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# ON ADELIC CHERN FORMS AND THE BOTT RESIDUE FORMULA

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**0. Introduction.** The Bott Residue Formula gained renewed attention recently due to its use in enumerative algebraic geometry (cf. [ES], [Ko]). If $X$ is a smooth projective variety over a field $k$ of characteristic 0, then Bott's formula makes sense purely algebraically, with the Chern classes taken in the algebraic De Rham cohomology $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X/k)$. In this paper we survey an algebraic proof of the formula using Beilinson adeles, which was discovered by R. Hübl and the author (see [HY]).

Suppose $v \in \Gamma(X, \mathcal{T}_X)$ is a global vector field with isolated, simple, $k$-rational zeroes (see Remark 3.2 for generalizations). Let $\mathcal{E}_1, \ldots, \mathcal{E}_m$ be locally free $\mathcal{O}_X$-modules. Suppose $\Lambda_i$ is an action of $v$ on $\mathcal{E}_i$, i.e. a differential operator $\Lambda_i : \mathcal{E}_i \to \mathcal{E}_i$ satisfying $\Lambda_i(ae) = v(a)e + a\Lambda_i(e)$ for local sections $a \in \mathcal{O}_X$, $e \in \mathcal{E}_i$. Suppose $Q(t_{i,j})$ is a homogeneous polynomial of degree $n = \dim X$ in the variables $t_{i,j}$ ($i = 1, \ldots, m$; $j = 1, \ldots, r_i$; $r_i := \operatorname{rank} \mathcal{E}_i$) which have degrees $\deg t_{i,j} = j$. For a zero $z$ of $v$ let us denote by $\Lambda_i|_z$ the restriction of $\Lambda_i$ to $\mathcal{E}_i|_z := \mathcal{E}_i \otimes k(z)$, which is a $k$-linear endomorphism. Also let us denote by $\operatorname{ad} v|_z$ the restriction of $\operatorname{ad} v$ to $\mathcal{T}_X \otimes k(z)$; this is invertible. We let $P_i$ denote the $i$th conjugation-invariant polynomial on matrices (of unspecified size). Finally let $\int_X : \mathrm{H}^{2n}_{\mathrm{DR}}(\mathrm{X/k}) \to \mathrm{k}$ be a canonical map (cap product with the fundamental class).

THEOREM 0.1. (Bott Residue Formula).

$$\int_X Q(c_j(\mathcal{E}_i)) = \sum_{v(z)=0} Q(P_j(\Lambda_i|_z)) \cdot \det(\operatorname{ad} v|_z)^{-1}$$

In Section 1 we discuss Beilinson's adeles and the sheaves $\mathcal{A}^{p,q}_X$, $\tilde{\mathcal{A}}^{p,q}_X$. These are analogues of the sheaves of smooth $(p,q)$-forms on a complex manifold. In Section 2 we define connections on the adelic sections $\tilde{\mathcal{A}}^0_X(\mathcal{E})$ of a vector bundle. Finally in Section 3 we prove

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Theorem 0.1. The proof is almost identical to Bott's proof in [Bo2]. In particular we use a projector $\omega \in \tilde{\mathcal{A}}_X^{1,0}$ to localize the integral to the zero locus of $v$.

I should mention other proofs of Bott's formula. Atiyah-Bott [AB] use a mix of analysis and topology. Carrel-Lieberman [CL] state their proof for complex manifolds, but it applies also to the purely algebraic setup.

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**1. Adeles.** Let $k$ be field of characteristic 0, and let $X$ be a smooth $n$-dimensional projective variety over $k$. According to Beilinson, to each quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ there is associated a cosimplicial sheaf $\underline{\mathbb{A}}^{\cdot}(\mathcal{M})$ on $X$, the *sheaf of adeles* (see [Be] and [Hr]). The definition of $\mathbb{A}^q(U,\mathcal{M}) = \Gamma(U, \underline{\mathbb{A}}^q(\mathcal{M}))$ is by a zig-zag process of direct and inverse limits, generalizing the classical adeles. (If $X$ is a smooth curve then the classical ring of adeles $\mathbb{A}(X)$ is just $\mathbb{A}^1_{\mathrm{red}}(X, \mathcal{O}_X)$.) One has a natural isomorphism $\underline{\mathbb{A}}^q(\mathcal{M}) \cong \underline{\mathbb{A}}^q(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M}$. Denote by $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})$ the standard normalization of $\underline{\mathbb{A}}^{\cdot}(\mathcal{M})$ (namely the common kernel of the codegeneracy maps), which is a complex of sheaves with coboundary operator $\partial$. Then each $\underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{M})$ is a flasque sheaf, and the natural map $\mathcal{M} \to \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})$ is a quasi-isomorphism.

The adeles $\mathbb{A}^q_{\mathrm{red}}(\mathcal{M})$ are a subsheaf of the product of the local factors $\prod_\xi \mathcal{M}_\xi$, where $\xi = (x_0, \ldots, x_q)$ runs over the set of reduced chains of length $q$ in $X$. For $\mathcal{M}$ coherent and $q = 0$ we simply have $\mathcal{M}_{(x)} = \widehat{\mathcal{M}}_x$, the $\mathfrak{m}_x$-adic completion.

Now if $D : \mathcal{M} \to \mathcal{N}$ is a differential operator between $\mathcal{O}_X$-modules, there is an induced operator $D : \underline{\mathbb{A}}^{\cdot}(\mathcal{M}) \to \underline{\mathbb{A}}^{\cdot}(\mathcal{N})$, compatible with the cosimplicial structure. Applying this to the De Rham complex $\Omega^{\cdot}_{X/k}$ we get a cosimplicial differential graded algebra (DGA)

$$\underline{\mathbb{A}}^{\cdot}(\Omega^{\cdot}_{X/k}) = \bigcup_{q \geq 0} \bigoplus_{p \geq 0} \underline{\mathbb{A}}^q(\Omega^p_{X/k}). \tag{1.1}$$

Definition 1.1. For $p, q \geq 0$ let $\mathcal{A}_X^{p,q} := \underline{\mathbb{A}}^q_{\mathrm{red}}(\Omega^p_{X/k})$. Then $\mathcal{A}_X^{\cdot\cdot}$ is a double complex, with commuting operators $\mathrm{d} : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p+1,q}$ and $\partial : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1}$, called the *De Rham-adele* double complex. Set $\mathrm{D}' := \mathrm{d}$, $\mathrm{D}'' := (-1)^p \partial$, $\mathrm{D} := \mathrm{D}' + \mathrm{D}''$ and $\mathcal{A}_X^i := \bigoplus_{p+q=i} \mathcal{A}_X^{p,q}$. Then $\mathcal{A}_X^{\cdot}$, with Alexander-Whitney product and the operator D, is a sheaf of DGAs on $X$.

Proposition 1.2. *The natural DGA map* $\Omega^{\cdot}_{X/k} \to \mathcal{A}_X^{\cdot}$ *is a quasi-isomorphism.*

The proposition implies that $\mathrm{H}^{\cdot}_{\mathrm{DR}}(\mathrm{X/k})$ with its cup product can be calculated as $\mathrm{H}^{\cdot}\Gamma(\mathrm{X}, \mathcal{A}_{\mathrm{X}}^{\cdot})$. However, the DGA $\mathcal{A}_X^{\cdot}$ is not (graded) commutative.

According to [Ye], for every maximal chain $\xi = (x_0, \ldots, x_n)$ in $X$ there is a residue map $\mathrm{Res}_\xi : \Omega^n_{X/k,\xi} \to k$. This induces

$$\int_X = \sum_\xi \mathrm{Res}_\xi : \mathrm{H}^{2n}\Gamma(\mathrm{X}, \mathcal{A}_{\mathrm{X}}^{\cdot}) \to \mathrm{k}. \tag{1.2}$$

$\int_X$ coincides with cap product with the fundamental class of $X$. Thus for $k = \mathbb{C}$ we get the usual integral (up to a factor of $2\pi\sqrt{-1}$).

For $\ell \geq 0$ let

$$\Delta^\ell := \mathrm{Spec}\, \mathbb{Q}[t_0, \ldots, t_\ell]/(t_0 + \cdots + t_\ell - 1)$$

be the standard rational $\ell$-simplex, and let $\Omega^{\cdot}(\Delta^\ell)$ be the De Rham complex on it, which is a DGA over $\mathbb{Q}$ generated by $t_0, \ldots, t_\ell$. Then $\Omega^{\cdot}(\Delta^{\cdot}) = \bigcup_{\ell \geq 0} \Omega^{\cdot}(\Delta^\ell)$ is a simplicial DGA. The definition below is extracted from [HS].

Definition 1.3. Let

$$\tilde{\mathcal{A}}_X^{p,q} \subset \prod_{\ell=0}^{\infty} \left( \underline{\mathbb{A}}^\ell(\Omega_{X/k}^p) \otimes_{\mathbb{Q}} \Omega^q(\Delta^\ell) \right)$$

be the subsheaf consisting of all sections $u = (u_0, \ldots, u_\ell, \ldots)$ such that

$$\begin{aligned} (\partial^i \otimes 1)u_\ell &= (1 \otimes \partial_i)u_{\ell+1} \\ (1 \otimes s_i)u_\ell &= (s^i \otimes 1)u_{\ell+1} \end{aligned}$$

for $0 \leq \ell$, $0 \leq i \leq \ell+1$. Here $\partial_i, s_i, \partial^i, s^i$ are the (co)simplicial operators. Set $\mathrm{D}' := \mathrm{d} \otimes 1$, $\mathrm{D}'' := (-1)^{\mathrm{p}} \otimes \mathrm{d}$, $\mathrm{D} := \mathrm{D}' + \mathrm{D}''$ and $\tilde{\mathcal{A}}_X^i := \bigoplus_{p+q=i} \tilde{\mathcal{A}}_X^{p,q}$. The sheaf of *Thom-Sullivan adeles* is the commutative DGA $(\tilde{\mathcal{A}}_X^{\cdot}, \mathrm{D})$.

Observe that for $p, q \geq 0$,

$$\tilde{\mathcal{A}}_X^{p,q} \subset \prod_{\ell \geq 0} \prod_{\xi = (x_0, \ldots, x_\ell)} \left( \Omega_{X/k,\xi}^p \otimes_{\mathbb{Q}} \Omega^q(\Delta^\ell) \right). \tag{1.3}$$

Usual integration on the real $\ell$ simplex $\Delta^\ell(\mathbb{R})$ yields a $\mathbb{Q}$-linear map $\int_{\Delta^\ell} : \Omega^{\cdot}(\Delta^\ell) \to \mathbb{Q}$, such that $\int_{\Delta^\ell}(\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_\ell) = \frac{1}{\ell!}$. By linearity this extends to a map of sheaves $\int_\Delta : \tilde{\mathcal{A}}_X^{\cdot} \to \underline{\mathbb{A}}^{\cdot}(\Omega_{X/k}^{\cdot})$.

Theorem 1.4. ([HS]). *$\int_\Delta$ sends $\tilde{\mathcal{A}}_X^{p,q}$ into $\mathcal{A}_X^{p,q}$, and commutes with the operators* $\mathrm{D}'$, $\mathrm{D}''$. *Therefore $\int_\Delta : \tilde{\mathcal{A}}_X^{\cdot} \to \mathcal{A}_X^{\cdot}$ is a homomorphism of DG $\Omega_{X/k}^{\cdot}$-modules. For every open set $U \subset X$ the resulting map in cohomology* $\mathrm{H}^{\cdot}(\mathrm{U}, \int_\Delta) : \mathrm{H}^{\cdot}(\mathrm{U}, \tilde{\mathcal{A}}_{\mathrm{X}}^{\cdot}) \to \mathrm{H}^{\cdot}(\mathrm{U}, \mathcal{A}_{\mathrm{X}}^{\cdot})$ *is an isomorphism of graded $k$-algebras.*

**2. Connections over Adeles.** Our construction is a fusion of ideas of Bott (in [Bo1]) and Parshin (in [Pa]). Let $\mathcal{E}$ be a locally free sheaf on $X$, and set $\tilde{\mathcal{A}}_X^{p,q}(\mathcal{E}) := \tilde{\mathcal{A}}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{E}$. Suppose we are given a family $\{\nabla_{(x)}\}_{x \in X}$, where

$$\nabla_{(x)} : \mathcal{E}_{(x)} \to \Omega_{X/k,(x)}^1 \otimes_{\mathcal{O}_{X,(x)}} \mathcal{E}_{(x)}$$

is a connection over the $k$-algebra $\mathcal{O}_{X,(x)}$. Let $\xi = (x_0, \ldots, x_\ell)$ be a chain in $X$. For $0 \leq i \leq \ell$ consider the $i$-th covertex map $\partial_{(i)}^{(0,\ldots,\ell)} : \Omega_{X/k,(x_i)}^{\cdot} \to \Omega_{X/k,\xi}^{\cdot}$. By extension of scalars, $\nabla_{(x_i)}$ induces a connection

$$\nabla_{\xi,i} : \mathcal{E}_\xi \to \Omega_{X/k,\xi}^1 \otimes_{\mathcal{O}_{X,\xi}} \mathcal{E}_\xi$$

over the algebra $\mathcal{O}_{X,\xi}$. Set

$$\nabla_\xi := \sum_{i=0}^{\ell} t_i \nabla_{\xi,i} : \mathcal{E}_\xi \to \Omega_{X/k,\xi}^1 \otimes_{\mathbb{Q}} \mathcal{O}(\Delta^\ell) \otimes_{\mathcal{O}_{X,\xi}} \mathcal{E}_\xi.$$

Proposition 2.1. *Given a family of connections $\{\nabla_{(x)}\}_{x \in X}$, there is a unique connection*

$$\nabla : \tilde{\mathcal{A}}_X^0(\mathcal{E}) \to \tilde{\mathcal{A}}_X^1(\mathcal{E})$$

*over the algebra* $\tilde{\mathcal{A}}^0_X$*, such that under the embedding* (1.3), $(\nabla u)_\xi = \nabla_\xi u$ *for every local* (*algebraic*) *section* $u \in \mathcal{E}$*.*

Definition 2.2. The curvature form associated to $\{\nabla_{(x)}\}_{x\in X}$ is

$$R = \nabla^2 \in \tilde{\mathcal{A}}^2_X(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})).$$

Given an invariant polynomial $P$, one has $\mathrm{DP(R)} = 0$. The resulting Chern-Weil homomorphism

$$\{\text{invariant polynomials}\} \to \mathrm{H}^{\cdot}(\mathrm{X}, \tilde{\mathcal{A}}^{\cdot}_{\mathrm{X}}) \cong \mathrm{H}^{\cdot}(\mathrm{X}, \mathcal{A}^{\cdot}_{\mathrm{X}}),$$

$P \mapsto [P(R)]$, is a homomorphism of $k$-algebras, independent of the connection $\nabla$.

Definition 2.3. The $i$-th Chern form of $\mathcal{E}$ with respect to the connection $\nabla$ is $\tilde{c}_i(\mathcal{E}, \nabla) := P_i(R) \in \Gamma(X, \tilde{\mathcal{A}}^{2i}_X)$.

Theorem 2.4. *The Chern classes* $c_i(\mathcal{E}) = [\int_\Delta \tilde{c}_i(\mathcal{E}, \nabla)] \in \mathrm{H}^{2i}_{\mathrm{DR}}(\mathrm{X})$ *satisfy the Whitney Sum Formula and commute with pullback. The map* $\mathrm{dlog} : \mathrm{Pic}\, X = \mathrm{H}^1(\mathrm{X}, \mathcal{O}^*_{\mathrm{X}}) \to \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X})$ *sends the class of an invertible sheaf* $[\mathcal{L}]$ *to* $c_1(\mathcal{L})$*. Thus for* $k = \mathbb{C}$ *we get the usual Chern classes* (*up to a factor of* $2\pi\sqrt{-1}$).

**3. Proof of the Formula.** Denote by $Z$ the zero scheme of $v$, which is a finite reduced scheme. Choose an open subset $U \subset X$ containing $Z$, and sections $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O}_X)$, such that the corresponding morphism $U \to \mathbf{A}^n_k$ is unramified, and the fibre over the origin is the scheme $Z$. This is possible since $X$ is projective. Thus $\mathcal{T}_X|_U$ is trivial, with a frame $(\frac{\partial}{\partial f_1}, \ldots, \frac{\partial}{\partial f_n})$. Moreover, we can choose $U$ such that $\mathcal{E}_i|_U$ are trivial, with frames $\underline{e}_i : \mathcal{O}^{r_i}_U \xrightarrow{\simeq} \mathcal{E}_i|_U$.

From here we continue along the lines of [Bo2], but of course we use adeles instead of smooth functions. The sheaf $\tilde{\mathcal{A}}^{p,q}_X$ plays the role of the sheaf of smooth $(p, q)$ forms on a complex manifold. The operator $\mathrm{D}''$ behaves like the anti-holomorphic derivative $\bar{\partial}$; specifically $\mathrm{D}''\alpha = 0$ for $\alpha \in \Omega^{\cdot}_{X/k}$.

Set $\mathcal{E} := \bigoplus_{i=1}^m \mathcal{E}_i$, $r := \sum r_i$, $\Lambda := \sum \Lambda_i$. Then $\underline{e} = (\underline{e}_1, \ldots, \underline{e}_m)$ is a frame for $\mathcal{E}|_U$. For each point $x \in U$ the isomorphism $\underline{e} : \mathcal{O}^r_{X,(x)} \xrightarrow{\simeq} \mathcal{E}_{(x)}$ induces a Levi-Civita connection $\nabla_{(x)}$ on $\mathcal{E}_{(x)}$. For $x \notin U$ choose an arbitrary connection $\nabla_{(x)}$. Let $R = \nabla^2 \in \tilde{\mathcal{A}}^2_X(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ be the resulting curvature form. Note that $R = \sum R_i$, and we can define

$$P(R) := Q(P_j(R_i)) \in \tilde{\mathcal{A}}^{2n}_X.$$

$R$ decomposes into bi-homogeneous parts $R = R^{2,0} + R^{1,1}$. We will work with $R^{1,1}$. Since $\tilde{\mathcal{A}}^{p,q}_X = 0$ for $p > n$, we get $P(R) = P(R^{1,1})$.

One shows, like in [Bo2], that

$$L := \Lambda - \iota(v) \circ \nabla \in \tilde{\mathcal{A}}^0_X(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$$

satisfies

$$-\iota(v)R^{1,1} = \mathrm{D}''\mathrm{L} \tag{3.1}$$

$$L|_z = \Lambda|_z. \tag{3.2}$$

Since $\nabla$ is algebraic on $U$, it follows that

$$L|_U \in \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})|_U. \tag{3.3}$$

For every point $z \in Z$ let

$$\Xi_z := \{\xi = (x_0, \dots, x_n) \mid x_n = z\}.$$

This set of chains is the analogue of a small ball around $z$. Let $\Xi := \bigcup_{z \in Z} \Xi_z$.

Given $\alpha = (\alpha_\xi) \in \mathcal{A}_X^{n,n}$, we say $\alpha$ is holomorphic (resp. has a simple pole) along a maximal chain $\xi = (x_0, \dots, x_n)$ if for every $a \in \mathcal{O}_{X,x_n}$ (resp. $a \in \mathfrak{m}_{x_n}$) one has $\operatorname{Res}_\xi a\alpha_\xi = 0$ (cf. [Ye] §4.2).

Denote the canonical pairing $\mathcal{T}_X \otimes \Omega^1_{X/k} \to \mathcal{O}_X$ by $\langle -, - \rangle$. It extends to a pairing $\tilde{\mathcal{A}}^0_X(\mathcal{T}_X) \otimes \tilde{\mathcal{A}}^0_X(\Omega^1_{X/k}) \to \tilde{\mathcal{A}}^0_X$.

Lemma 3.1. *There is a global section $\omega \in \tilde{\mathcal{A}}^{1,0}_X \cong \tilde{\mathcal{A}}^0_X(\Omega^1_{X/k})$ such that:*

(1) $\langle v, \omega \rangle = 1$ *on* $X - Z$.

(2) $\int_\Delta (\mathrm{D}''\omega)^{\mathrm{n}}$ *is holomorphic along any maximal chain* $\xi \notin \Xi$.

(3) $\int_\Delta (\mathrm{D}''\omega)^{\mathrm{n}}$ *has at most a simple pole along any* $\xi \in \Xi$. *Moreover, for any* $z \in Z$

$$\sum_{\xi \in \Xi_z} \operatorname{Res}_\xi \int_\Delta (\mathrm{D}''\omega)^{\mathrm{n}} = \det(\operatorname{ad} \mathrm{v}|_{\mathrm{z}})^{-1}.$$

The proof of the lemma is not difficult, but it is technical and we prefer to skip it. Let us just say that writing $v = \sum a_i \frac{\partial}{\partial f_i}$, $a_i \in \Gamma(U, \mathcal{O}_X)$, one can express $\omega$ in terms of the $a_i$.

Let $t$ be an indeterminate, and define

$$\begin{aligned} \eta &:= P(L + tR^{1,1}) \cdot \omega \cdot (1 - t\mathrm{D}''\omega)^{-1} \\ &= P(L + tR^{1,1}) \cdot \omega \cdot (1 + t\mathrm{D}''\omega + (\mathrm{t}\mathrm{D}''\omega)^2 + \cdots) \in \tilde{\mathcal{A}}^{\cdot}_{\mathrm{X}}[\mathrm{t}] \end{aligned} \tag{3.4}$$

(note that $(\mathrm{D}''\omega)^{\mathrm{n}+1} = 0$, so this makes sense). Writing $\eta = \sum_i \eta_i t^i$ we see that $\eta_i \in \tilde{\mathcal{A}}^{i+1,i}_X$. Just like in [Bo2], using formula (3.1) and Lemma 3.1, one shows that

$$\mathrm{D}''\eta_{\mathrm{n}-1} + \mathrm{P}(\mathrm{R}^{1,1}) = 0 \text{ on } \mathrm{X} - \mathrm{Z}. \tag{3.5}$$

*Proof of Theorem* 0.1. By definition $c_j(\mathcal{E}_i) = [\int_\Delta P_j(R_i)] \in \mathrm{H}^{2\mathrm{j}}_{\mathrm{DR}}(\mathrm{X})$. From Theorem 1.4 we see that

$$Q(c_j(\mathcal{E}_i)) = [\int_\Delta Q(P_j(R_i))] = [\int_\Delta P(R)].$$

As observed before $P(R) = P(R^{1,1}) \in \tilde{\mathcal{A}}^{2n}_X$. In view of formula (3.2) we must verify that

$$\int_X \int_\Delta P(R^{1,1}) = \sum_{z \in Z} P(L|_z) \det(\operatorname{ad} v|_z)^{-1}.$$

Now

$$\int_X \int_\Delta \mathrm{D}''\eta_{\mathrm{n}-1} = \int_{\mathrm{X}} \mathrm{D}'' \int_\Delta \eta_{\mathrm{n}-1} = 0$$

since $X$ is proper. Every maximal chain is either in $X - Z$ or in $\Xi$. Therefore, by (3.5)

$$\int_X \int_\Delta P(R^{1,1}) = \int_X \int_\Delta (P(R^{1,1}) + \mathrm{D}''\eta_{\mathrm{n}-1}) = \sum_{\xi \in \Xi} \operatorname{Res}_\xi \int_\Delta (\mathrm{P}(\mathrm{R}^{1,1}) + \mathrm{D}''\eta_{\mathrm{n}-1}).$$

By construction the connection $\nabla$ is integrable on $U$ (it is a Levi-Civita connection there with respect to the algebraic frame $\underline{e}$), therefore on $U$ one has: $R = 0$, $P(R^{1,1}) = 0$ and

$\mathrm{D}''\eta_{\mathrm{n}-1} = \mathrm{P(L)(D''}\omega)^{\mathrm{n}}$. The map $\int_\Delta$ is $\mathcal{O}_X$-linear, and by (3.3), $P(L)|_U \in \mathcal{O}_U$. Hence

$$\int_\Delta P(L)(\mathrm{D}''\omega)^{\mathrm{n}} = \mathrm{P(L)} \int_\Delta (\mathrm{D}''\omega)^{\mathrm{n}} \text{ on U.}$$

In view of Lemma 3.1 this concludes the proof. ■

REMARK 3.2. There are two easy extensions of Theorem 0.1.

(a) Dropping the assumption that the zeroes of $v$ are simple (cf. [HY]).

(b) Suppose $L \in \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ is a *semi-simple* endomorphism. Then there are well defined classes $P_j(L) \in \mathrm{H}^{2\mathrm{j}}_{\mathrm{DR}}(\mathrm{X/k})$, given by $[\int_\Delta P_j(L+R)]$ for an appropriate connection $\nabla$. For example $c_j(\mathcal{E}) = P_j(0_{\mathcal{E}})$ (cf. [Bo2]). If $L$ and $\Lambda$ commute the residue formula is:

$$\int_X P(L) = \sum_{v(z)=0} P((L+\Lambda)|_z) \cdot \det(\mathrm{ad}\, v|_z)^{-1}.$$

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