TOPOLOGY IN NONLINEAR ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 35
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1996

PARTIALLY DISSIPATIVE PERIODIC PROCESSES

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Abstract. Further extension of the Levinson transformation theory is performed for partially dissipative periodic processes via the fixed point index. Thus, for example, the periodic problem for differential inclusions can be treated by means of the multivalued Poincaré translation operator. In a certain case, the well-known Ważewski principle can also be generalized in this way, because no transversality is required on the boundary.

1. Introduction. The problem of transforming the existence of forced oscillations to the verification of some stability properties has been usually connected with the name of Norman Levinson, because of his pioneering article [21]. It is, for example, well-known (see e.g. [26, p. 172]) that, under some natural regularity assumptions, *the periodic dissipative first-order system admits a harmonic.* By the dissipativity we mean here the one in the sense of N. Levinson, i.e. when all solutions are (uniformly) ultimately bounded.

This classical result was formalized in the most abstract setting, i.e. in terms of

1991 *Mathematics Subject Classification*: Primary 34C25; Secondary 34A60, 55M25, 47H11.

Key words and phrases: periodic processes, fixed point index, partial dissipativity, differential inclusions, Poincaré translation operator, large-period forced oscillations.

Research of the first author partially supported by the grant no. 201/93/2311 of the Grant Agency of Czech Republic.

The paper is in final form and no version of it will be published elsewhere.

dissipative periodic processes, in [14], [15] (see also [13] and the references therein). Another interesting and rather abstract result of this type says (see [23]) that *a positively Lagrange-stable motion which is asymptotically stable limits to a periodic motion.* The historical survey of the Levinson transformation theory with an extensive bibliography can be found in [4], where the generalization to the nondissipative case has been probably done for the first time (for the quite equivalent reformulation in terms of guiding functions see also [2]). In fact, only the partially dissipative systems were there under the consideration, while, with respect to the remaining components, the origin was supposed to be a uniform repeller.

The main purpose of this paper is two-fold: (i) *to avoid* (*among others*) *some regularity conditions like uniqueness*, *assumed in* [4], (ii) *to make a further generalization of the Ważewski-type results* (for the comparison in terms of the Lefschetz index or the Conley index see e.g. a very recent paper [24] and the references therein).

Let us conclude that an increasing interest has been recently paid to the partial stability properties themselves (see e.g. two monographs [22], [25]) as well as to the further generalization of the concept of attractivity (see [7], [16], [17]) or repulsivity (see [8], [9]), mostly again with respect to the application to the existence of forced periodic oscillations (see e.g. [8], [18], [19], [20], [24], [27]).

The paper is organized as follows: in Section 2, the concept of fixed point index for multivalued decomposable mappings is introduced. The basic fixed point theorem is posed in Section 3, while the relationship with multivalued Poicaré operator is clarified in Section 4. The concluding remarks concerning the applications to ordinary differential equations and inclusions are given in Section 5.

2. Fixed point index for multivalued decomposable mappings. In this section we introduce the topological degree of so-called decomposable mappings that will be our main tool for carrying out the future results, concerning differential inclusions.

Definition 1. A nonempty subset A of a metric space X is of the R_δ *type* if it is the intersection of a decreasing sequence of compact contractible subsets of X.

Note that in case X is an ANR space, then A is R_δ, provided it is the intersection of a decreasing sequence of compact AR's. It is also convenient to point out that the product of two R_δ sets is an R_δ set as well.

Definition 2. A multivalued mapping $\varphi : X \to Y$ is called *upper-semi-continuous* (u.s.c.), provided for any open subset $B \subset Y$, the set $\{x \in X \mid \varphi(x) \subset B\}$ is open in X. φ is said to be *lower-semi-continuous* (l.s.c.), provided for any open subset $B \subset Y$, the set $\{x \in X \mid \varphi(x) \cap B \neq \emptyset\}$ is open in X.

Definition 3. A multivalued operator $\varphi : X \to Y$ is called *admissible*, provided X and Y are compact metric ANR's and φ is u.s.c. with R_δ values.

Definition 4. We say that a single-valued map $f : X \to Y$ is an ε*-approximation of* $\varphi : X \to Y$ *on its graph* if the following condition holds:

$$\forall x \in X \ \exists y \in X \quad d(x,y) < \varepsilon \qquad \text{and} \qquad \mathrm{dist}_Y(f(x), \varphi(y)) < \varepsilon.$$

The following approximation theorem (cf. [11]) is of a particular importance for defining a fixed point index.

THEOREM 1. *If* $\varphi : X \to Y$ *is admissible*, *then for every* $\varepsilon > 0$ *there exists a continuous map* $f_\varepsilon : X \to Y$, *that is an* ε*-approximation of* φ. *Moreover*, *there is an* $\varepsilon_0 > 0$ *such that every two* ε_0*-approximations of* φ *are homotopic.*

Assume we have a map $\varphi : X \to X$, having a decomposition:

$$D_\varphi :\ X = X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} X_2 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_n} X_n = X, \qquad \varphi = \varphi_n \ldots \varphi_2\varphi_1,$$

where each φ_i is admissible in the sense of Definition 3. In such case we say that φ is decomposable. Let A be an open subset of X with no fixed points of φ on its boundary and let f_i, i=1,…,n, be ε-approximations of φ_i for an $\varepsilon > 0$. Let us call the map: $f = f_n \ldots f_2 f_1$ the ε-decomposable approximation of φ. Using Theorem 1, one can show that there exists an $\varepsilon_0 > 0$ such that every two ε_0-decomposable approximations of φ are homotopic with the homotopy $\chi : X \times [0,1] \to X$ such that:

$$\forall t \in [0,1]\ \forall x \in \partial A \quad x \neq \chi(x,t).$$

Now, following [5], we define an index of φ over X with respect to A:

$$\operatorname{Ind}_X(D_\varphi, A) = \operatorname{ind}_X(f, A), \tag{1}$$

where ind indicates the ordinary fixed point index of maps on compact ANR's and f is an arbitrary ε_0-decomposable approximation of φ. The remarks above prove the correctness of (1).

Below we collect some properties of Ind:

THEOREM 2. *Let* $\varphi, \psi : X \to X$ *be the decomposable maps such that* $\operatorname{Ind}_X(D_\varphi, A)$ *exists.*

(i) (Existence) *If* $\operatorname{Ind}_X(D_\varphi, A) \neq 0$ *then* φ *has a fixed point in* A.

(ii) (Additivity) *If* A_j, $j = 1, \ldots, n$ *are open*, *disjoint subsets of* A *and all fixed points of* $\varphi_{|A}$ *lie in* $\bigcup_{j=1}^n A_j$ *then*, $\operatorname{Ind}_X(D_\varphi, A_j)$, $j = 1, \ldots, n$, *are defined and*:

$$\operatorname{Ind}_X(D_\varphi, A) = \sum_{j=1}^n \operatorname{Ind}_X(D_\varphi, A_j).$$

(iii) (Homotopy invariance) *Suppose that the decompositions* D_φ *and* D_ψ *are homotopic*, *that is*:

$$D_\varphi :\ X = X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} X_2 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_n} X_n = X, \qquad \varphi = \varphi_n \ldots \varphi_2\varphi_1,$$

$$D_\psi :\ X = X_0 \xrightarrow{\psi_1} X_1 \xrightarrow{\psi_2} X_2 \xrightarrow{\varphi_3} \cdots \xrightarrow{\psi_n} X_n = X, \qquad \psi = \psi_n \ldots \psi_2\psi_1,$$

and there is a decomposable homotopy: $\chi : X \times [0,1] \to X$:

$$D_\chi : X \times [0,1] = X_0 \times [0,1] \xrightarrow{\overline{\chi}_1} X_1 \times [0,1] \xrightarrow{\overline{\chi}_2} \cdots \xrightarrow{\overline{\chi}_{n-1}} X_{n-1} \times [0,1] \xrightarrow{\chi_n} X_n = X,$$

where $\chi = \chi_n\overline{\chi}_{n-1} \ldots \overline{\chi}_1$, $\overline{\chi}_i(x,\lambda) = \chi_i(x,\lambda) \times \{\lambda\}$ *for* $x \in X_{i-1}$, $\lambda \in [0,1]$, $i = 1, \ldots, n-1$, χ_i *are* u.s.c. *with* R_δ *values*, $\chi_i(\,\cdot\,, 0) = \varphi_i$, $\chi_i(\,\cdot\,, 1) = \psi_i$, $i = 1, \ldots, n$, *and* $x \notin \chi(x,\lambda)$ *for* $x \in \partial A$ *and* $\lambda \in [0,1]$. *Then* $\operatorname{Ind}_X(D_\psi, A)$ *is defined and* $\operatorname{Ind}_X(D_\psi, A) = \operatorname{Ind}_X(D_\varphi, A)$.

(iv) (Multiplicativity) *If* $\eta : Y \to Y$ *is decomposable and* $\mathrm{Ind}_Y(D_\eta, B)$ *exists*, *then*:

$$\mathrm{Ind}_{X\times Y}(D_\varphi \times D_\eta, A\times B) = \mathrm{Ind}_X(D_\varphi, A)\cdot \mathrm{Ind}_Y(D_\eta, B),$$

where:

$$D_\varphi : X = X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} X_2 \longrightarrow \cdots \xrightarrow{\varphi_n} X_n = X, \qquad \varphi = \varphi_n \ldots \varphi_2\varphi_1,$$
$$D_\eta : Y = Y_0 \xrightarrow{\eta_1} Y_1 \xrightarrow{\eta_2} Y_2 \longrightarrow \cdots \xrightarrow{\eta_n} Y_n = Y, \qquad \eta = \eta_n \ldots \eta_2\eta_1,$$

and

$$D_\varphi \times D_\eta : X\times Y = X_0\times Y_0 \overset{\varphi_1\times\eta_2}{\longrightarrow} X_1\times Y_1 \overset{\varphi_2\times\eta_2}{\longrightarrow} X_2\times Y_2 \longrightarrow \cdots$$
$$\cdots \overset{\varphi_n\times\eta_2}{\longrightarrow} X_n \times Y_n = X\times Y.$$

(v) (Units) *Suppose that* φ *is admissible. If* φ *is constant*, *that is*: *for every* $x\in X$ $\varphi(x) = B\subset X$ *then*:

$$\mathrm{Ind}_X(D_\varphi, A) = \begin{cases} 1 & \text{if } A\cap B\neq\emptyset,\\ 0 & \text{if } A\cap B = \emptyset.\end{cases}$$

One can see that the definition (1) depends on the decomposition D_φ. However, when it is clear, which decomposition we mean, we will usually write:

$$\mathrm{Ind}_X(\varphi, A)$$

instead of $\mathrm{Ind}_X(D_\varphi, A)$.

3. A fixed point theorem. The following theorem is an improved version of Lemma 1 in [4] and will be applied to the Poincaré operator, defined in the next section.

THEOREM 3. *Let* E_1 *and* E_2 *be two finite dimensional normed spaces. Assume we have*:

$$\varphi : [0,T]\times(E_1\times E_2)\to E_1,$$
$$\psi : [0,T]\times(E_1\times E_2)\to E_2$$

u.s.c. *mappings with* R_δ *values such that the following conditions hold*:

(i) *the maps* $\varphi_0 = \varphi(0,\cdot\,)$ *and* $\psi_0 = \psi(0,\cdot\,)$ *are projections onto the spaces*: E_1 *and* E_2, *respectively.*

Let $A\subset E_1$, $B\subset E_2$, A, B *are open*, *bounded and*:

(ii) $A\cdot[0,1] = A$, $B\cdot[0,1] = B$, (*that is*: A, B *are star-shaped with respect to the origins*),

(iii) $\varphi_T(\partial A\times \overline{B})\cap\overline{A} = \emptyset$, $\psi_T(\overline{A}\times\partial B)\subset B$,

(iv) $0\notin\varphi([0,T]\times(\partial A\times\{0\}))$.

Then the map $(\varphi_T,\psi_T) : E_1\times E_2\to E_1\times E_2$; $(\varphi_T,\psi_T)(x) = \varphi_T(x)\times\psi_T(x)$ *has at least one fixed point in the set* $\mathcal{R} = A\times B$.

P r o o f. Take K_1 and K_2 being the closed balls in E_1 and E_2, centered at the origins and large enough to contain the sets A and B, respectively. (We also demand that $\partial A\cap\partial K_1 = \partial B\cap\partial K_2 = \emptyset$.)

Set: $r_1 : E_1 \to K_1$, $r_2 : E_2 \to K_2$ to be the radial retractions onto K_1 and K_2, respectively. Consider the homotopy: $H : (K_1 \times K_2) \times [0,1] \to K_1 \times K_2$,

$$H((u,v),\lambda) = (r_1[(1-\lambda)u + \varphi_T(u,\lambda v)], r_2[\lambda\psi_T(\lambda u, v)]) \qquad u \in K_1,\ v \in K_2,\ \lambda \in [0,1],$$

where $\varphi_T = \varphi(T,\,\cdot\,)$ and $\psi_T = \psi(T,\,\cdot\,)$. This homotopy is decomposable in the sense of the fixed point index (compare Theorem 2 (iii)). It is not difficult to see that the map

$$S_1 : (K_1 \times K_2) \times [0,1] \to E_1,$$

$$S_1((u,v),\lambda) = (1-\lambda)u + \varphi_T(u,\lambda v)$$

is u.s.c. with R_δ-values, so the convex hull of its image is a compact ANR contained in E_1. The same is true for the mapping

$$S_2 : (K_1 \times K_2) \times [0,1] \to E_2,$$

$$S_2((u,v),\lambda) = \lambda\psi_T(\lambda u, v).$$

Hence,

$$D_H : (K_1 \times K_2) \times [0,1] \xrightarrow{S_1 \times S_2} \overline{\mathrm{co}}((S_1 \times S_2)((K_1 \times K_2) \times [0,1])) \xrightarrow{r_1 \times r_2} K_1 \times K_2$$

is a decomposition of H. The fact that H has no fixed point on the boundary of $\mathcal{R}$ follows from (ii) and (iii). By the homotopy invariance of fixed point index we obtain the following equality:

$$\mathrm{Ind}_{K_1 \times K_2}((r_1\varphi_T, r_2\psi_T), \mathcal{R}) = \mathrm{Ind}_{K_1 \times K_2}((r_1[\,\cdot\, + \varphi_T(\,\cdot\,,0)], 0), \mathcal{R}),$$

which by the multiplicativity and units properties is equal to:

$$\mathrm{Ind}_{K_1}(r_1[\,\cdot\, + \varphi_T(\,\cdot\,,0)], A).$$

Define the homotopy: $\overline{H} : K_1 \times [0,T] \longrightarrow K_1$ as

$$\overline{H}(u,t) = r_1[u + \varphi(t,(u,0))], \qquad u \in K_1,\ t \in [0,T].$$

Recalling (iv) and using the homotopy invariance property once again, this time for the index on K_1, we have:

$$\mathrm{Ind}_{K_1}(r_1[\,\cdot\, + \varphi_T(\,\cdot\,,0)], A) = \mathrm{Ind}_{K_1}(r_1[\,\cdot\, + \varphi_0(\,\cdot\,,0)], A),$$

which, by (i) is equal to:

$$\mathrm{Ind}_{K_1}(r_1[2 \cdot \mathrm{Id}_{K_1}], A) = \deg(-\mathrm{Id}_{E_1}, K, 0) = (-1)^j,$$

where deg denotes the Brouwer topological degree, K is a sufficiently small ball in E_1 and j indicates the dimension of E_1 (finite under the hypothesis).

We are now in a position to write the following equality:

$$\mathrm{Ind}_{K_1 \times K_2}((r_1\varphi_T, r_2\psi_T), \mathcal{R}) = (-1)^j. \tag{2}$$

By the existence property of the fixed point index, the result follows. □

Remark 1. Theorem 3 is also valid if we replace the assumption of E_1 to be finite dimensional by the compactness of the map φ. In both cases E_2 must be finite dimensional.

4. Multi-valued Poincaré operator. In this section we proceed with the study of the existence of periodic solutions to the problem:

$$\begin{cases} x'(t) \in S(t, x(t)), \\ x(0) = x(T), \end{cases} \tag{3}$$

where $S : [0, T] \times R^n \to R^n$ is a multivalued mapping (cf. [6], [10], [12]).

By the solution of (3) we mean the solution in the sense of Carathéodory, i.e. the absolutely continuous function $x : [0, T] \to R^n$, satisfying $x'(t) \in S(t, x(t))$ almost everywhere in $t \in [0, T]$ and such that $x(0) = x(T)$.

Consider the map $P : R^n \to C([0, T], R^n)$:

$$P(x_0) = \{x \in C([0, T], R^n) \mid x \text{ is a solution of: } x'(t) \in S(t, x(t)),\ x(0) = x_0\}$$

for $x_0 \in R^n$.

Below we present a generalization of the well-known theorem due to Aronszajn, dealing with the case of differential inclusions (see [10], [12]).

THEOREM 4. *If S is bounded*, u.s.c. *map with nonempty*, *compact*, *convex values*, *then P is* u.s.c. *with R_δ values.*

We define the evaluation map: $e_t : C([0, T], R^n) \to R^n$, $t \in [0, T]$,

$$e_t(x) = x(t), \qquad x \in C([0, T], R^n)$$

and have the following diagram:

$$R^n \xrightarrow{P} C([0, T], R^n) \xrightarrow{e_t} R^n. \tag{4}$$

The composition $Q_t = e_t P$ is called the Poincaré translation operator. Assuming that the conditions of Theorem 3 are satisfied and taking a compact, convex set $K \subset R^n$ and a retraction $r : R^n \to K$ onto K such that:

$$r(R^n \setminus K) \subset \partial K, \tag{5}$$

we obtain the decomposable map $rQ_T : K \to K$, to which we can apply the fixed point index theory, described in Section 2. Namely, by the existence property, we have:

THEOREM 5. *If $A \subset K$ is such that $\partial A \cap \partial K = \emptyset$ (the boundaries with respect to R^n) and* $\mathrm{Ind}_K(rQ_T, A)$ *is defined and different from* 0, *then* (3) *has a solution x with $x(0) \in A$.*

Consider a system of differential inclusions, given by a multivalued map $S : [0, T] \times R^n \times [0, 1] \to R^n$,

$$\begin{cases} x'(t) \in S(t, x(t), \lambda), \\ x(0) = x(T), \end{cases} \tag{6}$$

with $\lambda \in [0, 1]$. For every $\lambda \in [0, 1]$ we can set the map P_λ:

$$P_\lambda(x_0) = \{x \in C([0, T], R^n) \mid x \text{ is a solution of: } x'(t) \in S(t, x(t), \lambda),\ x(0) = x_0\}$$

for $x_0 \in R^n$, and the map $Q_t^\lambda = e_t P_\lambda$, which is the Poincaré operator for (6) and $\lambda \in [0, 1]$.

The following theorem (see [10]) corresponds to Theorem 4:

THEOREM 6. *If S is bounded*, u.s.c. *with nonempty*, *compact*, *convex values*, *then the map* $P : R^n \times [0,1] \to C([0,T], R^n)$,

$$P(x_0, \lambda) = P_\lambda(x_0),$$

is u.s.c. *with R_δ values.*

Assume that S is as in Theorem 4 and that P_1 splits in the following way:

$$P_1(x_0) = P_{11}(x_0) \times P_{12}(x_0),$$

$$P_{11} : R^n \to C([0,T], R^j), \qquad P_{12} : R^n \to C([0,T], R^k), \qquad j + k = n,$$

where P_{11} and P_{12} have R_δ values.

Define $\varphi(t, x_0) = e_t P_{11}(x_0)$ and $\psi(t, x_0) = e_t P_{12}(x_0)$, $x_0 \in R^n$, $t \in [0,1]$. Then, under the assumptions (ii)–(iv) of Theorem 2, one can obtain:

$$\operatorname{Ind}_{K_1 \times K_2}((r_1\varphi_T, r_2\psi_T), \mathcal{R}) = (-1)^j. \tag{7}$$

(K_1, K_2, r_1, r_2, $\mathcal{R}$ are as in Theorem 2).

The proof of this fact is exactly the same as the one of Theorem 2, although the maps φ and ψ are no more u.s.c. with R_δ values. Still, they are the compositions of such maps with the evaluation mappings, that makes the homotopies H and $\overline{H}$ decomposable in the sense of the fixed point index, as it was before.

Now, we reformulate (7):

$$(-1)^j = \operatorname{Ind}_{K_1 \times K_2}(r(\varphi_T, \psi_T), \mathcal{R}) = \operatorname{Ind}_K(rQ_T^1, \mathcal{R}),$$

where $r : R^n \to K = K_1 \times K_2$ is a retraction for which (5) is valid. Let $H : K \times [0,1] \to K$,

$$H(x, \lambda) = rQ_T^\lambda(x).$$

In view of Theorem 5, H is a decomposable homotopy, linking rQ_T^0 with rQ_T^1. If for each $\lambda \in [0,1)$ and every $x \in \partial\mathcal{R}$, $x \notin Q_T^\lambda(x)$, then by the homotopy invariance we obtain:

$$\operatorname{Ind}_K(rQ_T^0, \mathcal{R}) = (-1)^j,$$

which implies, in view of Theorem 4, that the inclusion $x'(t) \in S(t, x(t), 0)$ has at least one T-periodic solution with the initial value in $\mathcal{R}$.

Using the "a priori estimates" technique (see [10]), we can avoid the assumption of boundedness of the map S. Namely, let S satisfy all the assumptions above with the integrable boundedness instead of the boundedness condition.

Recall that S is said to be integrably bounded if there exists a Lebesgue integrable function $\mu \in L^1([0,T], R)$ such that:

$$\forall t \in [0,T]\ \forall x_0 \in R^n\ \forall \lambda \in [0,1]\ \forall y \in S(t, x_0, \lambda) : \|y\| < \mu(t).$$

We define $\overline{S} : [0,T] \times R^n \times [0,1] \to R^n$ by putting:

$$\overline{S}(t, x, \lambda) = \begin{cases} S(t, x, \lambda) & \text{if } \|x\| \le M,\ t \in [0,T],\ \lambda \in [0,1], \\ S(t, M \cdot x/\|x\|, \lambda) & \text{if } \|x\| \ge M,\ t \in [0,T],\ \lambda \in [0,1], \end{cases}$$

for a constant $M > 0$. Then $\overline{S}$ is as it was considered before. Moreover, one can prove (see [10]), that if M is large enough then the Poincaré operator P for the problem (6) coincides with the Poincaré operator for that system with S replaced by $\overline{S}$. Consequently, the inclusion $x'(t) \in S(t, x(t), 0)$ has a T-periodic solution x with $x(0) \in \mathcal{R}$.

Now, we turn to the case when S is no longer an u.s.c. map, but l.s.c. instead. In what follows, we will employ the result due to A. Bressan (cf. [6]).

THEOREM 7. *Let A be a closed subset of $R \times R^n$ and let $G : A \to R^n$ be a bounded,* l.s.c. *multifunction with nonempty, closed values. Then there exists an* u.s.c., *bounded map $F : A \to R^n$ with nonempty, compact, convex values such that every solution of $x'(t) \in F(t, x(t))$ is also a solution of $x'(t) \in G(t, x(t))$.*

Consider system (6) with $S : [0,T] \times R^n \times [0,1] \to R^n$ l.s.c., bounded and nonempty, closed-valued. From the proof of Theorem 7 (see [6]) it follows, that we can find a multivalued map $\overline{S} : [0,T] \times R^n \times [0,1] \to R^n$ u.s.c., bounded with nonempty, compact, convex values such that the Poincaré operator $\overline{P}$ (defined in Theorem 6) for the system (6) with S replaced by $\overline{S}$ is a selector of the Poincaré operator P for (6). Assume that the previous conditions for P hold and, additionally, $\overline{P}_1$ splits in the same way as P_1. Then $\overline{P}_0$ has a fixed point, which implies the existence of a T-periodic solution to the inclusion $x'(t) \in S(t, x(t), 0)$. An example of such a situation will be given in Section 5.

5. Applications to ordinary differential equations and inclusions. The main difficulty in applying the results obtained in Section 4 lies in giving the sufficient conditions for the right-hand side map of the system (6) to make the Poincaré operator P_1 split. Below we present three possible examples.

EXAMPLE 1 (Uniqueness case). S is such that the uniqueness of solutions to the problem:

$$\begin{cases} x'(t) \in S(t, x(t), \lambda), \\ x(0) = x_0 \end{cases}$$

is guaranteed for every $\lambda \in [0,1]$ and every initial value $x_0 \in R^n$. This case, with the additional restriction imposed on S to be single-valued is considered in details in [2], [4].

For the application to the large-period forced oscillations of the systems and higher-order equations see [1], [3], where the sufficient conditions, imposed on the right-hand sides, have been obtained in an effective way.

EXAMPLE 2 (U.s.c. splitting case). The mulivalued map S (u.s.c. with compact, convex, nonempty values) splits itself when $\lambda = 1$, that is:

$$S(t, (u, v), 1) = S_1(t, u) \times S_2(t, v) \qquad t \in [0,T],\ u \in R^j,\ v \in R^k,$$

where $S_1 : [0,T] \times R^j \to R^j$, $S_1 : [0,T] \times R^k \to R^k$.

This case will be treated in more general setting, with respect to obtaining the effective conditions, by the authors elsewhere.

EXAMPLE 3 (L.s.c. splitting case). $S : [0,T] \times R^{j+k} \to R^{j+k}$, $S_1 : [0,T] \times R^j \to R^j$, $S_2 : [0,T] \times R^k \to R^k$,

$$S(t, (u, v)) = S_1(t, u) \times S_2(t, v) \qquad t \in [0,T],\ u \in R^j,\ v \in R^k,$$

S is l.s.c. with nonempty, closed values. In such a case the Poincaré operators, namely the one for the inclusion $x'(t) \in S(t, x(t))$ and another for the inclusion corresponding to the above in the sense of Theorem 7, split. Also this case will be treated in details by the authors elsewhere.

In order to overcome the mentioned difficulties in a more general situation practically, the best tool seems to us the application of two bounding functions, one guaranteeing the partial retraction, while the second partial repulsivity, both on the boundary of $\mathcal{R}$ (see Theorem 5). This has also a lot to do with the so called “viability theory”, where such models are under intensive study, today.

The second possibility consists in the application of two guiding functions giving the same asymptotically (for a big multiple of the period T), but this time only the existence of subharmonics can be proved, because the appropriate fixed-point theorem in section 3 cannot be so easily modified as in [4].

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