TOPOLOGY IN NONLINEAR ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 35
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1996

MULTIFUNCTIONS OF TWO VARIABLES: EXAMPLES AND COUNTEREXAMPLES

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Abstract. A brief account of the connections between Carathéodory multifunctions, Scorza-Dragoni multifunctions, product-measurable multifunctions, and superpositionally measurable multifunctions of two variables is given.

Introduction. Singlevalued functions $f = f(t, u)$ of two variables t and u naturally arise as right-hand sides of differential equations

$$\dot{x}(t) = f(t, x(t)). \tag{1}$$

To get reasonable acting properties of the corresponding *Nemytskij operator* N_f defined by

$$N_f x(t) = f(t, x(t)), \tag{2}$$

one has to impose appropriate regularity conditions on the function f. Among these the most important ones in the classical theory are the *Carathéodory condition* (which means that f is measurable in the first and continuous in the second argument) or, equivalently, the *Scorza-Dragoni condition* (which means, loosely speaking, that f is continuous "up to small sets"). Further useful properties are the *product-measurability* of f, as well as the *superpositional measurability* of f (which means that the map $t \mapsto f(t, x(t))$ is measurable whenever the map $t \mapsto x(t)$ is).

Now, if $F = F(t, u)$ is a *multifunction* of two variables, the picture becomes more complicated. One reason for this is, for example, that the continuity in the definition of the Carathéodory and Scorza-Dragoni properties may be replaced by either upper semicontinuity or lower semicontinuity, and these semicontinuity properties are often not symmetric. As a consequence, several new phenomena occur which are "hidden" in the singlevalued case.

1991 *Mathematics Subject Classification*: 28B20, 47H04, 47H30, 54C60, 54C65.

The paper is in final form and no version of it will be published elsewhere.

The purpose of this brief survey is to provide a comparison of various regularity properties for multifunctions of two variables, building on a series of theorems (without proofs), examples, and counterexamples.

We remark that many contributions to the theory and applications of Carathéodory multifunctions, Scorza-Dragoni multifunctions, product-measurable multifunctions, and superpositionally measurable multifunctions of two variables are due to Polish mathematicians [10], [13], [14], [17]–[23], [25], [27], [30]–[41]. A detailed bibliography may be found in the forthcoming survey [1].

1. Regularity properties of multifunctions of two variables. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and $F : \Omega \times \mathbf{R}^m \to Cl(\mathbf{R}^n)$ a given multifunction. (By $Cl(X)$, $Cp(X)$, and $Cv(X)$ we denote the system of all nonempty closed, compact, and convex subsets, respectively, of a topological linear space X.) The following four regularity properties of F which are important in the theory of differential inclusions

$$\dot{x}(t) \in F(t, x(t)) \tag{3}$$

(see e.g. [7]) will be studied in what follows:

a) A multifunction $F : \Omega \times \mathbf{R}^m \to Cl(\mathbf{R}^n)$ is called *upper Carathéodory* (respectively *lower Carathéodory*) if $F(t, \cdot\,) : \mathbf{R}^m \to Cl(\mathbf{R}^n)$ is upper semicontinuous (respectively lower semicontinuous) for (almost) all $t \in \Omega$, and $F(\,\cdot\,, u) : \Omega \to Cl(\mathbf{R}^n)$ is measurable for all $u \in \mathbf{R}^m$. If F is both upper and lower Carathéodory, we call F simply a *Carathéodory multifunction*.

b) If Ω is a metric space and the σ-algebra $\mathcal{A}$ contains the Borel subsets of Ω, we say that a multifunction $F : \Omega \times \mathbf{R}^m \to Cl(\mathbf{R}^n)$ has the *upper Scorza-Dragoni property* (respectively *lower Scorza-Dragoni property*) if, given $\delta > 0$, one may find a closed subset Ω_δ of Ω such that $\mu(\Omega \setminus \Omega_\delta) \le \delta$, and the restriction of F to $\Omega_\delta \times \mathbf{R}^m$ is upper semicontinuous (respectively lower semicontinuous). If F has both the upper and lower Scorza-Dragoni property, we say that F has the *Scorza-Dragoni property*.

c) Next, by $\mathcal{A} \otimes \mathcal{B}(\mathbf{R}^m)$ we denote the minimal σ-algebra generated by the sets $A \in \mathcal{A}$ and the Borel subsets of $\mathbf{R}^m$. In what follows, the term *product-measurable* means measurability of $F : \Omega \times \mathbf{R}^m \to Cl(\mathbf{R}^n)$ with respect to $\mathcal{A} \otimes \mathcal{B}(\mathbf{R}^m)$.

d) Finally, given a (singlevalued!) function $x : \Omega \to \mathbf{R}^m$, we define $N_F x$ as family of all *"almost everywhere" selections* of the multifunction $F(\,\cdot\,, x(\,\cdot\,))$, i.e.

$$N_F x = \{y \mid y(t) \in F(t, x(t)) \text{ a.e. on } \Omega\}. \tag{4}$$

This defines the *Nemytskij operator* N_F which in the singlevalued case $F(t,u) = \{f(t,u)\}$ obviously coincides with (2). We say that the multifunction F is *superpositionally measurable* (or *sup-measurable*, for short) if, for any measurable function x, the multifunction $F(\,\cdot\,, x(\,\cdot\,))$ is also measurable (see [11]). If $F(\,\cdot\,, x(\,\cdot\,))$ contains only a measurable selection for each measurable function x, we call F *weakly sup-measurable*.

2. Relations between Carathéodory property and product-measurability. An important property of Carathéodory multifunctions is given in the following

THEOREM 1. *If* $F : \Omega \times \mathbf{R}^m \to Cp(\mathbf{R}^n)$ *is a Carathéodory multifunction, then* F *is product-measurable.*

As we shall see later (see Examples 3 and 4), an upper or lower Carathéodory multifunction need not be product-measurable. The converse of Theorem 1 is false:

EXAMPLE 1. Let $\Omega = [0,1]$ be equipped with the Lebesgue measure, and $F : \Omega\times\mathbf{R} \to Cp(\mathbf{R})$ defined by

$$F(t,u) = \begin{cases} \{0\} & \text{if } u = 0, \\ [0,1] & \text{otherwise.} \end{cases} \tag{5}$$

Then F is lower Carathéodory, but not upper Carathéodory, and hence not Carathéodory. Nevertheless, F is certainly product-measurable. □

Let $F : \Omega \times \mathbf{R}^m \to CpCv(\mathbf{R}^n)$ be a fixed multifunction. A *Carathéodory-Castaing representation* of F is, by definition, a sequence $(f_k)_k$ of Carathéodory functions $f_k : \Omega \times \mathbf{R}^m \to \mathbf{R}^n$ such that

$$F(t,u) = \overline{\{f_1(t,u), f_2(t,u), \ldots\}}\,. \tag{6}$$

We remark that a Carathéodory multifunction $F : \Omega \times \mathbf{R}^m \to CpCv(\mathbf{R}^n)$ always admits a Carathéodory-Castaing representation, but the converse is not true. In fact, the multifunction (5) is not Carathéodory, but admits many Carathéodory-Castaing representations.

One could expect that any *lower Carathéodory multifunction* $F : \Omega\times\mathbf{R}^m \to ClCv(\mathbf{R}^n)$ has a *Carathéodory selection*, simply by combining the Michael theorem [26] on continuous selections and the Kuratowski-Ryll-Nardzewski theorem [23] on measurable selections. However, this is false, as may be seen by the following rather sophisticated

EXAMPLE 2 ([17]). Let $\Omega = [0,1]$, $\mathcal{A}$ the σ-algebra generated by all singletons, and μ the counting measure on $\mathcal{A}$. Define a multifunction $F : \Omega \times \mathbf{R} \to CpCv(\mathbf{R})$ by

$$F(t,u) = \begin{cases} \{t\} & \text{if } t = u \text{ or } |t-u|^{-1} \in \mathbf{N}, \\ [0,1] & \text{otherwise.} \end{cases} \tag{7}$$

For fixed $t \in \Omega$ and $V \subseteq \mathbf{R}$, the large pre-image $F(t,\,\cdot\,)^{-1}_{-}(V) = \{u : u \in \mathbf{R}, F(t,u)\cap V \neq \emptyset\}$ is equal to Ω or $\Omega \setminus \{t, t\pm 1, t\pm 2, \ldots\}$, and hence $F(t,\,\cdot\,)$ is lower semicontinuous. For fixed $u \in \Omega$ and $V \subseteq \mathbf{R}$, in turn, $F(\,\cdot\,,u)^{-1}_{-}(V)$ is equal to Ω or $\Omega \setminus C$, where C is some subset of the countable set $\{t, t\pm \frac{1}{u}, t\pm\frac{2}{u}, \ldots\}$, and hence $F(\,\cdot\,,u)$ is measurable. Nevertheless, a straightforward but cumbersome computation shows that F does not admit a Carathéodory selection. □

To give a sufficient condition for the existence of Carathéodory selections of a lower Carathéodory multifunction, an additional definition is in order. We call $(\Omega,\mathcal{A},\mu)$ *m-projective* if, for any $D \in \mathcal{A}\otimes\mathcal{B}(\mathbf{R}^m)$, the projection $P_\Omega(D)$ of D onto Ω belongs to the σ-algebra $\mathcal{A}$, possibly up to some nullset. There are three important cases in which $(\Omega,\mathcal{A},\mu)$ is m-projective, viz. if the measure μ is σ-finite on Ω, if μ has the "direct sum property" (see e.g. [24]), or if μ is a Radon measure over a locally compact topological space Ω.

THEOREM 2 ([10]). *Let* $F : \Omega \times \mathbf{R}^m \to CpCv(\mathbf{R}^n)$ *be a multifunction such that* $F(t,\,\cdot\,)$ *is lower semicontinuous for (almost) all* $t \in \Omega$. *Assume that* $(\Omega,\mathcal{A},\mu)$ *is m-*

projective. Then F admits a Carathéodory-Castaing representation if and only if F is product-measurable.

Observe that in Theorem 2 one cannot replace the phrase "F admits a Carathéodory selection" by "F is a Carathéodory multifunction". In fact, in Example 1 the multifunction F is product-measurable, and the space $(\Omega, \mathcal{A}, \mu)$ is 1-projective.

Moreover, we point out that $(\Omega, \mathcal{A}, \mu)$ is also 1-projective in Example 2, since μ has the "direct sum property". The non-existence of Carathéodory selections of F is therefore due, according to Theorem 2, to the fact that F is not product measurable. Indeed, the large pre-image $F_-^{-1}(W)$ of the set $W = [0, \frac{1}{2}]$ cannot belong to $\mathcal{A} \otimes \mathcal{B}(\mathbf{R})$, since $P_\Omega(F_-^{-1}(W)) = [0, 1]$ does not belong to $\mathcal{A}$.

3. Relations between Carathéodory property and Scorza-Dragoni property. The following theorem is completely analogous to the singlevalued case:

THEOREM 3 ([15]). *A multifunction $F : \Omega \times \mathbf{R}^m \to Cp(\mathbf{R}^n)$ is Carathéodory if and only if F has the Scorza-Dragoni property.*

The natural question arises whether or not Theorem 3 is true also in the "semicontinuous variant". As a matter of fact, it is not difficult to prove that *every upper/lower Scorza-Dragoni multifunction is upper/lower Carathéodory.* The converse is false in both the "lower" and "upper" version. To see this, we give two examples.

EXAMPLE 3 ([29]). Let $\Omega = [0, 1]$ be equipped with the Lebesgue measure, $D \subset \Omega$ a non-measurable subset, and $F : \Omega \times \mathbf{R} \to Cp(\mathbf{R})$ defined by

$$F(t, u) = \begin{cases} \{0\} & \text{if } u = t \text{ and } t \in \Omega \setminus D, \\ \{1\} & \text{if } u = t \text{ and } t \in D, \\ [0, 1] & \text{otherwise.} \end{cases} \tag{8}$$

Then F is lower Carathéodory, but does not have the lower Scorza-Dragoni property. In fact, if the restriction of F to $\Omega_\delta \times \mathbf{R}$ (with $\mu(\Omega \setminus \Omega_\delta) \le \delta$) were lower semicontinuous, the same would be true for the restriction of F to the set $\{(t, t) : t \in \Omega_\delta\}$ which is impossible.

□

The next counterexample shows that also an upper Carathéodory multifunction need not have the upper Scorza-Dragoni property:

EXAMPLE 4 ([5]). Let $\Omega = [0, 1]$ be equipped with the Lebesgue measure, $D \subset \Omega$ a nonmeasurable set, and $F : \Omega \times \mathbf{R} \to Cp(\mathbf{R})$ defined by

$$F(t, u) = \begin{cases} [0, 1] & \text{if } u = t \text{ and } t \in D, \\ \{0\} & \text{otherwise.} \end{cases} \tag{9}$$

It is not hard to see that F is an upper Carathéodory multifunction. On the other hand, suppose that $\Omega_k \subseteq \Omega$ $(k = 1, 2, \ldots)$ is closed such that $\mu(\Omega \setminus \Omega_k) \le 1/k$, and the restriction F_k of F to $\Omega_k \times \mathbf{R}$ is upper semicontinuous. This implies, in particular, that the set $(F_k)_-^{-1}(\{1\}) = \{(t, t) \mid t \in D \cap \Omega_k\}$ is closed. Consequently, the set

$$D_+ = D \cap \left(\bigcup_{k \in \mathbf{N}} \Omega_k \right) = \bigcup_{k \in \mathbf{N}} (D \cap \Omega_k)$$

is measurable. On the other hand, since

$$\mu\Big(D \setminus \bigcup_{k\in\mathbf{N}} \Omega_k\Big) \le \mu\Big(\Omega \setminus \bigcup_{k\in\mathbf{N}} \Omega_k\Big) = 0\,,$$

the set

$$D_- = D \setminus \Big(\bigcup_{k\in\mathbf{N}} \Omega_k\Big)$$

is also measurable, contradicting our choice of $D = D_+ \cup D_-$. □

In view of these two examples, the problem arises to *characterize* those lower and upper Carathéodory multifunctions which have the lower resp. upper Scorza-Dragoni property. Such a characterization is in fact possible; roughly speaking, one has to add product-measurability:

THEOREM 4 ([12], [4]). *If $F : \Omega \times \mathbf{R}^m \to Cl(\mathbf{R}^n)$ is a product-measurable lower Carathéodory multifunction, then F has the lower Scorza-Dragoni property.*

For a parallel result for upper Carathéodory multifunctions one has to impose two additional conditions: the space $(\Omega, \mathcal{A}, \mu)$ has to be m-projective, and the multifunction F has to be compact-valued:

THEOREM 5 ([36]). *Let $F : \Omega \times \mathbf{R}^m \to Cp(\mathbf{R}^n)$ be an upper Carathéodory multifunction. Assume that $(\Omega, \mathcal{A}, \mu)$ is m-projective. Then F has the upper Scorza-Dragoni property if and only if F is product-measurable.*

By the way, there is an easy way to check if a given upper Carathéodory multifunction has the upper Scorza-Dragoni property, at least for compact-valued multifunctions. Let us say that a multifunction $F : \Omega \times \mathbf{R}^m \to Cl(\mathbf{R}^n)$ satisfies the *Filippov condition* if, for any open sets $U \subseteq \mathbf{R}^m$ and $V \subseteq \mathbf{R}^n$, the set

$$\Omega[U, V] = \{t \mid t \in \Omega, F(t, U) \subseteq V\} \tag{10}$$

is measurable, i.e. belongs to $\mathcal{A}$. As was shown in [8], [9], *an upper Carathéodory multifunction $F : \Omega \times \mathbf{R}^m \to Cp(\mathbf{R}^n)$ is upper Scorza-Dragoni if and only if F satisfies the Filippov condition.* In other words, the Filippov condition is precisely what is "missing" if an upper Carathéodory multifunction is not upper Scorza-Dragoni.

We close this section with an example which shows that Theorem 5 is, in contrast to Theorem 4, false if F assumes only closed values:

EXAMPLE 5 ([40]). Let $\Omega = [0, 1]$ be equipped with the Lebesgue measure, and $F : \Omega \times \mathbf{R} \to Cl(\mathbf{R}^2)$ defined by

$$F(t, u) = \{(\xi, t\xi) \mid \xi \in \mathbf{R}\}. \tag{11}$$

Then F is Carathéodory, since $F(t, \cdot\,)$ is constant for all $t \in [0, 1]$, and $F(\,\cdot\,, u)$ is measurable for all $u \in \mathbf{R}$, the graph $\Gamma(F(\,\cdot\,, u))$ being closed in $[0, 1] \times \mathbf{R}^2$. Moreover, it is easily checked that F satisfies the Filippov condition; in fact, for any open set $V \subseteq \mathbf{R}^2$ the set $\Omega[V]$ (see (10)) consists of all $t \in [0, 1]$ such that the straight line through the origin with slope t is entirely contained in V. □

Nevertheless, F cannot have the upper Scorza-Dragoni property, since $F(\,\cdot\,,u)$ is not upper semicontinuous on any subset $\Omega_\delta \subset \Omega$.

4. Relations between Carathéodory property and sup-measurability. In this section we discuss some sufficient conditions for the sup-measurability (or weak sup-measurability) of a multifunction F. First of all, we mention a basic sufficient condition which is one of the main reasons why the Carathéodory conditions are so important in the theory of differential equations and inclusions:

THEOREM 6. *If* $F : \Omega \times \mathbf{R}^m \to Cp(\mathbf{R}^n)$ *is a Carathéodory multifunction*, *then* F *is sup-measurable.*

Again, Theorem 6 is false for upper and lower Carathéodory multifunctions. We illustrate this first by means of a counterexample in the "upper" case:

EXAMPLE 6 ([28]). Let $\Omega = [0,1]$ be equipped with the Lebesgue measure, $D \subset \Omega$ a nonmeasurable subset, and $F : \Omega \times \mathbf{R}^m \to Cp(\mathbf{R})$ defined by

$$F(t,u) = \begin{cases} [0,1] & \text{if } u = t \text{ and } t \in \Omega \setminus D, \\ [0,1] & \text{if } u = t+1 \text{ and } t \in D, \\ \{1\} & \text{otherwise.} \end{cases} \tag{12}$$

Then F is upper Carathéodory, but not sup-measurable, since F maps the function $x(t) = t$ into the multifunction

$$F(t,t) = \begin{cases} [0,1] & \text{if } t \in \Omega \setminus D, \\ \{1\} & \text{if } t \in D, \end{cases} \tag{13}$$

which is not measurable. □

One could ask whether or not an upper Carathéodory multifunction is at least *weakly* sup-measurable; for instance, in Example 6 the non-measurable multifunction $F(t,t)$ admits the measurable selection $y(t) \equiv 1$. In fact, the following is true:

THEOREM 7 ([6]). *If* $F : \Omega \times \mathbf{R}^m \to Cp(\mathbf{R}^n)$ *is upper Carathéodory*, *then* F *is weakly sup-measurable.*

We turn now to the analogous problem for lower Carathéodory multifunctions. Surprisingly, a lower Carathéodory multifunction need not even be weakly sup-measurable:

EXAMPLE 7 ([29]). Let $\Omega = [0,1]$, $D \subset \Omega$ a non-measurable subset, and let $F : \Omega \times \mathbf{R} \to Cp(\mathbf{R})$ be defined as in Example 3. Then F is lower Carathéodory, but not weakly sup-measurable, since F maps the function $x(t) = t$ into the multifunction

$$F(t,t) = \begin{cases} \{0\} & \text{if } t \in \Omega \setminus D, \\ \{1\} & \text{if } t \in D, \end{cases} \tag{14}$$

which of course does not admit a measurable selection. □

Apart from the Carathéodory property, product-measurability is also a sufficient condition for sup-measurability:

THEOREM 8 ([38]). *If* $F : \Omega \times \mathbf{R}^m \to Cp(\mathbf{R}^n)$ *is product-measurable*, *then* F *is sup-measurable.*

It is worthwhile mentioning that the converse of Theorem 8 is false even in the singlevalued case. This was an open problem for many years and answered by means of a class of very exotic functions called "monsters" in the literature (see e.g. [16]). However, under additional assumptions one may show that certain subclasses of sup-measurable multifunctions are product-measurable. For example, the following holds:

THEOREM 9 ([39]). *If* $F : \Omega \times \mathbf{R}^m \to Cp(\mathbf{R}^n)$ *is upper Carathéodory and sup-measurable, and* $(\Omega, \mathcal{A}, \mu)$ *is m-projective, then* F *is product-measurable.*

5. Comparison of the preceding counterexamples. In the following table we collect the properties of all counterexamples considered so far.

F as given in	upper Car.	lower Car.	Filippov	upper S-D	lower S-D	product-meas.	sup-meas.	weakly sup-meas.
Ex. 1	no	yes	yes	no	yes	yes	yes	yes
Ex. 2	no	yes	no	—	—	no	yes	yes
Ex. 3	no	yes	no	no	no	no	no	no
Ex. 4	yes	no	no	no	no	no	no	yes
Ex. 5	yes	yes	yes	no	yes	yes	yes	yes
Ex. 6	yes	no	no	no	no	no	no	yes

All entries of this table are consequences of the few theorems given above. First of all, the space $(\Omega, \mathcal{A}, \mu)$ is 1-projective in all six examples. The properties of the corresponding multifunctions (5), (7), (8), (9), (11), and (12) may be deduced from the following reasoning.

1) The multifunction (5) is lower Carathéodory and product-measurable, but not upper Carathéodory, and hence not upper Scorza-Dragoni either. It has the lower Scorza-Dragoni property, by Theorem 4, and is sup-measurable, by Theorem 8.

2) The multifunction (7) is lower Carathéodory, but not upper Carathéodory, since it has no Carathéodory selection. For the same reason, it cannot be product-measurable, by Theorem 2. Studying this multifunction from the viewpoint of the Scorza-Dragoni property does not make sense, since the only closed subset $\Omega_\delta \subset \Omega$ with $\mu(\Omega \setminus \Omega_\delta) < 1$, say, is $\Omega_\delta = \Omega$. However, it is sup-measurable, as was observed in [38].

3) The multifunction (8) is lower Carathéodory, but not lower Scorza-Dragoni. As we have seen, it is not weakly sup-measurable, and hence neither upper Carathéodory, by Theorem 7 (or 6), nor upper Scorza-Dragoni. Moreover, Theorem 4 implies that it cannot be product-measurable.

4) The multifunction (9) is upper Carathéodory, but not upper Scorza-Dragoni. From Theorem 5 it follows that it is not product-measurable, and thus from Theorem 1 it follows that it is neither lower Carathéodory nor lower Scorza-Dragoni. Finally, it is weakly sup-measurable, by Theorem 7, but not sup-measurable, by Theorem 9.

5) The multifunction (11) is Carathéodory and lower Scorza-Dragoni, but not upper Scorza-Dragoni, as we have seen. The product-measurability and sup-measurability may be verified directly.

6) The multifunction (12) is upper Carathéodory, but not sup-measurable. Theorem 8 implies that it is not product-measurable, and thus Theorem 5 implies that it is not upper Scorza-Dragoni. Theorem 1 further shows that it is neither lower Carathéodory nor lower Scorza-Dragoni. Nevertheless, it is weakly sup-measurable, by Theorem 7.

6. Concluding remarks. All regularity conditions considered so far apply to the Nemytskij operator (4) between certain spaces of *measurable functions.* As a matter of fact, these conditions guarantee the *acting* of the operator (4), but in general do not imply further analytical properties which are needed to apply the basic principles of nonlinear analysis to the differential inclusion (3). A more detailed study of *boundedness* and *continuity properties* of the operator (4) may be found in the papers [2,3] or in the forthcoming survey article [1].

Another question which may be posed in this connection is the following. Suppose we are interested in conditions under which the Nemytskij operator (4) acts in spaces of *continuous functions*, rather than measurable functions. In the singlevalued case this problem may be solved by any first-year calculus student: the operator (2) maps continuous functions $t \mapsto x(t)$ into continuous functions $t \mapsto f(t, x(t))$ if and only if f is continuous on the product $\Omega \times \mathbf{R}^m$. (Here we assume that Ω is a compact domain without isolated points.) Let us briefly sketch the situation in the multivalued case.

Let us say that a multifunction $F : \Omega \times \mathbf{R}^m \to Cl(\mathbf{R}^n)$ is *superpositionally continuous* (or *sup-continuous*, for short) if, for any continuous function x, the multifunction $F(\,\cdot\,, x(\,\cdot\,))$ is also continuous (in the Hausdorff metric). If $F(\,\cdot\,, x(\,\cdot\,))$ contains only a continuous selection for each continuous function x, we call F *weakly sup-continuous.* In either case, we define the Nemytskij operator (4) as family of all *pointwise selections*, i.e.

$$N_F x = \{y \mid y(t) \in F(t, x(t)) \text{ on } \Omega\}. \tag{15}$$

The following result is an easy consequence of the classical Tietze-Uryson theorem and Michael's selection theorem:

THEOREM 10. *If* $F : \Omega \times \mathbf{R}^m \to Cp(\mathbf{R}^n)$ *is continuous*, *then* F *is sup-continuous. If* $F : \Omega \times \mathbf{R}^m \to ClCv(\mathbf{R}^n)$ *is lower semicontinuous*, *then* F *is weakly sup-continuous.*

It is easy to see that the lower semicontinuity of F in Theorem 10 is sufficient, but not necessary for the weak sup-continuity of F. Moreover, Theorem 10 is false for upper semicontinuous multifunctions:

EXAMPLE 8. Let $\Omega = [0, 1]$ and $F : \Omega \times \mathbf{R} \to \mathbf{R}$ defined by

$$F(t, u) = \begin{cases} \{0\} & \text{if } u < \frac{1}{2}, \\ [0, 1] & \text{if } u = \frac{1}{2}, \\ \{1\} & \text{if } u > \frac{1}{2}. \end{cases} \tag{16}$$

Then F is upper semicontinuous, but the multifunction $F(t, t)$ has no continuous selection.

□

It is interesting to compare Theorem 10 and Example 8 with Theorem 7 and Example 7. The roles of upper and lower semicontinuity are precisely reversed when passing from (weak) sup-measurability to (weak) sup-continuity.

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