

LINEARLY INVARIANT FAMILIES OF HOLOMORPHIC FUNCTIONS IN THE UNIT POLYDISC

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Abstract. In this paper we extend the definition of the linearly invariant family and the definition of the universal linearly invariant family to higher dimensional case. We characterize these classes and give some of their properties. We also give a relationship of these families with the Bloch space.

1. Introduction. Ch. Pommerenke has introduced ([1]) the notion of a linearly invariant family M as a class of functions f holomorphic in the unit disc $\Delta = \{z : z \in \mathbb{C}, |z| < 1\}$ such that

- 1) $f(0) = 0$, $f'(0) = 1$, $f'(z) \neq 0$ in Δ ,
- 2) for all $f \in M$ and $\theta \in \mathbb{R}$, $f(ze^{i\theta})e^{-i\theta} \in M$,
- 3) for all $f \in M$ and $a \in \Delta$ $f_a(z) := \frac{f(\frac{z+a}{1+\bar{a}z}) - f(a)}{f'(a)(1-|a|^2)} = z + \dots \in M$.

The number

$$\text{ord} f = \sup_{a \in \Delta} \frac{|f_a''(0)|}{2}$$

was called, by Ch. Pommerenke ([1]), the order of a locally univalent function f , and the number

$$\text{ord} M = \sup_{f \in M} \text{ord} f$$

- the order of the family M . Moreover,

$$\bigcup \{M : \text{ord} M \leq \alpha\} := U_\alpha$$

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was called the universal linearly invariant family.

Linearly invariant families play an important part in the theory of conformal mappings. Furthermore an interest in the families U_α grows, because of their relationship with the Bloch class ([2]).

The main goal of this paper is to extend the definition of the linearly invariant families onto the case of functions defined on the unit polydisc $\Delta^m \subset \mathbb{C}^m$, $m \geq 1$, and establish several properties

Let $\mathbb{T} = \{z : z \in \mathbb{C}, |z| = 1\}$ and \mathbb{T}^m be the unit torus. We will consider the class $H(\Delta^m)$ of all functions $f : \Delta^m \rightarrow \mathbb{C}$ holomorphic in Δ^m . The gradient of a holomorphic function f we denote by ∇f ; that is $\nabla f = (\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_m})$. For $z = (z_1, \dots, z_m) \in \mathbb{C}^m$ we define the norm

$$\|z\| = \max_{1 \leq j \leq m} |z_j|.$$

Let $\mathbf{O} = (0, \dots, 0) \in \mathbb{C}^m$. Recall that to every $a \in \Delta$ corresponds an automorphism ϕ_a of Δ : $\phi_a(z) = (a + z)/(1 + \bar{a}z)$, $z \in \Delta$. The same can be done in the polydisc Δ^m . For $a = (a_1, \dots, a_m) \in \Delta^m$ the Möbius mapping ϕ_a of Δ^m onto Δ^m we define by the formula

$$\phi_a(z) = (\phi_1(z_1), \dots, \phi_m(z_m)),$$

where

$$\phi_j(z_j) = \frac{z_j + a_j}{1 + \bar{a}_j z_j}, \quad j = 1, \dots, m.$$

Now, we are ready to introduce the linearly (Möbius) invariant family.

DEFINITION 1.1. Let $l = 1, \dots, m$ be fixed. The l -Möbius invariant family M_l is the class of all functions f , $f \in H(\Delta^m)$, such that

- 1) $f(\mathbf{O}) = 0$, $\frac{\partial f}{\partial z_l}(\mathbf{O}) = 1$, $\frac{\partial f}{\partial z_l}(z) \neq 0$, for $z \in \Delta^m$,
- 2) for all $f \in M_l$ and $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$, $f(ze^{i\theta})e^{-i\theta_l} \in M$, where $ze^{i\theta} = (z_1 e^{i\theta_1}, \dots, z_m e^{i\theta_m})$.
- 3) for all $f \in M_l$ and $a = (a_1, \dots, a_m) \in \Delta^m$,

$$f_a(z) := \frac{f(\phi_a(z)) - f(\phi_a(\mathbf{O}))}{\frac{\partial f}{\partial z_l}(a)(1 - |a_l|^2)} \in M_l.$$

EXAMPLES:

- (i) K_l - the class of functions $f \in H(\Delta^m)$ satisfying 1) of the above Definition and such that $f(\Delta^m)$ is a convex domain.
- (ii) S_l^* - the class of functions $f \in H(\Delta^m)$ satisfying 1) of the above Definition and such that there exists a point $w_f \in f(\Delta^m)$, such that the domain $f(\Delta^m)$ is starlike with respect to w_f .
- (iii) S_l^k , where $k = 1, \dots, m$ is fixed, - the class of all functions $f_k \in H(\Delta^m)$ satisfying 1) of the above Definition such that $F(z) = (f_1(z), \dots, f_m(z))$ is a univalent mapping of Δ^m into \mathbb{C}^m .

The following definition extends the Pommerenke's conception of the order of a function, ([1]).

DEFINITION 1.2. Let f satisfy the conditions 1) of Definition 1.1 and let $\frac{\partial f_a}{\partial z_l}(z) = 1 + c_1(f)z_1 + \dots + c_m(f)z_m + o(\|z\|)$. The l -order of the function f is defined as follows:

$$\text{ord}_l f = \sup_{a \in \Delta^m} \frac{1}{2} \|\nabla \frac{\partial f_a}{\partial z_l}(\mathbf{O})\| = \frac{1}{2} \sup_{a \in \Delta^m} \|(c_1(f_a), \dots, c_m(f_a))\|.$$

THEOREM 1.1 If $f \in M_l$, $a \in \Delta^m$, then

$$\begin{aligned} \text{ord}_l f &= \max_{1 \leq k \leq m} \sup_{z \in \Delta^m} \left| \frac{\frac{\partial^2 f}{\partial z_l \partial z_k}}{\frac{\partial f}{\partial z_l}(z)} \frac{1 - |z_k|^2}{2} - \bar{z}_k \delta_k^l \right| = \\ &\sup_{a \in \Delta^m} \left\| \frac{1}{2} \nabla \log \left(\frac{\partial f}{\partial z_l} \circ \phi_a \right) (\mathbf{O}) - (0, \dots, 0, \bar{a}_l, 0, \dots, 0) \right\|, \end{aligned}$$

where

$$\delta_k^l = \begin{cases} 1, & \text{for } k = l \\ 0, & \text{for } k \neq l. \end{cases}$$

Proof. Let us observe that

$$\frac{\partial f_a}{\partial z_l}(z) = \frac{(\frac{\partial f}{\partial z_l} \circ \phi_a)(z)}{\frac{\partial f}{\partial z_l}(a)(1 + \bar{a}_l z_l)^2}.$$

Then

$$\|\nabla \frac{\partial f_a}{\partial z_l}(\mathbf{O})\| = \max_{1 \leq k \leq m} \left| \frac{\frac{\partial^2 f}{\partial z_l \partial z_k}(a)}{\frac{\partial f}{\partial z_l}(a)} (1 - |a_k|^2) - 2\bar{a}_l \delta_k^l \right|.$$

The above gives the result. ■

Now, we introduce the order of a family M_l .

DEFINITION 1.3. The l -order of a l -Möbius invariant family M_l is defined as

$$\text{ord}_l M_l = \sup_{f \in M_l} \text{ord}_l f.$$

EXAMPLES:

- 1) $M_l = \{f(z) = \Phi(z_l) : \Phi \in U_\alpha\}$ is the l -Möbius invariant family of the l -order α .
- 2) Let $k \neq l$ and let $\Phi_k(z_k)$ be functions holomorphic in Δ such that $\Phi_k(0) = 0$. Then

$$M_l = \left\{ f(z) = \sum_{k=1}^{\infty} \lambda_k \Phi_k(z_k) : \Phi_l \in U_\alpha, \lambda_k \in \mathbb{C}, \lambda_l = 1 \right\}$$

is the l -Möbius invariant family of the l -order α .

- 3) Let

$$\Psi(z) = \frac{1}{2\alpha} \left[\prod_{k=1}^m \left(\frac{1+z_k}{1-z_k} \right)^\alpha - 1 \right].$$

Then for all $l = 1, \dots, m$ the class

$$\{\Psi_a(z e^{i\theta}) e^{-i\theta_l} : a \in \Delta^m, \theta \in \mathbb{R}^m\}$$

is the l -Möbius invariant family of the l -order α .

2. Universal linearly (Möbius) invariant family. In the next definition we introduce a universal linearly (Möbius) invariant family.

DEFINITION 2.1. The universal l -Möbius invariant family U_α^l of the l -order α is defined as the union of all families M_l such that $\text{ord}_l M_l \leq \alpha$; that is

$$U_\alpha^l = \cup \{M_l : \text{ord}_l M_l \leq \alpha\}.$$

THEOREM 2.1. For any $f \in U_\alpha^l$ and all $z \in \Delta^m$ we have

$$(2.1) \quad \left| \log((1 - |z_l|^2) \frac{\partial f}{\partial z_l}(z)) \right| \leq \alpha \log \prod_{k=1}^m \frac{1 + |z_k|}{1 - |z_k|},$$

$$(2.2) \quad \frac{1}{1 - |z_l|^2} \prod_{k=1}^m \left(\frac{1 - |z_k|}{1 + |z_k|} \right)^\alpha \leq \left| \frac{\partial f}{\partial z_l}(z) \right| \leq \frac{1}{1 - |z_l|^2} \prod_{k=1}^m \left(\frac{1 + |z_k|}{1 - |z_k|} \right)^\alpha.$$

The above inequalities are rendered by the functions

$$\Psi(z) = \frac{1}{2\alpha} \left[\prod_{k=1}^m \left(\frac{1 + z_k}{1 - z_k} \right)^\alpha - 1 \right],$$

for $\alpha \geq 1$ and real z_k .

Proof. From Theorem 1.1 we have

$$\left| \frac{\frac{\partial^2 f}{\partial z_l^2}(z)}{\frac{\partial f}{\partial z_l}(z)} \frac{1 - |z_l|^2}{2} - \bar{z}_l \right| \leq \alpha.$$

Thus for $z_l = r_l e^{i\theta_l}$ (if $z_l \neq 0$) we get

$$\left| \frac{\partial}{\partial r_l} \left[\log \left(\frac{\partial f}{\partial z_l}(z) (1 - r_l^2) \right) \right] \right| = \left| \frac{\frac{\partial^2 f}{\partial z_l^2}(z)}{\frac{\partial f}{\partial z_l}(z)} e^{i\theta_l} - \frac{2r_l}{1 - r_l^2} \right| \leq \frac{2\alpha}{1 - r_l^2}.$$

Now, let

$$z_l = (z_1, \dots, z_{l-1}, 0, z_{l+1}, \dots, z_m).$$

Then we obtain

$$(2.3) \quad \left| \log \left(\frac{\partial f}{\partial z_l}(z) (1 - r_l^2) \right) - \log \frac{\partial f}{\partial z_l}(z^l) \right| = \left| \int_0^{r_l} \frac{\partial}{\partial r_l} \left[\log \left(\frac{\partial f}{\partial z_l}(z) (1 - r_l^2) \right) \right] dr_l \right| \leq \int_0^{r_l} \frac{2\alpha}{1 - r_l^2} dr_l = \alpha \log \frac{1 + r_l}{1 - r_l}.$$

By Theorem 1.1 we have

$$\left| \frac{\frac{\partial^2 f}{\partial z_l \partial z_k}(z)}{\frac{\partial f}{\partial z_l}(z)} \right| \leq \frac{2\alpha}{1 - r_k^2}$$

for all $k \neq l$ and $z \in \Delta^m$.

Let $z^{l,k}$ be a point in Δ^m for which $z_l = z_k = 0$. Then

$$(2.4) \quad \left| \log \frac{\partial f}{\partial z_l}(z^l) - \log \frac{\partial f}{\partial z_l}(z^{l,k}) \right| = \left| \int_0^{r_k} \frac{\frac{\partial^2 f}{\partial z_l \partial z_k}(z^l)}{\frac{\partial f}{\partial z_l}(z^l)} e^{i\theta_k} dr_k \right| \leq \int_0^{r_k} \frac{2\alpha}{1 - r_k^2} dr_k = \alpha \log \frac{1 + r_k}{1 - r_k}.$$

Now, if $l \neq p \neq k$, $l \neq k$, then analogously to (2.4) we get

$$(2.5) \quad \left| \log \frac{\partial f}{\partial z_l}(z^{l,k}) - \log \frac{\partial f}{\partial z_l}(z^{l,k,p}) \right| \leq \alpha \log \frac{1+r_p}{1-r_p},$$

where a point $z^{l,k,p} \in \Delta^m$, and $z_l = z_k = z_p = 0$. Using the above scheme we obtain, in the end, an estimation of the type (2.5) of the expression

$$\left| \log \frac{\partial f}{\partial z_l}(z^*) - \log \frac{\partial f}{\partial z_l}(\mathbf{O}) \right|,$$

where a point $z^* \in \Delta^m$ and it has only one component different from zero. Summing (2.3), (2.4), (2.5), ... we obtain (2.1). If we put the function Ψ in (2.1), with $z_k = |z_k|$, for all $k = 1, \dots, m$ then we have the equality in (2.1).

For the proof of (2.2), let us observe that from (2.1) we obtain

$$\begin{aligned} \alpha \log \prod_{k=1}^m \frac{1-|z_k|}{1+|z_k|} - \log(1-|z_l|^2) &\leq \Re \left[\log \frac{\partial f}{\partial z_l}(z) - \log \frac{\partial f}{\partial z_l}(\mathbf{O}) \right] \leq \\ &\alpha \log \prod_{k=1}^m \frac{1+|z_k|}{1-|z_k|} - \log(1-|z_l|^2). \end{aligned}$$

For the function Ψ , with $z_k = \pm|z_k|$ for all $k = 1, \dots, m$, we have the equality in (2.2). ■

Remark. For $m = 1$ the above Theorem gives the well known result for the class U_α , ([1]).

COROLLARY. *In the Definition of U_α^l we have $\alpha \geq 1$, because $U_\alpha^l = \emptyset$ for $\alpha < 1$.*

Indeed, if we suppose that $\alpha < 1$, then from (2.2) it follows that $\lim_{|z_l| \rightarrow 1^-} \left| \frac{\partial f}{\partial z_l}(z) \right| = \infty$ for fixed rest components of $z = (z_1, \dots, z_l, \dots, z_m)$. The holomorphic, with respect to z_l , function $\frac{\partial f}{\partial z_l}$ is not equal zero. Thus $\min_{|z| < r} \left| \frac{\partial f}{\partial z_l}(z) \right|$ is attained on $\{|z_l| = r\}$ and this minimum tends to ∞ , if $r \rightarrow 1^-$. The above contradicts $\frac{\partial f}{\partial z_l}(\mathbf{O}) = 1$.

THEOREM 2.2. *The family U_α^l is the set of all functions holomorphic in Δ^m and satisfying the conditions 1), 2), 3) of Definition 1.1 and the condition (2.2) in a neighbourhood of \mathbf{O} .*

Proof. Let F be a family of functions satisfying the conditions mentioned in our theorem. It is enough to show that $F \subset U_\alpha(l)$. Let $f \in F$. Thus, by 2), for all z from a neighbourhood of \mathbf{O} we have

$$(2.6) \quad \alpha \log \prod_{k=1}^m \frac{1-r_k}{1+r_k} - \log(1-r_l^2) \leq \Re \log \frac{\partial f}{\partial z_l}(z) \leq \alpha \log \prod_{k=1}^m \frac{1+r_k}{1-r_k} - \log(1-r_l^2),$$

where $z_k = r_k e^{i\theta_k}$.

The above inequalities are true for a function f_a , for every $a \in \Delta^m$. For $z = \mathbf{O}$ both left and right expression of (2.6) are 0. Thus, after the differentiation, with respect to r_k , $k = 1, \dots, m$, of (2.6) in the point \mathbf{O} we get (if $k \neq l$)

$$-2\alpha \leq \Re \frac{\frac{\partial^2 f_a}{\partial z_l \partial z_k}(\mathbf{O})}{\frac{\partial f_a}{\partial z_l}(\mathbf{O})} \leq 2\alpha,$$

which is equivalent to

$$(2.7) \quad \left| \frac{\frac{\partial^2 f}{\partial z_l \partial z_k}(a)}{\frac{\partial f}{\partial z_l}(a)} (1 - |a_k|^2) \right| \leq 2\alpha.$$

Moreover, if $k = l$ we get

$$-2\alpha \leq \Re \left[\frac{\frac{\partial^2 f}{\partial z_l^2}(1 - |a_l|^2) - 2\bar{a}_l \frac{\partial f}{\partial z_l}(a)}{\frac{\partial f}{\partial z_l}(a)} e^{i\gamma_l} \right] \leq 2\alpha,$$

for all $\gamma_l \in \mathbb{R}$; which is equivalent to

$$(2.8) \quad \left| \frac{\frac{\partial^2 f}{\partial z_l^2}(a)}{\frac{\partial f}{\partial z_l}(a)} (1 - |a_l|^2) - 2\bar{a}_l \right| \leq 2\alpha.$$

From (2.7) and (2.8) it follows that $\text{ord}_l f \leq \alpha$. Thus $f \in U_\alpha^l$. ■

Remark. For $m = 1$ we get known result in the class U_α , ([3]).

Now, let for $x \in [0, 1)$, $q \in [-1, 1]$

$$\begin{aligned} \Xi(x, q) &= \int_0^x \frac{\sqrt{1 - q^2 t^2}}{1 - t^2} dt = \frac{1}{2} \sqrt{1 - q^2} \log \frac{\sqrt{1 - q^2 x^2} + x \sqrt{1 - q^2}}{\sqrt{1 - q^2 x^2} - x \sqrt{1 - q^2}} + q \arcsin x \leq \\ &\quad \frac{1}{2} \sqrt{1 - q^2} \log \frac{1 + x}{1 - x} + \arcsin x. \end{aligned}$$

Observe, that the function $\alpha \Xi(|z|, \frac{\sin \lambda}{\alpha})$ is increasing with respect to α .

In the paper [1] Ch. Pommerenke has obtained an estimate of $|\Re\{e^{-i\lambda} \log f'(z)\}|$ in the class U_α . Now, we give a similar result for the class U_α^l .

THEOREM 2.3. *For all $f \in U_\alpha^l$ and all real λ*

$$|\Re\{e^{-i\lambda} \log \frac{\partial f}{\partial z_l}(z)(1 - |z_l|^2)\}| \leq \alpha \left(\log \prod_{k \neq l} \frac{1 + |z_k|}{1 - |z_k|} + 2\Xi(|z_l|, \frac{|\sin \lambda|}{\alpha}) \right).$$

Proof. Let us denote

$$u(r_1, \dots, r_m) = \max_{|z_k| \leq r_k} \Re\{e^{-i\lambda} \log \frac{\partial f}{\partial z_l}(z_1, \dots, z_m)\}.$$

By the maximum principle for harmonic functions

$$u(r_1, \dots, r_m) = \Re\{e^{-i\lambda} \log \frac{\partial f}{\partial z_l}(r_1 e^{i\theta_1^{(0)}(r_1, \dots, r_m)}, \dots, r_m e^{i\theta_m^{(0)}(r_1, \dots, r_m)})\}.$$

Then

$$(2.9) \quad \begin{aligned} &-\frac{\partial}{\partial \theta_k} \left(\Re\{e^{-i\lambda} \log \frac{\partial f}{\partial z_l}(r_1 e^{i\theta_1}, \dots, r_m e^{i\theta_m})\} \right) \Big|_{\theta = \theta^{(0)}} = \\ &\Im\{e^{-i\lambda} \frac{\frac{\partial^2 f}{\partial z_l \partial z_k}}{\frac{\partial f}{\partial z_l}}(r_1 e^{i\theta_1^{(0)}}, \dots, r_m e^{i\theta_m^{(0)}}) e^{i\theta_k^{(0)}} r_k\} = 0; \end{aligned}$$

where $\theta^{(0)} = \theta^{(0)}(r_1, \dots, r_m) = (\theta_1^{(0)}(r_1, \dots, r_m), \dots, \theta_m^{(0)}(r_1, \dots, r_m))$. The function $u(r_1, \dots, r_m)$ increases with respect to every variable $r_k \in [0, 1)$. Thus, by the Lebesgue

theorem everywhere on $[0,1]$ there exists finite derivative $\frac{\partial u}{\partial r_k}(r_1, \dots, r_{k-1}, tr_k, \dots, r_m)$.
Then

$$\Re\{e^{-i\lambda} \log \frac{\partial f}{\partial z_l}(r_1 e^{i\theta_1^{(0)}}, \dots, r_{k-1} e^{i\theta_{k-1}^{(0)}}, t e^{i\theta_k^{(0)}}, r_{k+1} e^{i\theta_{k+1}^{(0)}}, \dots, r_m e^{i\theta_m^{(0)}})\} \leq$$

$$(2.10) \quad u(r_1, \dots, r_{k-1}, t, r_{k+1}, \dots, r_m),$$

with equality for $t = r_k$. From the above it follows, that for almost all r_k :

$$\Re\{e^{-i\lambda} \frac{\frac{\partial^2 f}{\partial z_l \partial z_k}}{\frac{\partial f}{\partial z_l}}(r_1 e^{i\theta_1^{(0)}}, \dots, r_k e^{i\theta_k^{(0)}}, \dots, r_m e^{i\theta_m^{(0)}}) e^{i\theta_k^{(0)}}\} =$$

$$\frac{\partial u}{\partial r_k}(r_1, \dots, r_k, \dots, r_m).$$

By (2.9) we get

$$\frac{\partial u}{\partial r_k}(r_1, \dots, r_k, \dots, r_m) = e^{-i\lambda} \frac{\frac{\partial^2 f}{\partial z_l \partial z_k}}{\frac{\partial f}{\partial z_l}}(r_1 e^{i\theta_1^{(0)}}, \dots, r_k e^{i\theta_k^{(0)}}, \dots, r_m e^{i\theta_m^{(0)}}) e^{i\theta_k^{(0)}}.$$

From the above and from Theorem 1.1 it follows that for almost all r_k (the rest of variables of the function u are fixed) we have

$$\left| \frac{\partial u}{\partial r_k}(r_1, \dots, r_k, \dots, r_m) \frac{1-r_k^2}{2} - r_k \delta_k^l e^{-i\lambda} \right| \leq \alpha,$$

$$\left(\frac{\partial u}{\partial r_k} \frac{1-r_k^2}{2} - r_k \delta_k^l \cos \lambda \right)^2 + r_k^2 \delta_k^l \sin^2 \lambda \leq \alpha^2,$$

and

$$\left| \frac{\partial u}{\partial r_k} - 2\delta_k^l \frac{r_k}{1-r_k^2} \cos \lambda \right| \leq 2 \frac{\sqrt{\alpha^2 - r_k^2 \delta_k^l \sin^2 \lambda}}{1-r_k^2}.$$

For $k = l$ we have

$$\left| \frac{\partial}{\partial r_l} [u + \cos \lambda \log(1-r_l^2)] \right| \leq 2 \frac{\sqrt{\alpha^2 - r_l^2 \sin^2 \lambda}}{1-r_l^2}.$$

Integrating, we obtain

$$|u(r_1, \dots, r_l, \dots, r_m) + \cos \lambda \log(1-r_l^2) - u(r_1, \dots, r_{l-1}, 0, r_{l+1}, \dots, r_m)| \leq$$

$$2\alpha \Xi(r_l, \frac{|\sin \lambda|}{\alpha});$$

and for $k \neq l$:

$$|u(r_1, \dots, r_{l-1}, 0, r_{l+1}, \dots, r_m) - u(r_1, \dots, r_{k-1}, 0, r_{k+1}, \dots, r_{l-1}, 0, r_{l+1}, \dots, r_m)| \leq$$

$$\int_0^{r_k} \left| \frac{\partial u}{\partial r_k}(r_1, \dots, r_{k-1}, s, r_{k+1}, \dots, r_{l-1}, 0, r_{l+1}, \dots, r_m) \right| ds \leq 2 \int_0^{r_k} \frac{\alpha}{1-s^2} ds =$$

$$\alpha \log \frac{1+r_k}{1-r_k}.$$

Having (m-1) similar inequalities with $k \neq l$ and summing them we obtain

$$|u(r_1, \dots, r_m) + \cos \lambda \log(1 - r_l^2) - u(\mathbf{O})| \leq \alpha \sum_{k \neq l} \log \frac{1 + r_k}{1 - r_k} + 2\alpha \Xi(r_l, \frac{|\sin \lambda|}{\alpha}).$$

Thus we get the Theorem. ■

For $\lambda = 2\pi n$, n an integer, we have Theorem 2.1, and for $e^{i\lambda} = i$ we get

COROLLARY.

$$\begin{aligned} |\arg \frac{\partial f}{\partial z_l}(z)| &\leq \alpha (\log \prod_{k \neq l} \frac{1 + |z_k|}{1 - |z_k|} + 2\Xi(|z_l|, \frac{1}{\alpha})) \leq \\ &\alpha \log \prod_{k \neq l} \frac{1 + |z_k|}{1 - |z_k|} + \sqrt{\alpha^2 - 1} \log \frac{1 + |z_l|}{1 - |z_l|} + 2 \arcsin |z_l|. \end{aligned}$$

For the proof see [1]. Here $\arg \frac{\partial f}{\partial z_l}(\mathbf{O}) = 0$ and it is continuous with respect to z . For $m = 1$ we get Pommerenke's result for U_α .

Remark. The above estimation is not rough.

To support this we give the following example of function.

$$\begin{aligned} \Psi_0(z) &= \frac{1}{2i\sqrt{\alpha^2 - 1}} \left[\prod_{k \neq l} \left(\frac{1 + z_k}{1 - z_k} \right)^{i\alpha} \left(\frac{1 + z_l}{1 - z_l} \right)^{i\sqrt{\alpha^2 - 1}} - 1 \right], \quad \alpha > 1. \\ (2.11) \quad \arg \Psi_0(r_1, \dots, r_m) &= \alpha \log \prod_{k \neq l} \frac{1 + r_k}{1 - r_k} + \sqrt{\alpha^2 - 1} \log \frac{1 + r_l}{1 - r_l}. \end{aligned}$$

Indeed

$$\sup_{z \in \Delta^m} \left| \frac{\frac{\partial^2 \Psi_0}{\partial z_l^2} \frac{1 - |z_l|^2}{2}}{\frac{\partial \Psi_0}{\partial z_l}} - \bar{z}_l \right| = \sup_{|z_l| < 1} \left| \frac{1 - |z_l|^2}{1 - z_l^2} (i\sqrt{\alpha^2 - 1} - 1 + 2z_l) - \bar{z}_l \right| = \alpha$$

(see:[1], page 128). For all $k \neq l$

$$\sup_{z \in \Delta^m} \left| \frac{\frac{\partial^2 \Psi_0}{\partial z_l \partial z_k} \frac{1 - |z_k|^2}{2}}{\frac{\partial \Psi_0}{\partial z_l}} \right| = \sup_{|z_k| < 1} \left| \frac{\alpha i}{1 - z_k^2} (1 - |z_k|^2) \right| = \alpha.$$

By the Theorem 1.1 ord ${}_l \Psi_0 = \alpha$ and the equality (3.11) is fulfilled.

3. Bloch class. Now, we introduce the Bloch class of holomorphic functions.

DEFINITION 3.1. A holomorphic function $g : \Delta^m \rightarrow \mathbb{C}$ is called a Bloch function if

$$\|g\|_B := |g(\mathbf{O})| + \max_{k=1, \dots, m} \sup_{z \in \Delta^m} \left| \frac{\partial g}{\partial z_k} (1 - |z_k|^2) \right| < \infty.$$

The set of all Bloch functions we will denote by $B := B(\Delta^m)$.

The following result give a condition which is equivalent to the definition of the Bloch function. For $m = 1$ the result was given by the authors in [2].

THEOREM 3.1. *Let $l = 1, \dots, m$ be fixed. Then the following conditions are equivalent*

- (i) $g \in B$,
- (ii) *There exists $f \in \bigcup_{\alpha < \infty} U_\alpha^l$ such that*

$$g(z) - g(\mathbf{O}) = \log \frac{\partial f}{\partial z_l}(z),$$

where $\alpha = \text{ord}_l f$. Moreover

$$2(\alpha - 1) \leq \|g(z) - g(\mathbf{O})\|_B \leq 2(\alpha + 1).$$

Proof. (i) \Rightarrow (ii) Let $g \in B$ and let F be a function such that

$$\frac{\partial F}{\partial z_l}(z) = \exp(g(z) - g(\mathbf{O})).$$

Now, let us consider a function f defined by the formula

$$f(z) = F(z) - F(\mathbf{O}).$$

One can see, that f satisfies 1) in Definition 1.1. Moreover

$$\log \frac{\partial f}{\partial z_l}(z) = \log \frac{\partial F}{\partial z_l}(z) = g(z) - g(\mathbf{O}).$$

Since

$$\alpha = \text{ord}_l f = \max_{k=1, \dots, m} \sup_{z \in \Delta^m} \left| \frac{\partial g}{\partial z_k}(z) \frac{1 - |z_k|^2}{2} - \bar{z}_k \delta_k^l \right|$$

we have

$$\frac{1}{2} \|g(z) - g(\mathbf{O})\|_B - 1 \leq \alpha \leq \frac{1}{2} \|g(z) - g(\mathbf{O})\|_B + 1$$

and $f \in U_\alpha^l$.

(ii) \Rightarrow (i) Let $f \in \bigcup_{\alpha < \infty} U_\alpha^l$ and $\text{ord}_l f = \alpha$. Let

$$g(z) = \log \frac{\partial f}{\partial z_l}(z).$$

We have $g(\mathbf{O}) = 0$. From Theorem 2.1 it follows that for every $k = 1, \dots, m$

$$\sup_{z \in \Delta^m} \left(\left| \frac{\partial g}{\partial z_k}(z) \right| (1 - |z_k|^2) \right) \leq 2\alpha + 2\delta_l^k \leq 2(\alpha + 1).$$

Thus $g \in B$ and

$$2(\alpha - 1) \|g\|_B \leq \max_{k=1, \dots, m} \sup_{z \in \Delta^m} \left(\left| \frac{\partial g}{\partial z_k}(z) \right| (1 - |z_k|^2) \right) \leq 2(\alpha + 1). \blacksquare$$

Now, let us give some properties of Bloch functions in terms of the order.

THEOREM 3.2. *Let g be a holomorphic function in Δ^m . Then $g \in B$ if and only if there exists a positive constant C_g such that for all $z \in \Delta^m$,*

$$(3.1) \quad \sup_{a \in \Delta^m} |g(\phi_a(z)) - g(a) - 2 \log(1 + \bar{a}_l z_l) + \log(1 - |z_l|^2)| \leq C_g \log \prod_{k=1}^m \frac{1 + |z_k|}{1 - |z_k|}.$$

The best value of the constant C_g is equal $\text{ord} \int_0^{z_1} \exp g(z_1, \dots, z_{l-1}, s, \dots, z_m) ds$.

Proof. Let us suppose that $g(\mathbf{O}) = 0$.

1⁰. Let $g \in B$. Then there exists a function $f \in U_\alpha^l$ such that $g(z) = \log \frac{\partial f}{\partial z_l}(z)$. Since

$$f_a(z) := \frac{f(\phi_a(z)) - f(\phi_a(\mathbf{O}))}{\frac{\partial f}{\partial z_l}(a)(1 - |a_l|^2)}$$

we get

$$\frac{\partial f_a}{\partial z_l}(z) = \frac{\frac{\partial f}{\partial z_l}(\phi_a(z))}{\frac{\partial f}{\partial z_l}(a)(1 + \overline{a_l}z_l)^2}.$$

By (2.1)

$$\log\left|(1 - |z_l|^2) \frac{\partial f_a(z)}{\partial z_l}\right| \leq \alpha \log \prod_{k=1}^m \frac{1 + |z_k|}{1 - |z_k|}.$$

Thus

$$\left| \log \frac{\frac{\partial f}{\partial z_l}(\phi_a(z))(1 - |z_l|^2)}{\frac{\partial f}{\partial z_l}(a)(1 + \overline{a_l}z_l)^2} \right| \leq \alpha \log \prod_{k=1}^m \frac{1 + |z_k|}{1 - |z_k|}$$

which is equivalent to (3.1).

2⁰. Now, let a holomorphic function g satisfies (3.1) (with $g(\mathbf{O}) = 0$). Let us consider a function

$$f(z) = \int_0^{z_l} \exp g(z_1, \dots, z_{l-1}, s, \dots, z_m) ds.$$

Then

$$\left| \log \frac{\frac{\partial f}{\partial z_l}(\phi_a(z))(1 - |z_l|^2)}{\frac{\partial f}{\partial z_l}(a)(1 + \overline{a_l}z_l)^2} \right| \leq \alpha \log \prod_{k=1}^m \frac{1 + |z_k|}{1 - |z_k|},$$

Thus

$$\left| \Re \left[\log \frac{\partial f}{\partial z_l}(\phi_a(z)) - \log \frac{\partial f}{\partial z_l}(a) - 2 \log(1 + \overline{a_l}z_l) \right] \right| \leq \alpha \log \prod_{k=1}^m \frac{1 + |z_k|}{1 - |z_k|} - \log(1 - |z_l|^2).$$

From the above inequality (differentiating with respect to $|z_k|$ in \mathbf{O}) for every k and l we get

$$-2\alpha \leq \Re \left\{ \frac{\frac{\partial^2 f(a)}{\partial z_l \partial z_k} (1 - |a_k|^2) e^{i\theta_k}}{\frac{\partial f}{\partial z_l}(a)} - 2\overline{a_l} e^{i\theta_k} \delta_k^l \right\} \leq 2\alpha,$$

Hence $\text{ord} f = \alpha$, $f \in U_\alpha^l$ and

$$\max_{k=1, \dots, m} \sup_{a \in \Delta^m} \left| \frac{\partial g}{\partial z_k}(a) \right| (1 - |a_k|^2) \leq 2(\alpha + 1).$$

Thus $g \in B$. ■

Now, we give corollaries.

COROLLARY 3.1. *The condition (3.1) we give in the following equivalent form:*

$$(3.1') \quad |g(\phi_a(z)) - g(a)| \leq \frac{K_g}{2} \log \prod_{k=1}^m \frac{1 + |z_k|}{1 - |z_k|},$$

where the best constant K_g is equal $\|g(z) - g(\mathbf{O})\|_B$.

Proof. If $g \in B$, then

$$|g(z) - g(z'_l)| = \left| \int_0^{z_l} \frac{\partial g}{\partial z_l}(z_1, \dots, z_{l-1}, \dots, z_l) ds \right| \leq \int_0^{|z_l|} \frac{K_g}{1-r^2} dr = \frac{K_g}{2} \log \frac{1+|z_l|}{1-|z_l|},$$

where $z'_l = (z_1, \dots, z_{l-1}, 0, \dots, z_m)$. Using this scheme we get

$$(3.2) \quad |g(z) - g(\mathbf{O})| \leq \frac{K_g}{2} \log \prod_{k=1}^m \frac{1+|z_k|}{1-|z_k|}.$$

If $g \in B$, then $g(\phi_a(z)) - g(a) \in B$. Thus, by (3.2) we get the result.

Now, let (3.1') be fulfilled. Then for all $a \in \Delta^m$:

$$|\Re\{g(\phi_a(z)) - g(a)\}| \leq \frac{K_g}{2} \log \prod_{k=1}^m \frac{1+|z_k|}{1-|z_k|}.$$

Differentiating with respect to $z_k = |z_k|e^{i\theta_k}$ in a neighbourhood of \mathbf{O} we get

$$|\Re\{\frac{\partial g}{\partial z_k}(a)(1-|a_k|^2)e^{i\theta_k}\}| \leq K_g,$$

for all $k = 1, \dots, m$. Thus

$$\left| \frac{\partial g}{\partial z_k}(a)(1-|a_k|^2) \right| \leq K_g,$$

for all $a \in \Delta^m$ and this ends the proof. ■

COROLLARY 3.2. *The following conditions are equivalent:*

- (i) $g \in B$
- (ii) *the family of functions $g(\phi_a(z)) - g(a)$ is finitely normal for $a \in \Delta^m$.*

Proof. Let $g \in B$. By (3.1) we get that $g(\phi_a(z)) - g(a)$ belongs to the class B for all $a \in \Delta^m$. Thus we have (ii).

Now, let (ii) is fulfilled. The for every sequence $a_n \in \Delta^m$ there exists a subsequence a_{n_p} such that a sequence $G_p(z) = g(\phi_{a_{n_p}}(z)) - g(a_{n_p})$ is uniformly convergent, in Δ^m to an analytic function (which is not equal ∞). For every k the function $\frac{\partial}{\partial z_k} G_p(z)$ is uniformly convergent to an analytic function. Thus for every k the function $\frac{\partial}{\partial z_k} g(\phi_a(z))$ is uniformly bounded (with respect to $a \in \Delta^m$) on compact sets $K \subset \Delta^m$. Then there is a constant K_g such that

$$\left| \frac{\partial}{\partial z_k} g(\phi_a(z)) \right| = \left| \frac{\partial g}{\partial z_k}(\phi_a(z)) \frac{1-|a_k|^2}{(1+\overline{a_k}z_k)^2} \right| \leq K_g.$$

Thus

$$\left| \frac{\partial g}{\partial z_k}(a)(1-|a_k|^2) \right| \leq K_g,$$

for all $a \in \Delta^m$. ■

COROLLARY 3.3. *Let $g \in B$, $\lambda \in [0, 2\pi]$. Then the function $g(z) - g(\mathbf{O})$ maps the polydisc $\{z : z \in \mathbb{C}, |z| \leq r\}^m$ into a domain with the boundary:*

$$\alpha e^{i\lambda} \left(\log \left(\frac{1+r}{1-r} \right)^{m-1} + 2\Xi \left(r, \frac{\sin \lambda}{\alpha} \right) \right) - \log(1-|r|^2).$$

4. Class \dot{U}_α^l . Now, we define a class of derivatives of functions from the class U_α^l and give some of its properties.

DEFINITION 4.1. $\dot{U}_\alpha^l = \{ \frac{\partial f}{\partial z_i} : f \in U_\alpha^l \}$.

The following result give a relationship between classes $U_\alpha^l(l)$ (thus between classes U_α^l) with different parameters l .

THEOREM 4.1. For all $\alpha \geq 2$ and all $l, k, n \in \{1, \dots, m\}$:

$$\dot{U}_{\alpha-1}^n \subset \dot{U}_\alpha^l \subset \dot{U}_{\alpha+1}^k.$$

PROOF. From the Definitions of U_α^l and $U_\alpha^l(l)$ it follows that $h \in \dot{U}_\alpha^l$ if and only if

- (i) $h(\mathbf{0}) = 1$, $h(z) \neq 0$ for $z \in \Delta^m$,
- (ii) for all $a \in \Delta^m$ and $t = [t_1, \dots, t_m] \in \mathbb{R}^m$, if $h \in \dot{U}_\alpha^l$, then

$$\frac{h(\phi_a(z))}{h(a)(1 + \bar{a}_l z)^2} \in \dot{U}_\alpha^l$$

and

$$h(ze^{it}) \in \dot{U}_\alpha^l, \quad \text{where} \quad ze^{it} = [z_1 e^{it_1}, \dots, z_m e^{it_m}],$$

- (iii) if $h(z) = 1 + c_1 z_1 + \dots + c_m z_m + o(\|z\|)$, then

$$\|(c_1, \dots, c_m)\| \leq 2\alpha.$$

Let us suppose that $h \in U_\alpha^l(l)$. Let $c_j(a)$ be a coefficient with z_j in the expansion of the function $\frac{h(\phi_a(z))}{h(a)}$. Since $(1 + \bar{a}_l z_l)^{-2} = 1 - 2\bar{a}_l z_l + 3(\bar{a}_l z_l)^2 + \dots$, then

$$\frac{h(\phi_a(z))}{h(a)(1 + \bar{a}_l z_l)^2} = 1 + c_1(a)z_1 + \dots + (c_l(a) - 2\bar{a}_l)z_l + \dots + c_m(a)z_m + o(\|z\|).$$

Using (iii) we get

$$\|(c_1(a), \dots, c_l(a) - 2\bar{a}_l, \dots, c_m(a))\| \leq 2\alpha$$

for all $a \in \Delta^m$. This is a necessary and sufficient condition that a function h holomorphic in Δ^m with $h(\mathbf{0}) = 1$ belongs to $U_\alpha^l(l)$. Thus, for $k \neq l$ and all $a \in \Delta^m$

$$\|(c_1(a), \dots, c_k(a) - 2\bar{a}_k, \dots, c_m(a))\| \leq 2(\alpha + 1),$$

and this means that $h \in U_{\alpha+1}(k)$ and moreover $\dot{U}_\alpha^l \subset \dot{U}_{\alpha+1}^k$. Let us observe that for $\alpha \geq 2$ we have $\dot{U}_{\alpha+1}^n \subset \dot{U}_\alpha^l$. ■

REMARK 4.1. Let us consider the function

$$h_o(z) = \frac{1}{(1 - z_k)^2} = \frac{\partial}{\partial z_k} \frac{z_k}{1 - z_k} \in \dot{U}_1^k.$$

For the function h_o we have that $c_k(a) = \frac{2(1 - |a_k|^2)}{1 - a_k}$, and $\sup_{a \in \Delta^m} |c_k(a)| = 4$. Thus $h_o \notin U_\alpha^l$ for $\alpha < 2$. Moreover, we see that in Theorem 4.1 the constant $\alpha + 1$ is the best.

THEOREM 4.2. For all $\alpha \in (1, \infty)$ the family \dot{U}_α^l is compact in the topology induced by locally uniform convergence in Δ^m .

Proof. From Theorems 1.1 and 2.1 it follows that for every compact set $K \subset \Delta^m$ and every function $h \in \dot{U}_\alpha^l$ there exists a constant $C(\alpha, K)$ such that

$$\sup_{z \in K} \frac{\|\nabla h(z)\|}{1 + |h(z)|^2} \leq C(\alpha, K).$$

From the result of R.M. Timoney ([4], theorem 3.10) it follows that \dot{U}_α^l is a normal family.

If $h_n \in \dot{U}_\alpha^l$ is a sequence which converges uniformly to h on a compact $K \subset \Delta^m$, then, by the Theorem 2.1, $h(\mathbf{O}) = 1$ and $h(z) \neq 0$ in Δ^m . Thus h is in \dot{U}_α^l , if the conditions (ii), (iii) in the proof of Theorem 4.1 are fulfilled. But these conditions are equivalent to the following one

$$\text{ord}_l \int_0^{z_l} h(s) ds_l \leq \alpha,$$

where $s = (s_1, \dots, s_m)$; which is equivalent to

$$\max_{k=1, \dots, m} \sup_{z \in \Delta^m} \left| \frac{\frac{\partial h}{\partial z_l}(z)}{h(z)} \frac{1 - |z_k|^2}{2} - \overline{z_k} \delta_k^l \right| \leq \alpha.$$

The above inequality is true for h_n and thus for h , too. ■

Remark 4.2. Let us observe that unlike the case $m = 1$, the families U_α^l are not compact in the topology induced by locally uniform convergence in Δ^m , for $m \geq 2$.

Indeed, let

$$Q(z^l) = Q(z_1, \dots, z_{l-1}, z_{l+1}, \dots, z_m)$$

be a function holomorphic in Δ^{m-1} such that $Q(\mathbf{O}) = 0$. Then for a fixed function $h \in \dot{U}_\alpha^l$ we have

$$f(z) = \int_0^{z_l} h(s) ds_l + Q(z^l) \in U_\alpha^l.$$

Now, we can take a sequence of functions $Q_n(z^l)$ (of the type of the function $Q(z^l)$) such that for all $z \in \Delta^{m-1} \setminus \{\mathbf{O}\}$, $Q_n(z^l) \rightarrow \infty$, if $n \rightarrow \infty$. But, it is not possible to choose (from the corresponding sequence f_n) a subsequence convergent to a function from U_α^l .

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