PARAMETER SPACES
BANACH CENTER PUBLICATIONS, VOLUME 36
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1996

PARAMETER SPACES FOR QUADRICS

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Abstract. The parameter spaces for quadrics are reviewed. In addition, an explicit formula for the number of quadrics tangent to given linear subspaces is presented.

1. Schubert's problem.

1.1. One century ago, in 1894, Schubert considered the following problem: Let P be a projective space. Assume there is given in P a finite number of linear subspaces in general position, say m_1 hyperplanes, m_2 codimension-2 planes, and in general, m_i codimension-i planes. Then, how many quadrics in P are tangent to the given linear subspaces?

In Schubert's problem, the quadrics are assumed to be non-singular. Assume $P = \mathbb{P}(E)$ where E is a vector space of rank r. Then a non-singular quadric in P corresponds to a regular symmetric $r \times r$ matrix up to multiplication by a scalar. The symmetric matrices form a vector space of rank $\binom{r+1}{2}$. Therefore, the set of non-zero symmetric matrices up to multiplication by a scalar is parametrized by a projective space of dimension $N := \binom{r+1}{2} - 1$. In this $\mathbb{P}^N$, the matrices with non-zero determinant form an open subset U. By construction, the points of U correspond to the non-singular quadrics in P, that is, U is a parameter space for the set of non-singular quadrics in P. The set of quadrics that are tangent to a given linear subspace of P form, in the parameter space U, a hypersurface. Therefore, in Schubert's problem it is natural to require that the number $\sum_i m_i$ of given linear subspaces is equal to the dimension N of the parameter space. Then the quadrics tangent to the given linear subspaces correspond in the parameter space to the points in the intersection of N hypersurfaces. It could be hoped that the intersection is finite; Schubert's problem is then to count the number of points in the intersection.

To solve the counting problem by enumerative techniques, a closed (or complete) parameter space is needed. By construction, the space U is an open (dense) subset of $\mathbb{P}^N$. A naive completion of U is then to take $\mathbb{P}^N$ as its closure. Clearly, the boundary points of U in $\mathbb{P}^N$ correspond to the singular quadrics in P. However, we cannot expect

1991 *Mathematics Subject Classification*: 14N10; 14M15.
Supported in part by the Danish Natural Science Research Council, grant 11–7428.
The paper is in final form and no version of it will be published elsewhere.

to solve Schubert's problem allowing singular quadrics as solutions. For instance, among the singular conics in a fixed projective plane are the double lines, corresponding to symmetric 3×3 matrices of rank 1. Viewed as a singular conic, a double line is tangent to any line. Hence, the set of conics tangent to any finite number of given lines will always contain the infinite set of double lines.

Schubert saw that it was possible to refine the notion of a limit point of U to obtain a different closure B of U. These refined limit points of U correspond to refined degenerations of non-singular quadrics. They are called complete quadrics. The refined closure B of U is then a parameter space for the complete quadrics. In the parameter space B, the complete quadrics tangent to a given linear subspace of P form a hypersurface. Moreover, given N linear subspaces in general position in P, the corresponding hypersurfaces of B intersect in a finite number of points.

In fact, Schubert considered a more general problem. He allowed the p-dimensional projective space P to vary in a fixed projective space Q subject to a given *Schubert condition:* fix in Q a flag of $r = p+1$ linear subspaces,

$$L_1 \subset L_2 \subset \cdots \subset L_r.$$

The corresponding Schubert condition on a p-plane P in Q is that $\dim P \cap L_i \geq i-1$ for $i = 1, \dots, r$. The Schubert condition is said to be of type $A = (a_1, \dots, a_r)$, where $a_i = \dim L_i$. The general problem considered by Schubert is the following: Given m_i codimension-i planes in Q and a Schubert condition of type $A = (a_1, \dots, a_r)$. How many quadrics in a variable p-plane P satisfying the given Schubert condition are tangent to the given linear subspaces?

1.2. To describe a naive parameter space for the general problem, assume that $Q = \mathbb{P}(V)$ where V is a vector space. Then the p-planes P in Q correspond to the rank-r quotients of V, where $r = p+1$. Thus the set of p-planes is parametrized by the Grassmannian $\mathrm{Grass}^r(V)$. In the Grassmannian, the p-planes P satisfying a given Schubert condition of type A form a subspace Ω of dimension equal to $\sum_{i=1}^r (a_i - i + 1)$. For a fixed p-plane P, the space of all quadrics in P is of dimension $\binom{r+1}{2} - 1$. Hence the space of quadrics in a variable P satisfying the given Schubert condition form a space of dimension $\sum a_i - \binom{r}{2} + \binom{r+1}{2} - 1 = \sum a_i + r - 1$. It is convenient to define

$$N(A) := \sum_{i=1}^{r} a_i + r - 1.$$

Thus the parameter space of all quadrics in a varying p-plane satisfying the given Schubert condition of type A is of dimension $N(A)$. In the parameter space, the quadrics tangent to a given linear subspace form a hypersurface. Therefore, in the general problem it is natural to require that the number of linear subspaces is equal to $N(A)$, that is,

$$\sum_i m_i = N(A).$$

In Schubert's notation, the number of quadrics satisfying the given Schubert condition of type $A = (a_1, \dots, a_r)$ and tangent to m_i codimension-i planes for $i = 1, \dots, q$ is denoted by the symbol,

$$(a_1, \dots, a_r)\mu_1^{m_1} \cdots m_q^{m_q}. \tag{1.2.1}$$

1.3. It was Schubert's ultimate goal to determine the number (1.2.1) explicitly as a function of the integers $a_1, \ldots, a_r$ and $m_1, \ldots m_q$. He did find a recursive procedure for the computation of the number. In the simplest case $q = 1$, that is, when all the given linear subspaces are hyperplanes, the number depends only on $a_1, \ldots, a_r$, since $m_1 = N(A)$. Schubert [18] defined a function $\psi_{a_1,\ldots,a_r}$ recursively, and proved the formula,

$$(a_1, \ldots, a_r)\mu_1^{N(A)} = \psi_{a_1,\ldots,a_r}.$$

He did not find an explicit formula for his function ψ, but he found other recursion formulas. An explicit formula was first found by Laksov–Lascoux–Thorup [14]. At the end of the paper we summarize some of the properties of the function ψ.

In terms of the function ψ, Schubert gave explicit formulas for the numbers (1.2.1) for $q = 2$ and for $q = 3$. In his paper [19], he considered the analogous problem for correlations. There he found a beautiful explicit expression for the function analogous to ψ, but he never published for correlations results corresponding to his formulas for $q = 2$ and $q = 3$ for quadrics. Giambelli [6] found for correlations a formula valid for all q under certain restrictions on the numbers m_i. In fact, Giambelli's formula for correlations is only valid without conditions on the m_i when $q = 2$. In [14], Giambelli's formula was reconsidered, and the analogous formula for quadrics was proved. But it should be emphasized that the analogous formula is only a generalization of Schubert's formula for $q = 2$; it does not encompass Schubert's formula for $q = 3$.

It is the purpose of the present paper to describe Schubert's problem in detail. We introduce the notion of complete quadrics, and the corresponding tangency conditions. We show how the application of modern intersection theory to the various parameter spaces leads to the determination of the number (1.2.1). In addition, we obtain a series of incidence formulas. Finally, we present some new explicit formulas for the numbers, specializing to Schubert's result for $q = 3$. Other closed formulas were found by Brion [1]. It should be emphasized that explicit formulas are only of theoretical interest. The recursive procedure described by Schubert has been verified by several authors, and in practice it might be easier to use than formulas. For instance, the tables of Schubert for the numbers have been verified and enlarged using a computer by DeConcini–Gianni–Procesi [3]. A history of the subject is found, among other places, in the papers of Kleiman [8,9,10] and Laksov [11,12]. It should also be noted that enumerative problems on quadrics different from the simple tangency conditions considered here require other parameter spaces for their solution, see for instance the papers on Halphen's theory by Casas–Xambó [2] and Procesi–Xambó [16].

2. Schubert conditions.

2.1. *Setup.* We work throughout over a field k of characteristic different from 2. Fix a projective space $Q = \mathbb{P}(V)$, associated to a vector space V over k. The notation is that of Grothendieck: $\mathbb{P}(V)$ is the set of linear hyperplanes in the vector space V, or equivalently, the set of surjective linear maps $V \to k$ up to multiplication by a scalar. In particular, the linear subspaces of Q are the projective spaces $\mathbb{P}(E)$ where E is a quotient vector space of V. It will be convenient to define the rank of $\mathbb{P}(E)$ to be the rank of E. Thus the dimension of $\mathbb{P}(E)$ is one less than the rank.

In addition, we fix a positive integer r and in Q a strictly increasing flag of r linear subspaces,

$$L_1 \subset L_2 \subset \cdots \subset L_r.$$

Set $p = r - 1$. The *Schubert condition* corresponding to the flag is the condition on a p-plane P in Q that $\dim P \cap L_i \geq i - 1$ for $i = 1, \dots, r$. By definition, the *type* of the Schubert condition is sequence $A = (a_1, \dots, a_r)$, where $a_i = \dim L_i$. We write $\|A\| = \sum_i a_i$ and, as in Section 1, $N(A) = \sum_i a_i + r - 1$.

The p-planes P in Q correspond to the rank-r quotients of V. Thus the set of p-planes is parametrized by the Grassmannian $\mathrm{Grass}^r(V)$. In the Grassmannian, the p-planes P satisfying a given Schubert condition of type A form a subscheme Ω, called the *Schubert subscheme*. It is well known that the dimension of the Schubert subscheme is equal to $\sum_{i=1}^{r}(a_i - i + 1)$, see for instance Kempf–Laksov [7, p. 158].

EXAMPLE 1. Consider lines ($r = 2$) in $Q = \mathbb{P}^3$. There are 6 types of Schubert conditions on a line P in $\mathbb{P}^3$:

(01) Given a point on a line: the line P is the given line.
(02) Given a point in a plane: the line P is in the given plane through the given point.
(03) Given a point: the line P goes through the given point.
(12) Given a line in a plane: the line P is in the given plane.
(13) Given a line: the line P intersects the given line.
(23) Given a plane: the line P can vary freely.

EXAMPLE 2. A Schubert condition of type $(0, 1, \dots, p)$ requires the p-plane P to be fixed. Hence, the number $(0, 1, \dots, p)\mu_1^{m_1} \cdots \mu_p^{m_p}$ in Schubert's notation (1.2.1) is the number of solutions to Schubert's simple problem. For instance, a fixed plane corresponds to the Schubert condition (012), and $N(012) = 5$. Hence $(012)\mu_1^m\mu_2^n$, for $m + n = 5$, is the number of conics through n given points and tangent to m given lines.

Consider planes ($r = 3$) in a fixed $\mathbb{P}^3$. A given point in a given plane defines a Schubert condition of type (023). It requires the plane P to go through the given point. We have $N(023) = 7$. For instance, $(023)\mu_2^7$ is the number of conics that lie in a plane through a given point and intersect seven given lines.

2.2. The notion of *incidence* will play an important role. Let $L = \mathbb{P}(V/K)$ be a linear subspace of Q. The codimension d of L in Q is the rank of the vector subspace K. Let P be a second linear subspace of Q, of dimension p. If $p < d$, then P and L will be called incident, if $L \cap P$ is non-empty. If $p \geq d$, then P and L will be called incident, if the codimension of $L \cap P$ in P is strictly smaller that d. In terms of vector spaces, say $P = \mathbb{P}(E)$ where E is a quotient of V, then the spaces P and L are incident if and only if the composite linear map, $K \to V \to E$, is not of maximal rank.

More generally, assume there is given a *rank-r flag* in Q, that is, a strictly increasing flag of linear subspaces,

$$P_1 \subset P_2 \subset \cdots \subset P_s = P,$$

where P is of rank r. Then L is said to be incident with the flag if L is incident with one of the spaces P_j.

Clearly, if the codimension d is less than rank r, then L is incident with the flag if and only if, for the first j such that $L \cap P_j \neq \emptyset$ we have that the codimension of $P_j \cap L$ in P_j is strictly less than d. If $d = r$, then L is incident with the flag if and only $L \cap P \neq \emptyset$.

If the flag is *complete*, that is, $s = r$ or equivalently, $\dim P_j = j - 1$ for all j, then L is incident with the flag if and only if $L \cap P_d \neq \emptyset$.

EXAMPLE 3. Consider a line L in $\mathbb{P}^3$. It is of codimension 2. A flag consisting of a point in a plane is incident with L if either L goes through the point or L lies in the plane.

Consider a complete rank-3 flag in $\mathbb{P}^3$. It consists of a point P_1 on a line P_2 in a plane P_3. A plane is incident with the flag if it contains the point P_1, a line is incident with the flag if it meets the line P_2 and a point is incident with the flag if it is contained in the plane P_3.

3. Quadrics and quadratic forms.

3.1. As the characteristic of the field k is different from 2, a symmetric form on a vector space E, that is, a linear map $u\colon \operatorname{Sym}^2 E \to k$ can be identified with the corresponding quadratic form on E. Moreover, we can identify $(\operatorname{Sym}^2 E)^*$ and $\operatorname{Sym}^2(E^*)$. Let P be a projective space, say $P = \mathbb{P}(E)$ where E is a vector space of rank r. By definition, a *quadric* in P is the subscheme defined by a nonzero equation of degree 2, that is, by a global section of $\mathcal{O}_P(2)$. The space of global sections is the symmetric square $\operatorname{Sym}^2(E)$. Hence a quadric can be viewed as a nonzero symmetric tensor $v \in \operatorname{Sym}^2 E$, up to a nonzero scalar. Quadrics may be singular. In fact, the *singular space* of the quadric defined by the tensor v is the linear subspace $\mathbb{P}(E/U)$, where U is the smallest k-linear subspace of V such that v belongs to $\operatorname{Sym}^2 U$. Note that v as a tensor in $\operatorname{Sym}^2 U$ is *regular*: As a linear map $v\colon k \to \operatorname{Sym}^2 U$, the dual map $v^*\colon \operatorname{Sym}^2 U^* \to k$ is a regular symmetric form on U^*, that is, the associated linear map $U^* \to U$ is an isomorphism. Its inverse, denoted $v^{-1}\colon U \to U^*$, corresponds then to a regular symmetric form $u\colon \operatorname{Sym}^2 U \to k$. The following result is the well known correspondence between singularity of the quadric and singularity of the quadratic form.

3.2. LEMMA. *Consider in $P = \mathbb{P}(E)$ a linear subspace $S = \mathbb{P}(E/U)$. Then the quadrics in P with S as singular space correspond bijectively to the non-singular forms $u\colon \operatorname{Sym}^2 U \to k$ modulo scalars. Moreover, if $L = \mathbb{P}(E/K)$ is a linear subspace disjoint from S, then the quadric defined by a non-singular form u is tangent to L, if and only if the restriction of u to the subspace $U \cap K$ is singular.*

Note in particular that if S is a hyperplane in P, that is, if U is of rank 1, then there is exactly one quadric in P with S as singular space.

3.3. DEFINITION. A *complete rank-r quadric* in Q consists of a rank-r flag of linear subspaces,

$$\emptyset = P_0 \subset P_1 \subset \cdots \subset P_s = P, \tag{3.3.1}$$

where P is of rank r, and, for $j = 1, \ldots, s$, a quadric in P_j with P_{j-1} as singular space. In particular, since P_0 is empty, the quadric in P_1 is non-singular. The complete quadric is called *non-singular* if $s = 1$. It is said to be *degenerated in rank q* if some P_j for $j < s$ is of rank q, and it is said to be *completely degenerated* if it is degenerated in every rank $q < r$. Clearly, completely degenerated complete quadrics correspond bijectively to complete flags.

A linear subspace L of codimension $d \le r$ in Q is said to be *tangent* to the complete quadric if either L is incident with the flag (3.3.1) or the first non-empty intersection $L \cap P_j$ for $j = 1, \ldots, s$ is tangent to the given quadric in P_j.

If the codimension d is equal to r, then L is tangent to the complete quadric if and only if $L \cap P \ne \emptyset$. Assume $d < r$. Then $d \le \dim P$, and hence $L \cap P \ne \emptyset$. Consider the smallest j such that $L \cap P_j \neq \emptyset$. Then L is incident with the flag if and only if the codimension of $L \cap P_j$ in P_j is strictly less than d. Assume that the codimension of $L \cap P_j$ in P_j is equal to d. Then L is tangent to the complete quadric if $L \cap P_j$ is tangent to the given quadric in P_j. Note that the linear subspace $L \cap P_j$ is disjoint from the singular space of the quadric in P_j, since L is disjoint from P_{j-1} by the choice of j.

Example 4. In $\mathbb{P}^3$ there are four types of rank-3 flags: a plane P, a line P_1 in a plane P, a point P_1 in a plane P, and a point P_1 on a line P_2 in a plane P. Correspondingly, there are four types of complete rank-3 quadrics in $\mathbb{P}^3$:

(1) A non-singular conic in a plane P,
(2) A non-singular quadric on a line P_1 (i.e., two different points on P_1) contained in a plane P.
(3) Two different lines in a plane P intersecting in a point P_1.
(4) A point P_1 on a line P_2 in a plane P.

Let L be a plane in $\mathbb{P}^3$. A quadric of type (1) is tangent to L if either $P = L$ or the intersection $L \cap P$ is tangent to the conic. A quadric of type (2) is tangent to L if one of the two points on P_1 belong to L. A quadric of type (3) or (4) is tangent to L if the point P_1 belongs to L.

Let L be a line in $\mathbb{P}^3$. A quadric of type (1) is tangent to L, if L intersects the conic (in particular, if L is contained in the plane P). A quadric of type (2) is tangent to L, if L meets the line P_1. A quadric of type (3) is tangent to L, if L meets one of the two lines. Finally, a quadric of type (4) is tangent to L if L meets the line P_2.

Note that tangency is also defined when L is a point. A point L of $\mathbb{P}^3$ is tangent to the quadric, if L belongs to the plane of the quadric.

3.4. *Translation into algebra.* A p-plane P of Q is a projective space $P = \mathbb{P}(E)$, where E is a rank-r quotient of the vector space V. In the flag (3.3.1), the subspace P_j of P is a projective space $\mathbb{P}(E/E_j)$ where E_j is a k-linear subspace of E. Hence the flag (3.3.1) corresponds to a flag of k-linear subspaces of E:

$$E = E_0 \supset E_1 \supset \cdots \supset E_s = (0). \tag{3.4.1}$$

It follows from 3.2 that a quadric in P_j with P_{j-1} as singular space corresponds to a nonsingular quadratic form $u_j\colon \operatorname{Sym}^2 E_{j-1}/E_j \to k$. Therefore, a complete rank-r quadric in $Q = \mathbb{P}(V)$ may be viewed algebraically as a rank-r quotient E of V, a flag of k-linear subspaces (3.4.1), and a sequence $u = (u_1, \ldots, u_s)$ consisting of non-singular quadratic forms up to scalar $u_j\colon \operatorname{Sym}^2 E_{j-1}/E_j \to k$. We will refer to the algebraic counterpart as the *complete quadratic form* $u = (u_1, \ldots, u_s)$ on E. Note that a complete quadratic form on E could have be defined inductively: u_1 is a non-singular quadratic form on E/E_1 and $u' := (u_2, \ldots, u_s)$ is a complete quadratic form on E_1.

3.5. Definition. Associated to a complete quadratic form $u = (u_1, \ldots, u_s)$ on E there are *exterior powers* $\bigwedge^d u$ for $d \leq r$. They are surjective forms,

$$\textstyle\bigwedge^d u\colon \operatorname{Sym}^2 \bigwedge^d E \to k,$$

defined as follows: Let t be the rank of E/E_1. Consider first the quadratic form $u_1\colon \operatorname{Sym}^2 E/E_1 \to k$. In a basis for E/E_1, the form u_1 is given by a symmetric matrix, and d'th exterior power of u_1 is the form $\bigwedge^d u_1\colon \operatorname{Sym}^2 \bigwedge^d (E/E_1) \to k$ defined by the matrix of d by d minors of the matrix of u_1. Since u_1 is nonsingular, the form $\bigwedge^d u_1$ is surjective for $d \le t$. In particular, when $d = t$, the exterior power $\bigwedge^t u_1$ is the determinant of u_1, viewed as a linear map $\det u_1\colon \det(E/E_1)^{\otimes 2} \to k$ of 1-dimensional vector spaces.

Now, for $d \le t$, the quadratic form $\bigwedge^d u$ is defined as the composition,

$$\operatorname{Sym}^2 \bigwedge^d E \to \operatorname{Sym}^2 \bigwedge^d E/E_1 \xrightarrow{\bigwedge^d u_1} k.$$

For $d \ge t$, there is a canonical surjective linear map,

$$\bigwedge^d E \to \det(E/E_1) \otimes \bigwedge^{d-t} E_1.$$

Its symmetric square is a linear map,

$$\operatorname{Sym}^2 \bigwedge^d E \to \det(E/E_1)^{\otimes 2} \otimes \operatorname{Sym}^2 \bigwedge^{d-t} E_1.$$

As $u' := (u_2, \dots, u_s)$ is a complete quadratic form on E_1, we may, by induction on s, assume that $\bigwedge^{d-t} u'$ is defined. Then define $\bigwedge^d u$ as the composition of the quadratic form $\det u_1 \otimes \bigwedge^{d-t} u'$ and the canonical map.

The following result is a consequence of Lemma 3.2.

3.6. Proposition. *Let E be a rank-r quotient of V. Given a complete quadric in $P = \mathbb{P}(E)$ corresponding to a complete quadratic form $u = (u_1, \dots, u_s)$ on E. Let $L = \mathbb{P}(V/K)$ be a linear subspace codimension d of Q, corresponding to a k-linear subspace K of rank d in V. Then L is tangent to the complete quadric, if and only if the following composition is zero:*

$$\left(\bigwedge^d K\right)^{\otimes 2} \to \operatorname{Sym}^2 \bigwedge^d V \to \operatorname{Sym}^2 \bigwedge^d E \xrightarrow{\bigwedge^d u} k.$$

4. Parameter spaces of quadrics.

4.1. Clearly, the set of non-zero quadratic forms $\operatorname{Sym}^2 E \to k$ up to scalars is parametrized by the projective space,

$$B_1 := \mathbb{P}(\operatorname{Sym}^2 E),$$

with its universal surjective form $u_1\colon \operatorname{Sym}^2 E_{B_1} \to \mathcal{O}_{B_1}(1)$. In particular, the open subset U of B_1 where the form u_1 is regular parametrizes the set of non-singular quadrics in $P = \mathbb{P}(E)$. It is well known, see for instance DeConcini–Procesi [4], Laksov [11,12,13], or Thorup–Kleiman [21], that a parameter space B for the set of complete quadrics can be constructed from U and B_1. The space B is obtained from B_1 by a finite sequence of blowing ups with centers lying over the complement of U. Alternatively, the exterior powers of u_1 define an embedding,

$$U \hookrightarrow \mathbb{P}(\operatorname{Sym}^2 \bigwedge^1 E) \times \cdots \times \mathbb{P}(\operatorname{Sym}^2 \bigwedge^r E), \tag{4.1.1}$$

and B may be described as the closure of its image. The map $B \to B_1$ is proper and smooth, and it is an isomorphism over U. The form u_1 on B_1 pulls back to a surjective form $u\colon \operatorname{Sym}^2 E_B \to \mathcal{L}$, where $\mathcal{L}$ is the pullback of $\mathcal{O}_{B_1}(1)$. The i'th exterior power of u, for $i \le r$,

$$\bigwedge^i u\colon \operatorname{Sym}^2 \bigwedge^i E_B \to \mathcal{L}^{\otimes i},$$

has as image an invertible subsheaf $\mathcal{M}_i$ of $\mathcal{L}^{\otimes i}$. In fact, if p_i denotes the map from B into the i'th factor $B_i := \mathbb{P}(\operatorname{Sym}^2 \bigwedge^i E)$ in (4.1.1), then $\mathcal{M}_i = p_i^* \mathcal{O}_{B_i}(1)$. Moreover, for the invertible sheaves $\mathcal{L}_i := \mathcal{M}_{i+1} \otimes \mathcal{M}_i^{\otimes -1}$ there are canonical injective maps, for $i = 1, \dots, r-1$,

$$\mathcal{L}_{i+1} \hookrightarrow \mathcal{L}_i.$$

In fact, it follows from Proposition 3.6 that the concepts of degeneracy and tangency are geometrically described on the parameter space as follows.

4.2. Proposition. *Let $B = B(E)$ be the scheme parametrizing the set of (complete) quadrics in $P = \mathbb{P}(E)$. Then the zero scheme of the inclusion,*

$$\mathcal{L}_{q+1} \to \mathcal{L}_q,$$

parametrizes the set of quadrics degenerated in rank q. Let $L = \mathbb{P}(E/K)$ be a linear subspace of codimension d in P. Then the zero scheme of the composition,

$$\left(\bigwedge^d K_B\right)^{\otimes 2} \to \operatorname{Sym}^2 \bigwedge^d E_B \to \mathcal{M}_d,$$

parametrizes the set of quadrics tangent to L.

4.3. *Setup.* The theory works over any base scheme. It yields, for any base scheme S and a locally free sheaf $\mathcal{E}$ of rank r on S, a corresponding parameter scheme $B(\mathcal{E})$. On B, there is an invertible quotient $\mathcal{M}_i$ of $\operatorname{Sym}^2 \bigwedge^i \mathcal{E}$ and with $\mathcal{L}_i := \mathcal{M}_{i+1} \otimes \mathcal{M}_i^{\otimes -1}$ there is an inclusion $\mathcal{L}_{i+1} \to \mathcal{L}_i$.

In particular, for Schubert's problem, let $G := \operatorname{Grass}^r(V)$ be the Grassmannian of rank-r quotients of V, and $\mathcal{E}$ the universal rank-r quotient of V_G. Then G parametrizes the p-planes P of $Q = \mathbb{P}(V)$. Moreover, the p-planes satisfying the given Schubert condition of type A are parametrized by the corresponding Schubert subscheme Ω of G. Take Ω as base scheme and form over Ω the parameter scheme $B(\mathcal{E}|\Omega)$. Then the latter scheme parametrizes the complete rank-r quadrics in Q satisfying the given Schubert condition. Although the Schubert subscheme Ω depends on the given Schubert condition, we shall usually indicate only the type A of the Schubert condition and write Ω_A for Ω and B_A for $B(\mathcal{E}|\Omega_A)$.

5. Intersection theory on the space of quadrics.

5.1. In the setup of Section 4, let $B = B_A$ be the parameter scheme corresponding to the given Schubert condition of type A. Then B maps to $B_1 = \mathbb{P}(\operatorname{Sym}^2 \mathcal{E}|\Omega)$ by a composition of blowups. In particular, the dimension of B is equal to the dimension of B_1. As $\mathcal{E}$ is of rank r, it follows that the relative dimension of B_1 over Ω is equal to $\binom{r+1}{2} - 1$. Moreover, the dimension of the Schubert scheme Ω is equal to $\sum_i a_i - \binom{r}{2}$. Hence the dimension of B_A is equal to $\sum a_i + r - 1$, or, with the notation of 2.1, the dimension is equal to $N(A)$.

Let $\mu_d := c_1(\mathcal{M}_d)$ be the first Chern class of the invertible sheaf $\mathcal{M}_d$. By Proposition 4.2, a d-dimensional k-linear subspace K of V defines a section of $\mathcal{M}_d$, and the zero scheme of the section parametrizes the set of quadrics tangent to $L = \mathbb{P}(V/K)$. Consider, in the group of cycles modulo rational equivalence on B_A, the following class,

$$\alpha = \mu_1^{m_1} \cdots \mu_q^{m_q}, \tag{5.1.1}$$

where $m_1 + \cdots + m_q = N(A)$. It follows that the class α is represented by the subscheme of B_A corresponding to complete quadrics that are tangent to m_i given codimension-i planes in general position in Q and lie in a p-plane satisfying the Schubert condition.

In other words, in Schubert's notation (1.2.1), the number $(A)\alpha$ is equal to the integral $\int_{B_A}\alpha$.

The integral of the class α can be obtained in two steps. First, push the class forward from B_A to the Schubert scheme Ω_A, and then integrate the resulting class. The first step is quite general. Consider any rank-r bundle $\mathcal{E}$ on a scheme S. Form the S-scheme $B = B(\mathcal{E})$. Take any class α which is a homogeneous polynomial,

$$\alpha = f(\mu_1, \dots, \mu_r),$$

in the first Chern classes $\mu_i = c_1(\mathcal{M}_i)$. Then, as is well known [14], the push forward of α to S is a linear combination of the Schur functions $s_J(\mathcal{E})$, indexed by strictly increasing sequences $J = (j_1, \dots, j_r)$. In fact, the coefficient to $s_J(\mathcal{E})$ depends only on J and the polynomial f defining the class α, and we denote it $\langle J, f\rangle$ or $\langle J, \alpha\rangle$. Hence there is an equation,

$$\int_{B(\mathcal{E})/S} \alpha = \sum_J \langle J, \alpha\rangle s_J(\mathcal{E}), \tag{5.1.2}$$

where the integral on the left hand side indicates the push forward from $B(\mathcal{E})$ to the base S.

5.2. Proposition. *The coefficient* $\langle A, \alpha\rangle$ *of* 5.1 *is equal to the number* $(A)\alpha$ *in Schubert's notation* (1.2.1). *Moreover, for the case* $\alpha = \mu_1^N$ *we have the equation,*

$$(A)\mu_1^{N(A)} = \psi_A,$$

where ψ *is the function of Schubert, defined in Section* 8.

Proof. Take $S = \Omega_A$ and $B = B_A$ in (5.1.2), and integrate the equation. On the left we obtain Schubert's number $(A)\alpha$. On the right, we obtain $\langle A, \alpha\rangle$, since by Giambelli's formula [5, p. 267], the integral over Ω_A of $s_J(\mathcal{E})$ is equal to 1 when $J = A$ and equal to 0 otherwise. Thus $(A)\alpha = \langle A, \alpha\rangle$.

Consider in particular the class $\alpha = \mu_1^N$. For any rank-r bundle $\mathcal{E}$ on any scheme S, the invertible sheaf $\mathcal{M}_1$ on $B = B(\mathcal{E})$ is the pullback of the tautological bundle $\mathcal{O}(1)$ on $B_1 := \mathbb{P}(\operatorname{Sym}^2 \mathcal{E})$. Consequently, by the Projection Formula, the push forward of α to B_1 is equal to $(c_1\mathcal{O}(1))^N$. Therefore, the following equation holds:

$$\int_{B(\mathcal{E})/S} \mu_1^N = \int_{\mathbb{P}(\operatorname{Sym}^2 \mathcal{E})/S} (c_1\mathcal{O}(1))^N.$$

The right hand side is, by definition, the $(N - e + 1)$'th Segre class of $\operatorname{Sym}^2 \mathcal{E}$, where $e = \binom{r+1}{2}$ is the rank of $\operatorname{Sym}^2 \mathcal{E}$, cf. [5, p. 46]. Therefore, by definition of the function ψ, cf. Section 8, we have the equation,

$$\int_{B(\mathcal{E})/S} \mu_1^N = \sum_{N(J)=N} \psi_J s_J(\mathcal{E}).$$

It follows that $\psi_J = \langle J, \mu_1^N\rangle$. Consequently, the asserted equation, $(A)\mu_1^{N(A)} = \psi_A$, follows from the first part of the Lemma. ∎

5.3. Definition. In the general setup of Section 4, the first Chern class $\mu_i := c_1(\mathcal{M}_i)$ is called the i'th *characteristic class*. For $i = 1, \dots, r-1$, the zero scheme of the inclusion $\mathcal{L}_{i+1} \to \mathcal{L}_i$ is a divisor D_i in B. Its first Chern class, $\delta_i := c_1(\mathcal{L}_i \otimes \mathcal{L}_{i+1}^{\otimes -1})$, is called the i'th *degeneration class*. By definition of the $\mathcal{L}_i$ we have that $c_1(\mathcal{L}_i) = \mu_{i+1} - \mu_i$. It is convenient to define $\lambda_i := c_1(\mathcal{L}_{i+1})$ for $i = 0, \dots, r-1$. Then, obviously, we have the *fundamental relations*,

$$\mu_i = \lambda_0 + \cdots + \lambda_{i-1} \quad \text{for } i = 1, \dots, r,$$
$$\delta_i = \lambda_{i-1} - \lambda_i \quad \text{for } i = 1, \dots, r-1.$$

In the sequel, a *class* will mean a class belonging to the ring generated by the characteristic classes μ_i, that is, a class is a homogeneous polynomial with integer coefficients in the classes μ_i. It follows from the first set of fundamental relations that a class alternatively may be viewed as a polynomial in the classes λ_i.

The following result and its corollary are crucial for our application of intersection theory to the parameter schemes. The result reflects the geometry of the degeneration divisor D_q into properties of Chern classes. A simple case of the result is found in [14, (6.2)(2), p. 175]; for a proof in general, see [20].

5.4. KEY RESULT. *Fix $q < r$. Consider two classes of the following forms,*

$$\alpha = f(\mu_1, \dots, \mu_q), \qquad \beta = g(\mu_{q+1} - \mu_q, \dots, \mu_r - \mu_q),$$

where f and g are polynomials in q and $r-q$ variables. Then the following equation holds:

$$\langle A, \alpha\beta\delta_q \rangle = \sum_{I \cup J = A} \operatorname{sign}(JI) \langle I, f \rangle \langle J, g \rangle\,,$$

where the sum is over all pairs of complementary subsequences (I, J) of A with q and $r-q$ elements respectively, and JI denotes the concatenated sequence.

5.5. COROLLARY. *Consider a class of the following form,*

$$\gamma = \lambda_0^{l_0} \lambda_{q_1}^{l_1} \cdots \lambda_{q_s}^{l_s} \delta_{q_1} \cdots \delta_{q_s},$$

where $1 \le q_1 < \cdots < q_s \le p$. Assume that the degree $l_0 + \cdots + l_s + s$ of γ is equal to $N(A)$. Then the following formula holds:

$$\langle A, \gamma \rangle = \sum_{J_0 \cup \cdots \cup J_s = A} \operatorname{sign}(J_s \cdots J_0) \psi_{J_0} \cdots \psi_{J_s},$$

where the sum is over all those shuffles $(J_0, \dots, J_s)$ of A for which the number of elements in $J_0 \cdots J_t$ is equal to q_{t+1} and $N(J_t) = l_t$ for $t = 0, \dots, s-1$.

Proof of the Corollary. An *s-shuffle* of A is a decomposition of the sequence A into an ordered set $(J_0, \dots, J_s)$ of $s+1$ subsequences. The concatenated sequence $J_s \cdots J_0$ is then a permutation of the sequence A.

Assume first that $s = 0$. Then the class γ is of the form $\lambda_0^{l_0}$, and by hypothesis $l_0 = N(A)$. The only 0-shuffle of A is $J_0 = A$. As $\lambda_0 = \mu_1$, the asserted formula reduces to the formula $\langle A, \mu_1^{N(A)} \rangle = \psi_A$ proved in 5.2. In general, the asserted formula follows by induction on s using the Key Result 5.4. ∎

6. Incidence formulas.

6.1. The class δ_q is the class of the divisor D_q of $B(\mathcal{E})$ corresponding to complete quadrics that are degenerated in rank q. The product class $\delta := \delta_1 \cdots \delta_{r-1}$ is represented by the intersection $D := D_1 \cap \cdots \cap D_{r-1}$. The intersection D parametrizes the set of completely degenerated quadrics, or equivalently, the set of complete flags in $\mathcal{E}$. Over D, there is a universal flag corresponding to (3.4.1),

$$\mathcal{E}_D = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_r = (0),$$

such that the successive quotients $\mathcal{E}_{j-1}/\mathcal{E}_j$ are invertible. Moreover, when restricted to D, the invertible module $\mathcal{M}_d$ is the square of the tensor product $\mathcal{E}_0/\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_{d-1}/\mathcal{E}_d$. Hence the Chern class μ_d is twice the Chern class of the tensor product.

In the notation of 4.3, assume that $B = B_A$. Then the subscheme D parametrizes the set of complete rank-r flags in Q satisfying the given Schubert condition. The subscheme D is of codimension $r-1$ in B. Hence the dimension of D is equal to $N(A)-r+1 = \|A\|$. Consider for $m_1 + \cdots + m_r = \|A\|$ the number of complete rank-r flags in Q that satisfy the given Schubert condition and are incident with m_i given codimension-i planes for $i = 1, \ldots, r$. In Schubert's notation [18, p. 171] the number is denoted by the symbol,

$$\eta(A)\mu_1^{m_1} \cdots \mu_r^{m_r}. \tag{6.1.1}$$

When restricted to D, the Chern class μ_d is equal to twice the Chern class of the tensor product $\mathcal{E}_0/\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_{d-1}/\mathcal{E}_d$. The Chern class of the tensor product is represented by the hypersurface in D consisting of flags that are incident with a given codimension-d plane in Q. It follows, with $\alpha := \mu_1^{m_1} \cdots \mu_r^{m_r}$, that the integral over D of α is equal to $2^{\|A\|}$ multiplied by Schubert's number (6.1.1). The integral over D of α is equal to the integral over B of $\alpha\delta$ and the latter integral is, by Lemma 5.2, equal to the coefficient $\langle A, \alpha\delta\rangle$. Hence the following equation holds:

$$\eta(A)\alpha = 2^{-\|A\|}\langle A, \alpha\delta\rangle\,. \tag{6.1.2}$$

Naturally, we extend Schubert's notation $\eta(A)\alpha$ by linearity to any class α.

6.2. Proposition. *Let $\alpha = \lambda_0^{l_0} \cdots \lambda_p^{l_p}$ be a monomial in the Chern classes λ_i. Assume that the degree $l_0+\cdots+l_p$ of α is equal to $\|A\|$. If the sequence $(l_0, \ldots, l_p)$ is a permutation of the sequence A, then*

$$\eta(A)\alpha = \operatorname{sign}(l_p, \ldots, l_0); \tag{6.2.1}$$

otherwise, $\eta(A)\alpha = 0$.

Proof. The product $\gamma := \alpha\delta$ is of the form in 5.5 with $s := p$ and $q_j := j$. Hence $\langle A, \alpha\delta\rangle$ is given by the sum in 5.5 over p-shuffles of A. The p-shuffles are the permutations $(j_0, \ldots, j_p)$ of the sequence A and the sum is over those permutations for which $N(j_t) = l_t$ for $t = 0, \ldots, p-1$, or equivalently, $j_t = l_t$ for $t = 0, \ldots, p-1$. It follows that the coefficient $\langle A, \alpha\delta\rangle$ vanishes unless $(l_0, \ldots, l_p)$ is a permutation of A. Moreover, if $(l_0, \ldots, l_p)$ is a permutation of A, then there is only one term in the sum, and we obtain the equation,

$$\langle A, \alpha\delta\rangle = \operatorname{sign}(l_p, \ldots, l_0)\psi_{l_0} \cdots \psi_{l_p}. \tag{6.2.2}$$

The function ψ in one variable is given by $\psi_l = 2^l$. Hence $\psi_{l_0} \cdots \psi_{l_p} = 2^{\|A\|}$ when $(l_0, \ldots, l_p)$ is a permutation of A. Thus (6.2.1) follows from (6.2.2) by dividing by $2^{\|A\|}$. ∎

6.3. Corollary. *Schubert's number* (6.1.1) *is given by following sum over all permutations $(b_1, \ldots, b_r)$ of the sequence A:*

$$\eta(A)\mu_1^{m_1} \cdots \mu_1^{m_r} = \sum_{b_1,\ldots,b_r} \operatorname{sign}(b_1, \ldots, b_r)C_{b_1,\ldots,b_r}, \tag{6.3.1}$$

where $C_{b_1,\dots,b_r}$ is the following product of r binomial coefficients,

$$C_{b_1,\dots,b_r} = \binom{m_r}{b_1}\binom{m_{r-1}+m_r-b_1}{b_2}\cdots\binom{m_1+\cdots+m_r-b_1-\cdots-b_{r-1}}{b_r}.$$

Note that the last factor in the product is equal to 1*, because $b_1,\dots,b_r$ is a permutation of A and $m_1+\cdots+m_r=\|A\|$.*

Proof. Since $\mu_i = \lambda_0+\cdots+\lambda_{i-1}$, the assertion follows from the proposition by expanding the left hand side of (6.3.1) as a polynomial in the classes λ_j. ■

Example 5. Consider complete rank-2 flags in $Q=\mathbb{P}^n$. Each flag consists of a point P_1 on a line P_2. The Schubert condition for the flag to vary freely is of type $A=(n-1,n)$, and $\|A\|=2n-1$. Consider the number of flags that are incident with 1 hyperplane and $2n-2$ codimension-2 planes. In Schubert's notation, the number is $\eta(n-1,n)\mu_1\mu_2^{2n-2}$. Thus, by 6.3, the number is the difference,

$$\binom{2n-2}{n-1}-\binom{2n-2}{n}=\frac{1}{n}\binom{2n-2}{n-1}. \tag{6.3.2}$$

Note that a flag $P_1\subset P_2$ is incident with a hyperplane H, if and only if the point P_1 belongs to H. In other words, when H is given, then for the general flag incident with H, the point P_1 of the flag is simply the intersection of P_2 and H. Hence, the number (6.3.2) is also equal to the number of lines that are incident with $2n-2$ codimension-2 planes in $\mathbb{P}^n$. For instance, in $\mathbb{P}^3$ there are 2 lines that are incident with four given lines.

6.4. The formula in 6.3 is due to Schubert [18, § 4]. By expanding the binomial coefficients in the product $C_{b_1,\dots,b_r}$ in terms of factorials we obtain, since $(b_1,\dots,b_r)$ is a permutation of A, a fraction with the denominator $a_1!\cdots a_r!$. For special sequences of exponents $(m_1,\dots,m_r)$ the expression can be simplified. For instance, for the sequence $(r-1,r-2,\dots,1,m)$, where $m=\|A\|-\binom{r}{2}$, the following formula of Schubert [17, p. 117] is obtained:

$$\eta(A)\mu_1^{r-1}\mu_2^{r-2}\cdots\mu_{r-1}\mu_r^m=\frac{m!}{a_1!\cdots a_r!}\Delta(a_1,\dots,a_r), \tag{6.4.1}$$

where $\Delta(a_1,\dots,a_r)=\prod_{j>i}(a_j-a_i)$. As in Example 5, when a Schubert condition of type A is given in $\mathbb{P}^n$, the number (6.4.1) is equal to the number of rank-r planes in $\mathbb{P}^n$ that satisfy the given Schubert condition and are incident with m given codimension-r planes. For example, in $\mathbb{P}^3$ take $r=2$ and $A=(23)$. Then $m=4$ and we recover the result of Example 5.

7. Tangency formulas.

7.1. It is not hard to see from the fundamental relations that any class α has an expansion as a linear combination of classes γ of the form considered in 5.5, for $s=0,\dots,p$. In fact there is an explicit formula for expressing any polynomial in the classes μ_i (or in the classes λ_j) as a linear combination of the classes γ. Hence, corresponding to the equation of 5.5, there is an explicit formula for the intersection coefficient $\langle A,\alpha\rangle$. When the expansion is applied to a monomial in the classes λ_j, the following result is obtained from 5.5, see [20]:

7.2. THEOREM. *Let $\alpha = \lambda_0^{l_0} \cdots \lambda_p^{l_p}$ be a monomial in the Chern classes λ_i. Assume that $l_0+\cdots+l_p = N(A)$. Then the intersection coefficient $\langle A, \alpha\rangle$ is equal to the following expression:*

$$\sum_{1\le q_1<\cdots<q_s\le p} (-1)^s \sum_{A=J_0\cup\cdots\cup J_s}{}' \operatorname{sign}(J_s\cdots J_0)\,\psi_{J_0}\cdots\psi_{J_s}\,. \tag{7.2.1}$$

The outer sum is over all strictly increasing sequences $1\le q_1<\cdots<q_s\le p$ for $s = 0,1,\dots,p$. The inner sum is over all shuffles $(J_0,\dots,J_s)$ of A such that, for $t = 1,\dots,s$, the number of elements in $J_0\cdots J_{t-1}$ is equal q_t and the following inequality holds:

$$N(J_t\cdots J_s) < l_{q_t} + l_{q_t+1} + \cdots + l_p \quad \textit{for } t = 1,\dots,s. \tag{7.2.2}$$

7.3. In the sum (7.2.1), for $s = 0$ there is only the single term ψ_A. In general, for $s > 0$, there is a huge number of s-shuffles of A, but the inequalities (7.2.2) limit the number of s-shuffles that contribute to the sum. For instance, assume for some $q < p$ that $l_{q+1} = \cdots = l_p = 0$. If $q_s > q$, then for $t = s$ the right side of (7.2.2) is zero and the left side is positive. Hence no s-shuffle satisfying the conditions if $q_s > q$. Therefore, in the sum (7.2.1) the summation may be restricted to sequences $q_1<\cdots<q_s \le q$.

Clearly, if α is a polynomial in the classes λ_i, then the value of the coefficient $\langle A, \alpha\rangle$ can be obtained as a linear combination of the values given in Theorem 7.2 when α is a monomial. In particular, for $\alpha = \mu_1^{m_1}\cdots\mu_r^{m_r}$ it is possible, see [20], to obtain an explicit formula for $\langle A, \alpha\rangle$ by expanding $\mu_i^{m_i} = (\lambda_0 + \cdots + \lambda_{i-1})^{m_i}$. In particular, the following result is obtained from Theorem 7.2.

7.4. THEOREM. *Consider a class of α of the form,*

$$\alpha = \mu_1^{m_1}\cdots\mu_q^{m_q}\mu_{q+1}^{m}\mu_{q+2}^{n}$$

(where $0 \le q < p$). Set $l := m_1 + \cdots + m_q$. Assume that

$$l + m + n = N(A).$$

In addition, assume that the following $q-1$ inequalities are satisfied:

$$\sum_{i=1}^{h} m_i > \sum_{i=1}^{h} a_{r-i+1} + h - 1 \quad \textit{for} \quad h = 1,\dots,q-1. \tag{7.4.1}$$

Then the intersection coefficient $\langle A, \alpha\rangle$ is equal to $1^{m_1}2^{m_2}\cdots q^{m_q}$ multiplied by the following expression,

$$\psi_A(q+1)^m(q+2)^n - \sum_{I\cup J=A} \operatorname{sign}(JI)\psi_I\psi_J D_{I,J}$$
$$- \sum_{K\cup L=A} \operatorname{sign}(LK)\psi_K\psi_L E_{K,L} + \sum_{I\cup a\cup L=A} \operatorname{sign}(LaI)\psi_I\psi_a\psi_L F_{I,a,L}.$$

The three sums are over shuffles of A, the first over pairs (I, J) with q and $r-q$ elements respectively, the second over pairs (K, L) with $q+1$ and $r-q-1$ elements respectively, and the third over triples (I, a, L) with q, 1 and $r-q-1$ elements respectively. The terms in the sums are given by the following expressions:

$$\begin{aligned}
D_{I,J} &= \sum_{j=0}^{n} \binom{n}{j} \sum_{t=0}^{N(I)-l} \binom{m+j}{t} q^t, \\
E_{K,L} &= \sum_{j=0}^{N(K)-l-m} \binom{n}{j} (q+1)^{m+j}, \\
F_{I,a,L} &= \sum_{j=0}^{N(I,a)-l-m} \binom{n}{j} \sum_{t=0}^{N(I)-l} \binom{m+j}{t} q^t.
\end{aligned}$$

7.5. Consider first the case $q = 1$ of Theorem 7.4. Then formula (7.4.1) is an expression for the intersection coefficient,

$$\langle A, \mu_1^l \mu_2^m \mu_3^n \rangle,$$

where $l + m + n = N(A)$, and there is no restriction for its validity. The formula is essentially the formula proved by Schubert [18, §9]. Indeed, it is easily checked that the sums D and E correspond to the sums denoted similarly by Schubert. Moreover, the sum F is over all pairs (b, a) with 2 elements of A and L is the complementary subsequence. Clearly, every subset $\{b, a\}$ with 2 elements yields two terms $F_{b,a,L}$ and $F_{a,b,L}$ with opposite signs in the formula. The difference $F_{b,a,L} - F_{a,b,L}$, for $b < a$, corresponds to the term denoted $F_{b,a}$ by Schubert. Thus, replacing in F the term $F_{b,a,L}$ by the latter difference and restricting the summation to pairs (b, a) such that $b < a$, it follows that F corresponds to the sum denoted similarly by Schubert.

Consider next the case $n = 0$ of Theorem 7.4. Then the upper limits for the summation over j in the sums E and F are negative since $N(A) = l+m$. Hence $E = F = 0$. Moreover, the upper limit for the summation over t in the sum D is equal to $m - \|J\| - (r - q)$, because J, I is a shuffle of A. So the expression for the sum D reduces to the following:

$$D_{I,J} = \sum_{t=0}^{m-\|J\|-(r-q)} \binom{m}{t} q^t.$$

Therefore, the formula of Theorem 7.4 for $n = 0$ reduces to the formula of Laksov–Lascoux–Thorup [14, p. 176].

Example 6. For $m + n = 5$, how many plane conics go through n given points and are tangent to m given lines? By 5.2, the answer is the coefficient $c_{m,n} := \langle 012, \mu_1^m \mu_2^n \rangle$. Hence Theorem 7.4 applies with $A = (012)$ and $q := 0$. As $q = 0$, the terms D and F vanish. So, the answer is given by the expression,

$$c_{m,n} = \psi_{012} 2^n - \big(\psi_0 \psi_{12} E_0(m) - \psi_1 \psi_{02} E_1(m) + \psi_2 \psi_{01} E_2(m)\big),$$

where $E_k(m) := \sum_{j=0}^{k-m} \binom{n}{j}$. Obviously, $\psi_{012} = 1$ and the sums $E_k(m)$ vanish for $m = 3, 4, 5$. Evaluating the sums $E_k(m)$ for $m = 0, 1, 2$ and the necessary values of ψ, we obtain the well known table,

$$\begin{aligned}
&c_{5,0} = 1, \quad c_{4,1} = 2, \quad c_{3,2} = 2^2 = 4, \\
&c_{2,3} = 2^3 - \psi_2 \psi_{01} = 4, \\
&c_{1,4} = 2^4 - (5\psi_2 \psi_{01} - \psi_1 \psi_{02}) = 2, \\
&c_{0,5} = 2^5 - (16\psi_2 \psi_{01} - 6\psi_1 \psi_{02} + \psi_0 \psi_{12}) = 1.
\end{aligned}$$

Curiously enough, if we apply instead Theorem 7.4 with $q := 1$ and $n := 0$, then we obtain immediately the last three entries of the table. At any rate, the symmetry in the table is no surprise since, by duality, $c_{m,n} = c_{n,m}$.

EXAMPLE 7. By definition, a *cone* in $\mathbb{P}^p$ is a complete rank-r quadric (i.e., of maximal rank) degenerated in rank 1. The condition for a p-plane to be fixed is of type $A = (01\ldots p)$. So the parameter space B of complete quadrics of maximal rank is of dimension equal to $N(A) = p(p+3)/2$. The cones are parametrized by the hypersurface D_1 in B, of dimension equal to $N := p(p+3)/2 - 1$. Hence, the number of cones that are tangent to N codimension-2 planes in $\mathbb{P}^p$ is equal to the coefficient,

$$\langle 01\ldots p, \mu_2^N \delta_1\rangle. \tag{7.5.1}$$

As $\mu_2 = \lambda_0 + \lambda_1$, we obtain the expansion of μ_2^N as a linear combination of monomials, $\mu_2^N = \sum_k \binom{N}{k}\lambda_0^k\lambda_1^{N-k}$. Hence, the coefficient (7.5.1) is equal to the corresponding linear combination of coefficients $\langle A, \lambda_0^k\lambda_1^{N-k}\delta_1\rangle$. The class $\lambda_0^k\lambda_1^{N-k}\delta_1$ is of the type in 5.5, with $s := 1$ and $q_1 := 1$. The sum in 5.5 is over 1-shuffles (J_0, J_1) of A such J_0 is a singleton and $N(J_0) = k$. As $A = (01\ldots p)$, there are no such shuffles if $k > p$. If $k \le p$, then the shuffle is $(k\,|\,01\ldots\hat{k}\ldots p)$, and we obtain, using the values of ψ from the appendix, the equation,

$$\langle A, \lambda_0^k\lambda_1^{N-k}\delta_1\rangle = (-1)^{p-k}\psi_k\psi_{0,1,\ldots,\hat{k},\ldots,p} = (-1)^{p-k}2^k\binom{p+1}{k+1}.$$

Consequently, we obtain for the number of cones (7.5.1) the following expression,

$$\langle 01\ldots p, \mu_2^N\delta_1\rangle = \sum_{k=0}^{p}(-1)^{p-k}2^k\binom{N}{k}\binom{p+1}{k+1}. \tag{7.5.2}$$

In particular, in $\mathbb{P}^3$ the number of cones tangent to 8 given lines is equal to

$$-\binom{4}{1} + 2\binom{8}{1}\binom{4}{2} - 4\binom{8}{2}\binom{4}{3} + 8\binom{8}{3} = 92.$$

7.6. The parameter scheme B parametrizing quadrics of maximal rank in $\mathbb{P}^p$ is of dimension $N := N(01\ldots p) = p(p+3)/2$. In the parameter scheme, the quadrics tangent to a given quadric form a hypersurface representing the class 2α where $\alpha = \mu_1 + \cdots + \mu_p$. Consequently, the number of quadrics in $\mathbb{P}^p$ that are tangent to N given quadrics in general position is equal to the number,

$$2^N\langle 01\ldots p, (\mu_1 + \cdots + \mu_p)^N\rangle. \tag{7.6.1}$$

By the fundamental relations in 5.3, $\alpha = p\lambda_0 + \cdots + 2\lambda_{p-2} + \lambda_{p-1}$. Therefore, using the Multinomial Theorem and Theorem 7.2, the following explicit formula is obtained [20]:

7.7. PROPOSITION. *The number of quadrics in $\mathbb{P}^p$ tangent to $N = p(p+3)/2$ quadrics in general position is given by the expression,*

$$2^N \sum_{1\le q_1<\cdots<q_s\le p-1} (-1)^s {\sum}' \operatorname{sign}(J_s\cdots J_0)\,\psi_{J_0}\cdots\psi_{J_s}C_{J_0,\ldots,J_s}. \tag{7.7.1}$$

The inner sum is over all s-shuffles $(J_0,\ldots,J_s)$ of the sequence $(01\ldots p)$ such that, for $t = 1,\ldots,s$, the number of elements in $J_0\cdots J_{t-1}$ is equal q_t. The number $C_{J_0,\ldots,J_s}$ is the *restricted sum,*

$$\sum_{i_1,\dots,i_p}' \binom{N}{i_1,\dots,i_p} 1^{i_p} 2^{i_{p-1}} \cdots p^{i_1},$$

where the sum is over all sets $(i_1,\dots,i_p)$ *of non-negative integers satisfying the following* s *inequalities,*

$$i_1 + i_2 + \cdots + i_{q_t} \leq N(J_0 \cdots J_{t-1}) \quad \text{for } t = 1,\dots,s.$$

Example 8. For plane conics, $p = 2$ and $N = 5$. For $s = 0$, the contribution in the sum (7.7.1) is equal to $(2+1)^5 = 3^5$. If $s > 0$, then, since $1 \leq q_1 < \cdots < q_s < 2$, it follows that $s = 1$ and $q_1 = 1$. Hence, the only shuffles contributing to the sum are the three 1-shuffles $(0|12)$, $(1|02)$, and $(2|01)$. The corresponding numbers C are $C_{0|12} = 1$, $C_{1|02} = 1 + \binom{5}{1}2 = 11$, and $C_{2|01} = 1 + \binom{5}{1}2 + \binom{5}{2}2^2 = 51$. Hence, the number of conics tangent to 5 given conics is the well known number,

$$2^5(3^5\psi_{012} - (\psi_0\psi_{12} - 11\psi_1\psi_{02} + 51\psi_2\psi_{01})) = 2^5 \cdot 3 \cdot 34 = 3264.$$

Naturally, the value could have been obtained directly from the values of $c_{m,n} = \langle 012, \mu_1^m \mu_2^n \rangle$ for $m + n = 5$ given in Example 6.

8. Appendix: Schubert's function.

8.1. Schubert [18] defined his function ψ recursively. For the applications in enumerative geometry, it is more natural to define it in terms of symmetric polynomials. Let $x_1,\dots,x_r$ be a sequence of r independent variables. For any strictly increasing sequence $I = (i_1,\dots,i_r)$ of r non-negative integers denote by $s_{i_1,\dots,i_r}$ the corresponding Schur function in the variables $x_1,\dots,x_r$, see [14, p. 182]. Then, by definition, $\psi_I = \psi_{i_1,\dots,i_r}$ is the integer coefficient to the Schur function $s_{i_1,\dots,i_r}$ in the expansion of $\prod_{i\leq j}(1-(x_i+x_j))^{-1}$, that is,

$$\prod_{i\leq j} \frac{1}{1-(x_i+x_j)} = \sum_I \psi_I s_I,$$

where the sum is over all strictly increasing sequences $I = (i_1,\dots,i_r)$ of non-negative integers. The value ψ_J on an arbitrary sequence $J = (j_1,\dots,j_r)$ of non-negative integers is equal to 0 if J has two equal entries and equal to $\operatorname{sign}(J)\psi_I$ if J is a permutation of a strictly increasing sequence I.

The following properties of the functions ψ are proved in [14, Appendix]:

8.2. The functions ψ are given by the explicit formula,

$$\psi_I = \sum_J \det E_J^I, \tag{8.2.1}$$

where the sum is over all strictly increasing sequences $J = (j_1,\dots,j_r)$ of non-negative integers. The (infinite) matrix E is Pascal's triangular matrix,

$$E := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & \dots \\ 1 & 3 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

given by $E^i_j := \binom{i}{j}$ for $i, j = 0, 1, 2, \ldots$. The determinant $\det E^I_J$ in (8.2.1) is the minor of E obtained by selecting row entries from I and column entries from J.

8.3. The functions ψ in one and two variables are given by

$$\psi_i = 2^i \quad \text{and} \quad \psi_{i,j} = \sum_{i<k\le j} \binom{i+j}{k}. \tag{8.3.1}$$

Moreover, for $r > 2$, the functions ψ are determined by the functions ψ_i and $\psi_{i,j}$ through the following recurrence formula:

$$\psi_{i_1,\ldots,i_r} = \begin{cases} \sum_{k=1}^r (-1)^{k-1} \psi_{i_k}\, \psi_{i_1,\ldots,\widehat{i_k},\ldots,i_r}, & \text{if } r \text{ is odd.} \\ \sum_{k=2}^r (-1)^k \psi_{i_1,i_k}\, \psi_{i_2,\ldots,\widehat{i_k},\ldots,i_r}, & \text{if } r \text{ is even.} \end{cases} \tag{8.3.2}$$

8.4. The functions ψ are given by the following explicit formula:

$$\psi_{i_1,\ldots,i_r} = \begin{cases} \mathrm{Pf}(\psi_{i_k,i_l})_{k,l=1,\ldots,r}\,, & \text{if } r \text{ is even,} \\ \mathrm{Pf}(\psi_{i_k,i_l})_{k,l=0,\ldots,r}\,, & \text{if } r \text{ is odd,} \end{cases} \tag{8.4.1}$$

where Pf denotes the Pfaffian and where, for odd r, the undefined entry ψ_{i_0,i_l} is interpreted as ψ_{i_l}.

8.5. The following recurrence formulas hold for all strictly increasing sequences $0 \le i_1 < \cdots < i_r$:

$$r\psi_{i_1,\ldots,i_r} - 2\sum_{k=1}^r \psi_{i_1,\ldots,i_k-1,\ldots,i_r} = 0, \quad \text{if } i_1 > 0. \tag{8.5.1}$$

$$r\psi_{0,i_2,\ldots,i_r} - 2\sum_{k=2}^r \psi_{0,i_2,\ldots,i_k-1,\ldots,i_r} = \psi_{i_2,\ldots,i_r}. \tag{8.5.2}$$

8.6. The recurrence formulas in 8.3 and 8.5 were proved by Schubert, who in fact took 8.3 as the definition of ψ. The formulas given in 8.4 appeared in [15].

Schubert was unable to prove the formula,

$$\psi_{0,1,\ldots,r-1} = 1, \tag{8.6.1}$$

directly from his recursive definition, but had to appeal to the geometric interpretation of the function ψ. The formula (8.6.1) follows immediately from the explicit formula (8.2.1). Similarly, it follows from (8.2.1) that

$$\psi_{0,1,\ldots,\widehat{i},\ldots,r} = \binom{r+1}{i+1}. \tag{8.6.2}$$

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