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INTERSECTION-THEORETICAL COMPUTATIONS ON $\overline{\mathcal{M}}_g$

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Introduction. In this paper we explore several concrete problems, all more or less related to the intersection theory of the moduli space of (stable) curves, introduced by Mumford [Mu 1].

In Section 1 we only intersect divisors with curves. We find a collection of necessary conditions for ample divisors, but the question whether these conditions are also sufficient is very much open.

The other sections are concerned with moduli spaces of curves of low genus, but we use the ring structure of the Chow ring. In Sections 2, 3 we find necessary conditions for very ample divisors on $\overline{\mathcal{M}}_2$ and $\overline{\mathcal{M}}_3$.

The intersection numbers of the kappa-classes are the subject of the Witten conjecture, proven by Kontsevich. In Section 4 we show how to compute these numbers for $g = 3$ within the framework of algebraic geometry.

Finally, in Section 5 we compute λ^9 on $\overline{\mathcal{M}}_4$. This also gives the value of λ_{g-1}^3 (for $g = 4$), which is relevant for counting curves of higher genus on manifolds [BCOV]. Another corollary is a different computation of the class of the Jacobian locus in the moduli space of 4-dimensional principally polarized abelian varieties; in a sense this gives also a different proof that the Schottky locus is irreducible in dimension 4.

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1. Necessary conditions for ample divisors on $\overline{\mathcal{M}}_g$. Let $g \geq 2$ be an integer and put $h = [g/2]$. Cornalba and Harris [C-H] determined which divisors on $\overline{\mathcal{M}}_g$ of the form $a\lambda - b\delta$ are ample: this is the case if and only if $a > 11b > 0$. Divisors of this form are numerically effective (nef) if and only if $a \geq 11b \geq 0$. (More generally, the ample cone is the interior of the nef cone and the nef cone is the closure of the ample cone ([Ha], p. 42)). Here $\delta = \sum_{i=0}^{h} \delta_i$ with $\delta_i = [\Delta_i]$ for $i \neq 1$ and $\delta_1 = \frac{1}{2}[\Delta_1]$.

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Arbarello and Cornalba [A-C] proved that the $h+2$ divisors $\lambda, \delta_0, \delta_1, \ldots, \delta_h$ form for $g \geq 3$ a $\mathbb{Z}$-basis of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ (the Picard group of the moduli functor), using the results of Harer and Mumford (we work over $\mathbb{C}$). As pointed out in [C-H] it would be interesting to determine the nef cone in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ for $g \geq 3$. (For $g = 2$ the answer is given by the result of [C-H], because of the relation $10\lambda - \delta_0 - 2\delta_1 = 0$.)

In [Fa 1], Theorem 3.4, the author determined the nef cone for $g = 3$. The answer is: $a\lambda - b_0\delta_0 - b_1\delta_1$ is nef on $\overline{\mathcal{M}}_3$ if and only if $2b_0 \geq b_1 \geq 0$ and $a - 12b_0 + b_1 \geq 0$. That a nef divisor necessarily satisfies these inequalities, follows from the existence of one-dimensional families of curves for which $(\deg \lambda, \deg \delta_0, \deg \delta_1)$ equals $(1, 12, -1)$ resp. $(0, -2, 1)$ resp. $(0, 0, -1)$. Such families are easily constructed: for the first family, take a simple elliptic pencil and attach it to a fixed one-pointed curve of genus 2; for the second family, take a 4-pointed rational curve with one point moving and attach a fixed two-pointed curve of genus 1 to two of the points and identify the two other points; for the third family, take a 4-pointed rational curve with one point moving and attach two fixed one-pointed curves of genus 1 to two of the points and identify the two other points.

That a divisor on $\overline{\mathcal{M}}_3$ satisfying the inequalities is nef, follows once we show that λ, $12\lambda - \delta_0$ and $10\lambda - \delta_0 - 2\delta_1$ are nef. It is well-known that λ is nef. Using induction on the genus one shows that $12\lambda - \delta_0$ is nef: on $\overline{\mathcal{M}}_{1,1}$ it vanishes; for $g \geq 2$, writing $12\lambda - \delta_0 = \kappa_1 + \sum_{i=1}^{h} \delta_i$ one sees that $12\lambda - \delta_0$ is positive on every one-dimensional family of curves where the generic fiber has at most nodes of type δ_0; if on the other hand the generic fiber has a node of type δ_i for some $i > 0$, one partially normalizes the family along a section of such nodes and uses the induction hypothesis (cf. the proof of Proposition 3.3 in [Fa 1], which unfortunately proves the result only for $g = 3$). Finally, the proof that $10\lambda - \delta_0 - 2\delta_1$ is nef on $\overline{\mathcal{M}}_3$ is ad hoc (see the proof of Theorem 3.4 in [Fa 1]).

All we do in this section is come up with a couple of one-dimensional families of stable curves for which we compute the degrees of the basic divisors. The naive hope is that at least some of these families are extremal (cf. [C-H], p. 475), but the author hastens to add that there is at present very little evidence to support this.

The method of producing families is a very simple one: we start out trying to write down all the families for which the generic fiber has $3g - 4$ nodes. This turns out to be a bit complicated. However, the situation greatly simplifies as soon as one realizes that the only one-dimensional moduli spaces of stable pointed curves are $\overline{\mathcal{M}}_{0,4}$ and $\overline{\mathcal{M}}_{1,1}$: for the computation of the basic divisor classes on these families, one only needs to know the genera of the pointed curves attached to the moving 4-pointed rational curve resp. the moving one-pointed curve of genus 1 as well as the types of the nodes one gets in this way. In other words, the fixed parts of the families can be taken to be general.

We now consider the various types of families obtained in this way and compute on each family the degrees of the basic divisor classes. Each family gives a necessary condition for the divisor $a\lambda - \sum_{i=0}^{h} b_i\delta_i$ to be nef. In order to write this condition, it will be convenient to define $\delta_i = \delta_{g-i}$ and $b_i = b_{g-i}$ for $h < i < g$.

A) In the case of $\overline{\mathcal{M}}_{1,1}$, there is very little choice: we can only attach a (general) one-pointed curve of genus $g - 1$. Taking a simple elliptic pencil for the moving part, we get—as is well-known—the following degrees: $\deg \lambda = 1$, $\deg \delta_0 = 12$, $\deg \delta_1 = -1$ and $\deg \delta_i = 0$ for $1 < i \leq h$. This gives the necessary condition $a - 12b_0 + b_1 \geq 0$.

B) The other families are all constructed from a 4-pointed smooth rational curve with one of the points moving and the other three fixed; when the moving point meets one of the fixed points, the curve breaks up into two 3-pointed smooth rational curves glued at one point. We have to examine the various ways of attaching general curves to this 4-pointed rational curve. E.g., one can attach one curve, necessarily 4-pointed and of genus $g-3$. All nodes are of type δ_0 and the 3 degenerations have an extra such node. Therefore deg $\delta_0 = -4+3 = -1$, while the other degrees are zero; one obtains the necessary condition $b_0 \geq 0$.

C) Now attach a 3-pointed curve of genus i and a 1-pointed curve of genus $j \geq 1$, with $i+j = g-2$. One checks deg $\delta_0 = -3+3 = 0$ and deg $\delta_j = -1$, the other degrees vanish. One obtains $b_j \geq 0$ for $j \geq 1$. Thus all b_i are non-negative for a nef divisor.

R e m a r k. If one uses the families above, one simplifies the proof of Theorem 1 in [A-C] a little bit.

D) If we attach two-pointed curves of genus $i \geq 1$ and $j \geq 1$, with $i+j = g-2$, we find deg $\delta_0 = -4+2 = -2$ and deg $\delta_{i+1} = 1$. So for $2 \leq k \leq h$ we find the condition $2b_0 - b_k \geq 0$.

E) Attaching a two-pointed curve of genus i and two one-pointed curves of genus j and k, with $i,j,k \geq 1$ and $i+j+k = g-1$, we find that two of the degenerations have an extra node of type δ_0 while the third has an extra node of type δ_{j+k}. Therefore deg $\delta_0 = -2+2 = 0$. It is cumbersome to distinguish the various cases that occur for the other degrees, but is also unnecessary: one may simply write the resulting necessary condition in the form $b_j + b_k - b_{j+k} \geq 0$, for j,k with $1 \leq j \leq k$ and $j+k \leq g-2$.

F) Attaching 4 one-pointed curves of genera $i,j,k,l \geq 1$, with $i+j+k+l = g$, we get the necessary condition $b_i + b_j + b_k + b_l - b_{i+j} - b_{i+k} - b_{i+l} \geq 0$.

G) If we identify two of the 4 points to each other and attach a two-pointed curve of genus $g-2$ to the remaining two points, we obtain the necessary condition $2b_0 - b_1 \geq 0$.

H) As in G), but now we attach 1-pointed curves of genera $i,j \geq 1$ to the remaining two points, with $i+j = g-1$. The resulting condition is $b_i + b_j - b_1 \geq 0$.

The only other possibility is to identify the first with the second and the third with the fourth point. This gives a curve of genus 2, so this is irrelevant. We have proven the following theorem.

THEOREM 1. *Assume $g \geq 3$. A numerically effective divisor $a\lambda - \sum_{i=0}^{h} b_i\delta_i$ in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ satisfies the following conditions:*
a) $a - 12b_0 + b_1 \geq 0$;
b) *for all $j \geq 1$,*
$$2b_0 \geq b_j \geq 0;$$
c) *for all j,k with $1 \leq j \leq k$ and $j+k \leq g-1$,*
$$b_j + b_k \geq b_{j+k};$$
d) *for all i,j,k,l with $1 \leq i \leq j \leq k \leq l$ and $i+j+k+l = g$,*
$$b_i + b_j + b_k + b_l \geq b_{i+j} + b_{i+k} + b_{i+l}.$$

Here $b_i = b_{g-i}$ for $h < i < g$, as before. The conditions in the theorem are somewhat redundant. E.g., it is easy to see that condition (c) implies the non-negativity of the b_i with $i \geq 1$.

As we have seen, the conditions in the theorem are sufficient for $g = 3$. The proof proceeded by determining the extremal rays of the cone defined by the inequalities and analyzing the (three) extremal rays separately. It may therefore be of some interest to find (generators for) the extremal rays of the cone in the theorem. We have done this for low genus:

$$g=4: \quad \begin{cases} \lambda \\ 12\lambda-\delta_0 \\ 10\lambda-\delta_0-2\delta_1 \\ 10\lambda-\delta_0-2\delta_1-2\delta_2 \\ 21\lambda-2\delta_0-3\delta_1-4\delta_2 \end{cases}$$

$$g=5: \quad \begin{cases} \lambda \\ 12\lambda-\delta_0 \\ 10\lambda-\delta_0-2\delta_1-\delta_2 \\ 10\lambda-\delta_0-2\delta_1-2\delta_2 \\ 32\lambda-3\delta_0-4\delta_1-6\delta_2 \end{cases}$$

$$g=6: \quad \begin{cases} \lambda \\ 12\lambda-\delta_0 \\ 10\lambda-\delta_0-2\delta_1-2\delta_2 \\ 10\lambda-\delta_0-2\delta_1-2\delta_3 \\ 10\lambda-\delta_0-2\delta_1-2\delta_2-2\delta_3 \\ 32\lambda-3\delta_0-4\delta_1-6\delta_2-6\delta_3 \\ 98\lambda-9\delta_0-10\delta_1-16\delta_2-18\delta_3 \end{cases}$$

Unfortunately, we have not been able to discover a general pattern. (There are 10 extremal divisors for $g=7$, 20 extremal divisors for $g=8$ and 21 extremal divisors for $g=9$.) It is easy to see that λ, $12\lambda-\delta_0$ and $10\lambda-2\delta+\delta_0$ are extremal in every genus. It should be interesting to know the answer to the following question.

Question.

a) *Is* $10\lambda-2\delta+\delta_0$ *nef for all* $g\geq 4$*?*

b) *Are the conditions in the theorem sufficient?*

Note that an affirmative answer to the first question implies the result of [C-H] mentioned above, since $12\lambda-\delta_0$ is nef. Note also that a divisor satisfying the conditions in the theorem is non-negative on every one-dimensional family of curves whose general member is smooth. This follows easily from [C-H, (4.4) and Prop. (4.7)]. (I would like to thank Maurizio Cornalba for reminding me of these results.)

2. Necessary conditions for very ample divisors on $\overline{\mathcal{M}}_2$. We know which divisors on $\overline{\mathcal{M}}_2$ are ample: it is easy to see that λ and δ_1 form a $\mathbb{Z}$-basis of the functorial Picard group $\mathrm{Pic}(\overline{\mathcal{M}}_2)$; then $a\lambda+b\delta_1$ is ample if and only if $a>b>0$, as follows from the relation $10\lambda=\delta_0+2\delta_1$ and the fact that λ and $12\lambda-\delta_0$ are nef.

Therefore it might be worthwhile to study which divisors are very ample on the *space* $\overline{\mathcal{M}}_2$. Suppose that $D=a\lambda+b\delta_1$ is a very ample divisor. Then for every k-dimensional subvariety V of $\overline{\mathcal{M}}_2$ the intersection product $D^k\cdot[V]$ is a positive integer, the degree of $[V]$ in the embedding of $\overline{\mathcal{M}}_2$ determined by $|D|$. We work this out for the subvarieties that we know; we use Mumford's computation [Mu 1] of the Chow ring (with $\mathbb{Q}$-coefficients) of $\overline{\mathcal{M}}_2$. The result may be formulated as follows:

$$A^*(\overline{\mathcal{M}}_2)=\mathbb{Q}[\lambda,\delta_1]/(\delta_1(\lambda+\delta_1),\lambda^2(5\lambda-\delta_1)).$$

The other piece of information we need is on p. 324 of [Mu 1]: $\lambda^3=\frac{1}{2880}p$. However, one should realize that the identity element in $A^*(\overline{\mathcal{M}}_2)$ is $[\overline{\mathcal{M}}_2]_Q = \frac{1}{2}[\overline{\mathcal{M}}_2]$, which means that

$$\lambda^3 \cdot [\overline{\mathcal{M}}_2] = \frac{1}{1440}.$$

Therefore

$$D^3 \cdot [\overline{\mathcal{M}}_2] = \frac{a^3 + 15a^2b - 15ab^2 + 5b^3}{1440}.$$

One of the requirements is therefore that the integers a and b are such that the expression above is an integer. It is not hard to see that this is the case if and only if

$$60|a \qquad \text{and} \qquad 12|b.$$

It turns out that these conditions imply that $D^2 \cdot [\Delta_0]$ and $D^2 \cdot [\Delta_1]$ are integers. Also $D^2 \cdot 4\lambda$ is an integer, but $D^2 \cdot 2\lambda$ is an integer if and only if $8|(a + b)$. Therefore, if for some integer k the class $(4k + 2)\lambda$ is the fundamental class of an effective 2-cycle, then a very ample D satisfies $8|(a + b)$. We do not know whether such a k exists; clearly, $20\lambda = [\Delta_0] + [\Delta_1]$ is effective; the fundamental class of the bi-elliptic divisor turns out to be $60\lambda + 3\Delta_1$.

Turning next to one-dimensional subvarieties, the conditions $60|a$ and $12|b$ imply that $D \cdot [\Delta_{00}]$ and $D \cdot [\Delta_{01}]$ are integers as well.

PROPOSITION 2. *A very ample divisor $a\lambda + b\delta_1$ on the moduli space $\overline{\mathcal{M}}_2$ satisfies the following conditions:*

a) *$a, b \in \mathbb{Z}$ and $a > b > 0$;*

b) *$60|a$ and $12|b$.*

COROLLARY 3. *The degree of a projective embedding of $\overline{\mathcal{M}}_2$ is at least* 516.

P r o o f. We need to determine for which a and b satisfying the conditions in the proposition the expression $5(b-a)^3+6a^3$ attains its minimum value. Clearly this happens exactly for $b = 12$ and $a = 60$. If $60\lambda + 12\delta_1$ is very ample, the degree of $\overline{\mathcal{M}}_2$ in the corresponding embedding is $(5(b - a)^3 + 6a^3)/1440 = 516$. ∎

R e m a r k. It is interesting to compare the obtained necessary conditions with the explicit descriptions of $\overline{\mathcal{M}}_2$ given by Qing Liu ([Liu]). The computations we have done (in characteristic 0) indicate that $60\lambda+60\delta_1$ maps $\overline{\mathcal{M}}_2$ to a copy of $\mathfrak{X}$ (loc. cit., Théorème 2), that $60\lambda + 36\delta_1$ maps $\overline{\mathcal{M}}_2$ to the blowing-up of $\mathfrak{X}$ with center $\mathcal{J}_{\mathbb{Q}}$ (loc. cit., Corollaire 3.1) and that $60\lambda + 48\delta_1$ is very ample, realizing $\overline{\mathcal{M}}_2$ as the blowing-up of $\mathfrak{X}$ with center the ideal generated by I_4^3, J_{10}, H_6^2 and $I_4^2H_6$ (loc. cit., Corollaire 3.2).

3. Necessary conditions for very ample divisors on $\overline{\mathcal{M}}_3$. In this section we compute necessary conditions for very ample divisors on the moduli space $\overline{\mathcal{M}}_3$. As we mentioned in Section 1, a divisor $D = a\lambda - b\delta_0 - c\delta_1 \in \mathrm{Pic}(\overline{\mathcal{M}}_3)$ with $a, b, c \in \mathbb{Z}$ is ample if and only if $a - 12b + c > 0$ and $2b > c > 0$. The necessary conditions for very ample D are obtained as in Section 2: for a k-dimensional subvariety V of $\overline{\mathcal{M}}_3$, the intersection product $D^k \cdot [V]$ should be an integer. We use the computation of the Chow ring of $\overline{\mathcal{M}}_3$ in [Fa 1]. The computations are more involved than in the case of genus 2; also, we know the fundamental classes of more subvarieties.

First we look at the degree of $\overline{\mathcal{M}}_3$:

$$\begin{aligned} D^6 = (a\lambda - b\delta_0 - c\delta_1)^6 = {} & \tfrac{1}{90720}a^6 - \tfrac{1}{576}a^4c^2 - \tfrac{1}{18}a^3b^3 + \tfrac{1}{48}a^3bc^2 + \tfrac{35}{3456}a^3c^3 \\ & + \tfrac{5}{8}a^2b^2c^2 - \tfrac{43}{96}a^2bc^3 + \tfrac{13}{512}a^2c^4 + \tfrac{203}{20}ab^5 - \tfrac{145}{12}ab^3c^2 \\ & + \tfrac{25}{4}ab^2c^3 - \tfrac{31}{48}abc^4 + \tfrac{149}{7680}ac^5 - \tfrac{4103}{72}b^6 + 55b^4c^2 \\ & - \tfrac{505}{18}b^3c^3 + \tfrac{65}{16}b^2c^4 - \tfrac{91}{384}bc^5 + \tfrac{5}{1024}c^6, \end{aligned}$$

as follows from [Fa 1], p. 418. The requirement that this is in $\mathbb{Z}_2$ implies, firstly, that $2|c$, secondly, that $2|a$ *and* $4|c$, thirdly, that $2|b$. Looking in $\mathbb{Q}_3$ we get, firstly, that $3|a$, secondly, that $3|b$. Modulo 5 we get $5|a$ or $5|(a+3b+c)$. Finally, working modulo 7 we find that $7|a$ should hold.

Writing $a = 42a_1$, $b = 6b_1$ and $c = 4c_1$, with $a_1, b_1, c_1 \in \mathbb{Z}$, the condition $D^6 \cdot [\overline{\mathcal{M}}_3] \in \mathbb{Z}$ becomes $5|a_1$ or $5|(3a_1+2b_1+c_1)$. Interestingly, unlike the case of genus 2, these conditions are not the only necessary conditions we find.

For instance, the condition $D^5 \cdot \delta_0 \in \mathbb{Z}$ translates in $3|c_1$; then $[\Delta_1] = 2\delta_1$ gives no further conditions; but the hyperelliptic locus, with fundamental class $[\mathcal{H}_3] = 18\lambda - 2\delta_0 - 6\delta_1$, improves the situation modulo 5: necessarily $5|(3a_1+2b_1+c_1)$. It follows that $D^5 \cdot \lambda$ is an integer, so all divisors have integer-valued degrees.

In codimension 2, writing $c_1 = 3c_2$ with $c_2 \in \mathbb{Z}$, the condition $D^4 \cdot [\Delta_{01a}] \in \mathbb{Z}$ translates in

$$5|a_1 \qquad \text{or} \qquad 5|c_2 \qquad \text{or} \qquad 5|(a_1+c_2) \qquad \text{or} \qquad 5|(a_1+3c_2).$$

The (boundary) classes $[\Delta_{00}]$, $[\Delta_{01b}]$, $[\Delta_{11}]$, $[\Xi_0]$, $[\Xi_1]$ and $[\mathrm{H}_1]$ ([Fa 1], pp. 340 sqq.) give no further conditions.

In codimension 3, the class $[(i)] = 8[(i)]_Q$ forces $2|a_1$. Write $a_1 = 2a_2$ with $a_2 \in \mathbb{Z}$. Somewhat surprisingly, the class $[\mathrm{H}_{01a}] = 4\eta_0$ (loc. cit., pp. 386, 388) gives the condition $5|(a_2+2c_2)$. Consequently, combining the various conditions modulo 5, we obtain

$$5|a_2 \qquad \text{and} \qquad 5|b_1 \qquad \text{and} \qquad 5|c_2.$$

Finally, we checked that the 12 cycles in codimension 4 and the 8 cycles in codimension 5 (loc. cit., pp. 346 sq.) do not give extra conditions.

PROPOSITION 4. *A very ample divisor $a\lambda - b\delta_0 - c\delta_1$ on the moduli space $\overline{\mathcal{M}}_3$ satisfies the following conditions:*

a) *$a, b, c \in \mathbb{Z}$ with $a - 12b + c > 0$ and $2b > c > 0$;*
b) *$420|a$ and $30|b$ and $60|c$.*

COROLLARY 5. *The degree of a projective embedding of $\overline{\mathcal{M}}_3$ is at least*

$$650924662500 = 2^2 \cdot 3^2 \cdot 5^5 \cdot 7 \cdot 826571.$$

Proof. We need to minimize the expression given for the degree of $\overline{\mathcal{M}}_3$ while fulfilling the conditions in the proposition. Write $a = 420A$, $b = 30B$ and $c = 60C$. One shows that in the cone given by $7A - 6B + C \geq 0$ and $B \geq C \geq 0$ the degree is minimal along the (extremal) ray $(A, B, C) = (5x, 7x, 7x)$ (corresponding to $10\lambda - \delta_0 - 2\delta_1$). Comparing the value for $(A, B, C) = (5, 7, 7)$ with that for $(A, B, C) = (2, 2, 1)$, one concludes $A \leq 5$, $B \leq 7$ and $C \leq 7$. This leaves only a few triples in the interior of the cone; the minimum degree is obtained for $(A, B, C) = (2, 2, 1)$, corresponding to $840\lambda - 60\delta$.

Remark. In [Fa 1], Questions 5.3 and 5.4, we asked whether the classes X (resp. Y) are multiples of classes of complete subvarieties of $\overline{\mathcal{M}}_3$ of dimension 4 (resp. 3) having empty intersection with Δ_1 (resp. Δ_0). We still do not know the answers, but we verified that X and $-Y = 504\lambda_3$ are effective:

$$X = \tfrac{1}{15}\delta_{00} + \tfrac{1}{6}\delta_{01a} + \tfrac{11}{15}\delta_{01b} + 8\delta_{11} + \tfrac{3}{14}\xi_0 + \tfrac{48}{35}\xi_1 + \tfrac{40}{21}\eta_1;$$
$$-Y = \tfrac{1}{2}[(a)]_Q + [(b)]_Q + [(c)]_Q + \tfrac{11}{30}[(d)]_Q + \tfrac{2}{5}[(f)]_Q + 2[(g)]_Q + \tfrac{2}{3}\eta_0.$$

(For the notation, see [Fa 1], pp. 343, 386, 388.)

4. Algebro-geometric calculation of the intersection numbers of the tautological classes on $\overline{\mathcal{M}}_3$. Here we show how to compute the intersection numbers of the classes κ_i $(1 \le i \le 6)$ on $\overline{\mathcal{M}}_3$ in an algebro-geometric setting. These calculations were done originally in May 1990 to check the genus 3 case of Witten's conjecture [Wi], now proven by Kontsevich [Ko]. We believe that there is still an interest, though, in finding methods within algebraic geometry that allow to compute the intersection numbers of the kappa- or tau-classes. For instance, the identity

$$K^{3g-2} = \langle \tau_{3g-2} \rangle = \langle \kappa_{3g-3} \rangle = \frac{1}{(24)^g \cdot g!}$$

(in cohomology) should be understood ([Wi], between (2.26) and (2.27)).

In [Fa 1] the 4 intersection numbers of κ_1 and κ_2 were computed; using the identity $\kappa_1 = 12\lambda - \delta_0 - \delta_1$, we can read these off from Table 10 on p. 418:

$$\kappa_1^6 = \tfrac{176557}{107520}, \quad \kappa_1^4\kappa_2 = \tfrac{75899}{322560}, \quad \kappa_1^2\kappa_2^2 = \tfrac{32941}{967680}, \quad \kappa_2^3 = \tfrac{14507}{2903040}.$$

To compute the other intersection numbers, we need to express the other kappa-classes in terms of the bases introduced in [Fa 1]. The set-up is as in [Mu 1], §8 (and §6): if C is a stable curve of genus 3, ω_C is generated by its global sections, unless

a) C has 1 or 2 nodes of type δ_1, in which case the global sections generate the subsheaf of ω_C vanishing in these nodes;
b) C has 3 nodes of type δ_1, i.e., C is a $\mathbb{P}^1$ with 3 (possibly singular) elliptic tails, in which case $\Gamma(\omega_C)$ generates the subsheaf of ω_C of sections vanishing on the $\mathbb{P}^1$.

(See [Mu 1], p. 308.) Let $Z \subset \overline{\mathcal{C}}_3$ be the closure of the locus of pointed curves with 3 nodes of type δ_1 and with the point lying on the $\mathbb{P}^1$. Working over $\overline{\mathcal{C}}_3 - Z$ we get

$$0 \to \mathcal{F} \to \pi^*\pi_*\omega_{\overline{\mathcal{C}}_3/\overline{\mathcal{M}}_3} \to I_{\Delta_1^*} \cdot \omega_{\overline{\mathcal{C}}_3/\overline{\mathcal{M}}_3} \to 0$$

with $\mathcal{F}$ locally free of rank 2. Working this out as in [Fa 1], p. 367 we get

$$0 = c_3(\mathcal{F}) = \pi^*\lambda_3 - K \cdot \pi^*\lambda_2 + K^2 \cdot \pi^*\lambda_1 - K^3$$
$$- (\pi^*\lambda_1 - K) \cdot [\Delta_1^*]_Q + i_{1,*}(K_1 + K_2)$$

modulo $[Z]$. Multiplying this with K and using that ω^2 is trivial on $[\Delta_1^*]$, we get

(1) $$0 = K \cdot c_3(\mathcal{F}) = K \cdot \pi^*\lambda_3 - K^2 \cdot \pi^*\lambda_2 + K^3 \cdot \pi^*\lambda_1 - K^4 + {*}K \cdot [Z].$$

It is easy to see that $K^2 \cdot [Z] = 0$, so we also get

(2) $$0 = K^2 \cdot \pi^*\lambda_3 - K^3 \cdot \pi^*\lambda_2 + K^4 \cdot \pi^*\lambda_1 - K^5,$$

(3) $$0 = K^3 \cdot \pi^*\lambda_3 - K^4 \cdot \pi^*\lambda_2 + K^5 \cdot \pi^*\lambda_1 - K^6,$$

(4) $$0 = K^4 \cdot \pi^*\lambda_3 - K^5 \cdot \pi^*\lambda_2 + K^6 \cdot \pi^*\lambda_1 - K^7.$$

Pushing-down to $\overline{\mathcal{M}}_3$ we get

(1′) $$0 = 4\lambda_3 - \kappa_1\lambda_2 + \kappa_2\lambda_1 - \kappa_3 + N \cdot [(i)]_Q,$$

(2′) $$0 = \kappa_1\lambda_3 - \kappa_2\lambda_2 + \kappa_3\lambda_1 - \kappa_4,$$

$$(3') \qquad 0 = \kappa_2\lambda_3 - \kappa_3\lambda_2 + \kappa_4\lambda_1 - \kappa_5,$$

$$(4') \qquad 0 = \kappa_3\lambda_3 - \kappa_4\lambda_2 + \kappa_5\lambda_1 - \kappa_6.$$

To get κ_3 from $(1')$ we use two things. Firstly, one computes $Y = -504\lambda_3$, as mentioned at the end of Section 3. This follows since both Y and λ_3 are in the one-dimensional subspace of $A^3(\overline{\mathcal{M}}_3)$ of classes vanishing on all subvarieties of Δ_0. The factor -504 is computed using $\lambda^4 = 8\lambda\lambda_3$ or $\lambda_3 \cdot [(i)]_Q = \frac{1}{6}\lambda^3 \cdot [(i)]_Q$. Secondly, to compute N, one uses that κ_3 vanishes on the classes $[(b)]_Q$, $[(c)]_Q$, $[(f)]_Q$, $[(g)]_Q$, $[(h)]_Q$ and $[(i)]_Q$. This gives 6 relations in N of which 3 are identically zero; the other 3 all imply $N = 1$.

The formulas above allow one to express the kappa-classes in terms of the bases of the Chow groups given in [Fa 1]. We give the formula for κ_3 (from which the other formulas follow):

$$\begin{aligned}\kappa_3 = {} & \tfrac{1}{280}[(a)]_Q + \tfrac{31}{840}[(b)]_Q + \tfrac{19}{420}[(c)]_Q + \tfrac{1}{1260}[(d)]_Q + \tfrac{1}{35}[(e)]_Q \\ & + \tfrac{19}{840}[(f)]_Q + \tfrac{29}{84}[(g)]_Q + \tfrac{11}{35}[(h)]_Q + \tfrac{93}{35}[(i)]_Q + \tfrac{11}{252}\eta_0.\end{aligned}$$

This gives the following intersection numbers:

$$\kappa_1^3\kappa_3 = \tfrac{4073}{161280}, \quad \kappa_1\kappa_2\kappa_3 = \tfrac{149}{40320}, \quad \kappa_3^2 = \tfrac{131}{322560}, \quad \kappa_1^2\kappa_4 = \tfrac{2173}{967680},$$
$$\kappa_2\kappa_4 = \tfrac{971}{2903040}, \quad \kappa_1\kappa_5 = \tfrac{1}{5760}, \quad \kappa_6 = \tfrac{1}{82944}.$$

5. A few intersection numbers in genus 4. Kontsevich's proof of Witten's conjecture enables one to compute the intersection numbers of the kappa-classes on the moduli space of stable curves of arbitrary genus. There are many more intersection numbers that one would like to know, see e.g. [BCOV], (5.54) and end of Appendix A. As a challenge, we pose the following problem:

PROBLEM. *Find an algorithm that computes the intersection numbers of the divisor classes* $\lambda, \delta_0, \delta_1, \ldots, \delta_{[g/2]}$ *on* $\overline{\mathcal{M}}_g$.

These numbers are known for $g = 2$ [Mu 1] and $g = 3$ [Fa 1]. Note that the problem includes the computation of κ_1^{3g-3}.

PROPOSITION 6. *Denote by* h_g *the intersection number* $\lambda^{2g-1} \cdot [\overline{\mathcal{H}}_g]_Q$, *where* $\overline{\mathcal{H}}_g$ *is the closure in* $\overline{\mathcal{M}}_g$ *of the hyperelliptic locus. Then*

$$h_1 = \frac{1}{96};$$

$$h_g = \frac{2}{2g+1}\sum_{i=1}^{g-1} i(i+1)(g-i)(g-i+1)\binom{2g-2}{2i-1}h_i h_{g-i} \qquad \textit{for} \qquad g \geq 2.$$

Proof. This follows from [C-H], Proposition 4.7, which expresses λ on $\overline{\mathcal{H}}_g$ in terms of the classes of the components of the boundary $\overline{\mathcal{H}}_g - \mathcal{H}_g$. It is easy to see that $\lambda^{2g-2}\xi_i = 0$ for $0 \leq i \leq [(g-1)/2]$. Also,

$$\lambda^{2g-2}\delta_j[\overline{\mathcal{H}}_g]_Q = (2j+2)(2g-2j+2)\binom{2g-2}{2j-1}h_j h_{g-j},$$

because $\lambda = \pi_j^*\lambda + \pi_{g-j}^*\lambda$ on $\Delta_j \cap \overline{\mathcal{H}}_g$. Normalizing h_1 to $\frac{1}{96}$, which reflects the identity $\lambda = \frac{1}{24}p$ on $\overline{\mathcal{M}}_{1,1}$ and the fact that an elliptic curve has four 2-torsion points, we get the formula. ∎

This gives for instance $h_2 = \frac{1}{2880}$, $h_3 = \frac{1}{10080}$ and $h_4 = \frac{31}{362880}$. So this already gives the value of λ^3 on $\overline{\mathcal{M}}_2$, and the value of λ^6 on $\overline{\mathcal{M}}_3$ follows very easily: we only need that $[\mathcal{H}_3]_Q = 9\lambda$ in $A^1(\mathcal{M}_3)$, because clearly $\lambda^5\delta_0 = \lambda^5\delta_1 = 0$. We get $\lambda^6 = \frac{1}{90720}$.

Proposition 7. $\lambda^9 = \frac{1}{113400}$ *on* $\overline{\mathcal{M}}_4$.

Proof. We need to know the class $[\overline{\mathcal{H}}_4]$ modulo the kernel in $A^2(\overline{\mathcal{M}}_4)$ of multiplication with λ^7. We computed this class using the test surfaces of [Fa 2]; of the 14 classes at the bottom of p. 432, only κ_2, λ^2 and δ_1^2 are not in the kernel of $\cdot\lambda^7$, and the result is:

$$[\overline{\mathcal{H}}_4] \equiv 3\kappa_2 - 15\lambda^2 + \tfrac{27}{5}\delta_1^2 \pmod{\ker(\cdot\lambda^7)}.$$

We also have the relation ([Fa 2], p. 440)

$$60\kappa_2 - 810\lambda^2 + 24\delta_1^2 \equiv 0 \pmod{\ker(\cdot\lambda^7)}.$$

Thus $[\overline{\mathcal{H}}_4] \equiv \frac{51}{2}\lambda^2 + \frac{21}{5}\delta_1^2$. We compute

$$\begin{aligned}\lambda^7\delta_1^2 &= \tbinom{7}{1}(\lambda\cdot[\overline{\mathcal{M}}_{1,1}]_Q)(\lambda^6\cdot(-K_{\overline{\mathcal{M}}_{3,1}/\overline{\mathcal{M}}_3})\cdot[\overline{\mathcal{M}}_{3,1}])\\ &= 7\cdot\tfrac{1}{24}\cdot\tfrac{-4}{90720}\\ &= \tfrac{-1}{77760}.\end{aligned}$$

Therefore

$$\lambda^9 = \tfrac{2}{51}\left(2\cdot\tfrac{31}{362880} + \tfrac{21}{5}\cdot\tfrac{1}{77760}\right) = \tfrac{1}{113400}.$$

Also

$$\lambda^7\kappa_2 = \tfrac{169}{1360800}.$$

The hardest part of this proof is the computation of (three of) the coefficients of the class $[\overline{\mathcal{H}}_4]$. We present the test surfaces we need to compute these coefficients. Write

$$\begin{aligned}[\overline{\mathcal{H}}_4] = 3\kappa_2 - 15\lambda^2 + c\lambda\delta_0 + d\lambda\delta_1 + e\delta_0^2 + f\delta_0\delta_1 + g\delta_0\delta_2\\ + h\delta_1^2 + i\delta_1\delta_2 + j\delta_2^2 + k\delta_{00} + l\delta_{01a} + m\gamma_1 + n\delta_{11}.\end{aligned}$$

The class $[\mathcal{H}_4] \in A^2(\mathcal{M}_4)$ was computed by Mumford ([Mu 1], p. 314).

a) Take test surface (α) from [Fa 2], p. 433: two curves of genus 2 attached in one point; on both curves the point varies. We have $[\overline{\mathcal{H}}_4]_Q = 6\cdot 6 = 36$ and $\delta_2^2 = 8$. Thus $j = 9$.

b) Test surface (ζ): curves of type δ_{12}, vary the elliptic tail and the point on the curve of genus 2. We have $[\overline{\mathcal{H}}_4] = 0$, $\delta_0\delta_2 = -24$ and $\delta_1\delta_2 = 2$. Thus $i = 12g$.

c) Test surface (μ): curves of type δ_{02}, vary the elliptic curve in a simple pencil with 3 disjoint sections and vary the point on the curve of genus 2. Then $\delta_0\delta_2 = -20$ and $\delta_2^2 = 4$. To compute $[\overline{\mathcal{H}}_4]$ we use a trick. Consider the pencil of curves of genus 3 which we get by replacing the one-pointed curve of genus 2 with a fixed one-pointed curve of genus 1. On that pencil $\lambda = 1$, $\delta_0 = 12 - 1 - 1 = 10$, $\delta_1 = -1$, thus $[\overline{\mathcal{H}}_3]_Q = 9\lambda - \delta_0 - 3\delta_1 = 2$. So on the test surface we get $[\overline{\mathcal{H}}_4]_Q = 2\cdot 6 = 12$. Therefore $-20g + 36 = 24$ so $g = \frac{3}{5}$ and $i = \frac{36}{5}$.

d) This test surface is taken from [Fa 3], pp. 72 sq. We take the universal curve over a pencil of curves of genus 2 as in [A-C], p. 155, and we attach a fixed one-pointed curve of genus 2. As in [Fa 3] we have $\lambda = 3(G - \Sigma)$, $\delta_0 = 30(G - \Sigma)$, $\delta_2 = -2G + \Sigma$. Since $G^2 = 2$, $G\Sigma = 0$ and $\Sigma^2 = -2$ we have $\delta_0\delta_2 = -60$ and $\delta_2^2 = 6$. To compute κ_2 we use the same trick as above: replacing the fixed one-pointed curve of genus 2 by one of genus 1, we get a test surface of curves of genus 3. This will not affect the computation of κ_2; using the formulas of [Fa 1] we find $\kappa_2 = 6$. Also $\delta_0\Sigma = 2\gamma_1$ here, thus $\gamma_1 = 30$. Since $[\overline{\mathcal{H}}_4] = 0$, we get $0 = 18 - 60g + 6j + 30m = 30m + 36$ so $m = -\frac{6}{5}$.

e) Test surface (λ) from [Fa 2]: curves of type δ_{12}, vary both the j-invariant of the middle elliptic curve and the (second) point on it. We have $\delta_0\delta_2 = -12$, $\delta_1\delta_2 = 1$, $\delta_2^2 = 1$, $\kappa_2 = 1$, $\delta_{01a} = 12$ and $\gamma_1 = 12$. Since $[\overline{\mathcal{H}}_4] = 0$, we get $0 = 3 - 12g + i + j + 12l + 12m = 12l - \frac{12}{5}$ so $l = \frac{1}{5}$.

f) Test surface (κ): curves of type δ_{12}, vary a point on the middle elliptic curve and vary the elliptic tail. Then $\delta_0\delta_2 = -12$, $\delta_1\delta_2 = 1$, $\delta_{01a} = -12$, $\delta_{11} = -1$. Since $[\overline{\mathcal{H}}_4] = 0$, we find $0 = -12g + i - 12l - n$ so $n = -\frac{12}{5}$.

g) The final test surface we need is (γ) from [Fa 2]: we attach fixed elliptic tails to two varying points on a curve of genus 2. Then $\delta_1^2 = 16$, $\delta_2^2 = -2$, $\kappa_2 = 2$, $\delta_{11} = 6$. When the two varying points are distinct Weierstrass points, we get hyperelliptic curves. So $[\overline{\mathcal{H}}_4]_Q = 6 \cdot 5 = 30$ and we get $60 = 6 + 16h - 2j + 6n = 16h - \frac{132}{5}$ so $h = \frac{27}{5}$, as claimed.

This finishes the proof of Proposition 7. ∎

We can now evaluate the contribution from the constant maps for $g = 4$ (cf. [BCOV], §5.13, (5.54)):

Corollary 8. *$\lambda_3^3 = \frac{1}{43545600}$ on $\overline{\mathcal{M}}_4$.*

Proof. As explained in [Mu 1], §5, we have on $\overline{\mathcal{M}}_4$ the identity

$$(*) \qquad (1 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)(1 - \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4) = 1.$$

One checks that this implies $\lambda_3^3 = \frac{1}{384}\lambda_1^9$, which finishes the proof.

Corollary 9 (Schottky, Igusa). *The class of $\mathcal{M}_4$ in $\mathcal{A}_4$ equals 8λ.*

Proof. Since $(*)$ holds also on the toroidal compactification $\widetilde{\mathcal{A}}_4$, we get $\lambda_1^{10} = 384\lambda_1\lambda_3^3 = 768\lambda_1\lambda_2\lambda_3\lambda_4$. But it follows from Hirzebruch's proportionality theorem [Hi 1, 2] that

$$\lambda_1\lambda_2\lambda_3\lambda_4 = \prod_{i=1}^{4} \frac{|B_{2i}|}{4i} = \frac{1}{1393459200},$$

hence $\lambda^{10} = \frac{1}{1814400}$ on $\widetilde{\mathcal{A}}_4$. Using Theorem 1.5 in [Mu 2] we see that the class of $\mathcal{M}_4$ in $\mathcal{A}_4$ is a multiple of λ. Denote by $t : \mathcal{M}_4 \to \mathcal{A}_4$ the Torelli morphism and denote by $\mathcal{J}_4$ its image, the locus of Jacobians. Proposition 7 tells us that $t^*\lambda^9 = \frac{1}{113400}$. Applying t_* we get $[\mathcal{J}_4] \cdot \lambda^9 = \frac{1}{113400}$, hence $[\mathcal{J}_4] = 16\lambda$, hence $[\mathcal{J}_4]_Q = 8\lambda$, as claimed. (The subtlety corresponding to the fact that a general curve of genus $g \geq 3$ has only the trivial automorphism, while its Jacobian has two automorphisms, appears also in computing λ^6 on $\overline{\mathcal{M}}_3$ resp. on $\widetilde{\mathcal{A}}_3$: we have already seen that $t^*\lambda^6 = \frac{1}{90720}$; applying t_* we get $[\mathcal{J}_3] \cdot \lambda^6 = \frac{1}{90720}$; since $[\mathcal{J}_3] = 2[\widetilde{\mathcal{A}}_3]_Q$, we get $\lambda^6 = \frac{1}{181440}$, which is also what one gets using the proportionality theorem.)

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