset \Omega$.

With the above notation we obtain

THEOREM 3.1. *Let Ω and D be domains in $\mathbb{C}^{2n}$ and $\mathbb{C}^n$, respectively, and $f \in H(U_1^n)$, $f(0) = \tilde{1}$. Suppose that there exists $\psi_n \in \psi_n^n(\tilde{1})$ such that*

$$(f(z), [Df(z)][z]') \in \Omega \quad \text{and} \quad \psi_n(f(z), [Df(z)][z]'; z) \in D \quad \text{for all } z \in U_1^n.$$

Then Re $f(z) > 0$ *in* U_1^n.

Proof. It is clear that if we prove $f \prec g$, where $g(z)$ is given by (2), then Re $f(z) > 0$, $z \in U_1^n$. If this subordination does not hold, then using the same reasons as in the proof of Theorem 2.3, we deduce that there exist points $z_0 \in U_1^n$, $\zeta_0 \in \partial B_1^n \setminus A$ such that

$$f(z_0) = \left(\frac{1+\zeta_0^1}{1-\zeta_0^1}, \ldots, \frac{1+\zeta_0^n}{1-\zeta_0^n}\right) \quad \text{and} \quad |[Df(z_0)][z_0]'| \geq \frac{1}{2}s, \quad s \geq 1.$$

Let $u = f(z_0)$ and $v = [Df(z_0)][z_0]'$; then it is clear that $(u, v) \in G_n^s(\zeta_0, \tilde{1})$, so using the definition of the class $\psi_n^n(\tilde{1})$ we conclude that $\psi_n(u, v; z_0) \notin D$, but this contradicts the hypothesis. Therefore f is subordinate to g, as desired.

An immediate application of Theorem 2.1 is given by the following

THEOREM 3.2. *Let M and N be positive numbers, let a, b be functions which satisfy the inequality $|a(z) + mb(z)| \geq N/M^2$ for all $z \in U_1^n$ and $m \geq 1$. Let $f \in H(U_1^n)$, $f(0) = 0$, and suppose that*

$$|a(z)f(z) + b(z)[Df(z)][z]'| < N/M \quad \textit{for all} \quad z \in U_1^n.$$

Then $|f(z)| < M$ in U_1^n.

Proof. If we suppose that the relation $|f(z)| < M$ does not hold in U_1^n, then, using the continuity of the Euclidean norm and the relation $f(0) = 0$, we deduce that there exists a point $z_0 \in U_1^n$ with the property

$$M = |f(z_0)| = \max\{|f(z)| : ||z|| \leq ||z_0||\}.$$

Now it is sufficient to apply the conclusion of Theorem 2.1 to see that

$$|a(z_0)f(z_0) + b(z_0)[Df(z_0)][z_0]'| \geq N/M,$$

but this is a contradiction with the hypothesis. Hence $|f(z)| < M$ for all $z \in U_1^n$.

For $a(z) \equiv 0$ in U_1^n, we obtain

COROLLARY 3.1. *Let M and N be positive numbers and let f be a holomorphic mapping on U_1^n with $f(0) = 0$. Suppose that $b : U_1^n \to \mathbb{C}$ is a function which satisfies the conditions*

$$|b(z)[Df(z)][z]'| < N/M \quad \textit{and} \quad |b(z)| \geq N/M^2 \quad \textit{for all} \quad z \in U_1^n.$$

Then $|f(z)| < M$ in U_1^n.

Another application of Theorem 2.3 is given in

THEOREM 3.3. *Let $f \in H(U_1^n)$ and g be a biholomorphic mapping on $\bar{B}_1^n$ with $g(0) = f(0)$. Suppose that $|[Df(z)][z]'| < M$ for all $z \in U_1^n$, where $M = \inf_{|\zeta|=1} |[Dg(\zeta)]^{-1}|^{-1}$. Then $f \prec g$.*

Proof. If this subordination does not hold, then, by Theorem 2.3, there exist the points $z_0 \in U_1^n$, $\zeta_0 \in \partial B_1^n$ and a real number s, $s \geq 1$, such that

$$f(z_0) = g(\zeta_0) \quad \text{and} \quad |[Df(z_0)][z_0]'| \geq s|[Dg(\zeta_0)]^{-1}|^{-1},$$

so $|[Df(z_0)][z_0]'| \geq M$ which contradicts the hypothesis. Hence f is subordinate to g.

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