TOPOLOGY IN NONLINEAR ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 35
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1996

NONRADIAL SOLUTIONS OF NONLINEAR NEUMANN PROBLEMS IN RADIALLY SYMMETRIC DOMAINS

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1. Introduction. In recent years, the following semilinear elliptic Neumann problem has been studied extensively

$$(1.1)_\lambda \begin{cases} -\Delta u + \lambda u = u^p, u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary, $\lambda > 0$ and $p > 1$ are constants and ν is the unit outer normal to $\partial\Omega$.

Concerning the existence, multiplicity, and qualitative properties of solutions of $(1.1)_\lambda$ many interesting results have appeared; especially after Ni and Takagi ([NT1]) first discovered the spike-layer structure on the shape of least energy solutions for the subcritical problems, a lot of work has been devoted to the study of qualitative properties of solutions of $(1.1)_\lambda$. For more references, we refer to [NT2] and [Wz5], in which both the subcritical exponent case (i.e. $1 < p < \frac{N+2}{N-2}$) and the critical exponent case (i.e. $p = \frac{N+2}{N-2}$) are surveyed.

In this paper, we shall focus on the case where Ω is a spherically symmetric domain, especially on the case where Ω is a ball domain. We are mainly interested in the existence and the shape of *nonradial* solutions of $(1.1)_\lambda$. When we replace the Neumann boundary condition by the Dirichlet boundary condition the well known Gidas-Ni-Nirenberg result ([GNN]) asserts that any positive solutions must be radially symmetric. However, we shall see that contrary to its Dirichlet counterpart, $(1.1)_\lambda$ possesses many *nonradial solutions* when Ω is a ball domain.

In [Wz6], we have presented an approach to this problem to construct multi-peaked solutions for $(1.1)_\lambda$ with the critical Sobolev exponent when Ω is a symmetric domain. We

1991 *Mathematics Subject Classification*: Primary 35J65; Secondary 35J05, 58E05.
Partially supported by an NSF grant and a faculty research grant at Utah State University.

The paper is in final form and no version of it will be published elsewhere.

shall review the main ideas and methods from our work [Wz6] in Section 2 and present some extensions of our results using similar methods.

Let us start with some preliminaries here. We define an energy functional associated with $(1.1)_\lambda$

$$E_\lambda(u) = \int_\Omega (|\nabla u|^2 + \lambda u^2)dx, \quad u \in V(\Omega), \tag{1.2}$$

where

$$V(\Omega) = \{u \in W^{1,2}(\Omega) | \int_\Omega |u|^{p+1}dx = 1\}. \tag{1.3}$$

It is easy to check that positive critical points of $E_\lambda(u)$ correspond to solutions of $(1.1)_\lambda$. In fact, if u is a critical point of $E_\lambda(u)$ in $V(\Omega)$ and u is positive in Ω, then $w = [E_\lambda(u)]^{\frac{1}{p-1}}u$ is a solution of $(1.1)_\lambda$.

A natural idea of finding critical points of $E_\lambda(u)$ would be to minimize the functional in a suitable space. This turns out to be the case for single-peaked solutions (e.g. [AM], [Wx], [Wz1-2]) as one minimizes the functional $E_\lambda(u)$ in $V(\Omega)$. This idea also can be carried out for double-peaked solutions ([Wz4]) when one minimizes $E_\lambda(u)$ in $V_e(\Omega) := \{u \in V(\Omega) \mid u(-x) = u(x)\}$ when Ω is an antipodal invariant domain. However, in order to find more nonradial solutions (basically we shall distinguish solutions by the number of peaks of the solutions), when we work in some more general symmetric subspaces this global minimization method does not seem to work as well as in the above mentioned situations. This will be demonstrated in Section 4 (see Proposition 4.1 and Remark 4.1).

In order to get multi-peaked solutions in a class of symmetric domains including ellipsoid domains, we have presented an approach to the problem in [Wz6]. The idea is to seek a "local minimum" of the energy functional instead of a global minimum. By carefully constructing some special subsets in $V(\Omega)$ we are able to target the solutions we want to obtain. Our new approach has several advantages. We can locate the peaks of the solutions from the construction that we use. We can get multiplicity results of multi-peaked solutions for $(1.1)_\lambda$ by distincting the location of the peaks. Also the procedure of proving multi-peakedness is simpler than that we used in [Wz4] for proving double-peakedness. Moreover, we do not need any dimensional restrictions like in [Wz4] for double-peaked solutions.

In Section 2, we shall concentrate on the critical exponent case, presenting some results from [Wz6] with extensions. Then we shall prove in Section 3 that the same methods would also apply to the subcritical exponent problem to construct multi-peaked solutions. Finally we close up the paper in Section 4 by making several remarks about symmetry properties of some minimization problems related to $(1.1)_\lambda$.

2. The critical exponent case. From now on we shall assume that $\Omega = B_1(0) := B$, the unit ball in $\mathbf{R}^N$ centered at 0. We are interested in nonradial solutions of

$$(2.1)_\lambda \begin{cases} -\Delta u + \lambda u = u^p, u > 0 & \text{in } B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B \end{cases}$$

where p is, in this section, the critical Sobolev exponent, i.e. $p = \frac{N+2}{N-2}$. We assume $N \geq 3$ in this section. We shall seek nonradial symmetric solutions of $(2.1)_\lambda$.

First we introduce some symmetries in $\mathbf{R}^N$. We write

$$\mathbf{R}^N = \mathbf{R}^2 \times \mathbf{R}^{N-2}. \tag{2.2}$$

Let $k \geq 2$ be an integer and let Z_k be the cyclic group of order k. We let $G_1' = \{Id, T, T^2, \ldots, T^{k-1}\}$ be a representation of Z_k in $\mathbf{O}(2) \subset \mathbf{O}(N)$ (i.e. T is the generator of the action). We make the following assumption.

$(S1)$. $\mathrm{Fix}_{G_1'} = \{0\} \times \mathbf{R}^{N-2}$, and for any $x \in \mathbf{R}^N \setminus (\{0\} \times \mathbf{R}^{N-2})$, the G_1'-orbit of x contains exactly k points.

Then without loss of generality we may assume

$$T(x^1, x^2, x^3, \cdots, x^N) = T(z, x^3, \cdots, x^N) = (e^{i\frac{2\pi}{k}} z, x^3, \cdots, x^N) \tag{2.3}$$

where we have written (x^1, x^2) as $z \in \mathbf{C}$.

Let G_1'' be the representation of Z_2 in $\mathbf{O}(2) \subset \mathbf{O}(N)$ given by $G_1'' = \{Id, \mathcal{R}\}$ with

$$\mathcal{R}(x^1, x^2, x^3, \cdots, x^N) = (x^1, -x^2, x^3, \cdots, x^N) \tag{2.4}$$

i.e. the reflection with respect to the plane perpendicular to $(0, 1, 0, \cdots, 0)$. We define $G_1 := G_1' \times G_1''$. Finally let $G_2 := \mathbf{O}(N-2) \subset \mathbf{O}(N)$ such that $\mathrm{Fix}_{G_2} = \mathbf{R}^2 \times \{0\}$, and define a representation of a subgroup of $\mathbf{O}(N)$ by

$$G := G_1 \times G_2. \tag{2.5}$$

Remark 2.1. Under the assumption $(S1)$, for any $x \in \mathbf{R}^N$, the G-orbit of x contains at least k points, and indeed there are points in $\mathbf{R}^2 \times \{0\}$ whose orbits contain exactly k points.

THEOREM 2.1. *Let G be given as above satisfying $(S1)$. Then there exists $\lambda_k > 0$ such that for all $\lambda > \lambda_k$, $(2.1)_\lambda$ possesses a nonconstant solution u_λ satisfying the following.*
(i). u_λ is exactly $G-$invariant, i.e. for any $g \in \mathbf{O}(N)$, $u_\lambda(gx) = u_\lambda(x)$ for any $x \in B$ if and only if $g \in G$.
(ii). Let S be the best Sobolev constant, then

$$\lim_{\lambda \to \infty} E_\lambda \left(\frac{u_\lambda}{||u_\lambda||_{L^{p+1}(B)}} \right) = k^{\frac{2}{N}} 2^{-\frac{2}{N}} S. \tag{2.6}$$

(iii). u_λ is k-peaked on ∂B in the sense that it attains its maximum over $\overline{B}$ at exactly k points in $\overline{B}$ which all lie on $\partial B \cap (\mathbf{R}^2 \times \{0\})$. In fact, these k points are given by $T^j P$ for $j = 1, ..., k$ with the G-orbit of P containing exactly k points.
(iv).

$$\lim_{\lambda \to \infty} ||\nabla u_\lambda - \sum_{j=1}^{k} \nabla U_{\epsilon_\lambda, T^j P}||_{L^2(B)} = 0, \tag{2.7}$$

where P is given in (iii), $\epsilon_\lambda = [u_\lambda(P)]^{-\frac{p-1}{2}}$, and $U_{\epsilon,P}(x) = \epsilon^{-\frac{N-2}{2}} U(\frac{x-y}{\epsilon})$ with $U(x) = [\frac{N(N-2)}{N(N-2)+|x|^2}]^{\frac{N-2}{2}}$ being the positive solution of

$$-\Delta u = u^p \quad in\ \mathbf{R}^N. \tag{2.8}$$

R e m a r k 2.2. When $k = 2$ we may use a real representation of Z_2 as G_1, given by $G_1 := \{Id, T\}$ with

$$T(x^1, x^2, \cdots, x^N) = (-x^1, x^2, \cdots, x^N).$$

Define $G_2 := \mathbf{O}(N-1)$ and $G := G_1 \times G_2$. Then we may have a similar result to Theorem 2.1 above concluding that for λ large there exists a nonconstant solution u_λ and that this solution is exactly $G-$invariant and $2-$peaked on $\partial B \cap (\mathbf{R}^1 \times \{0\})$ which contains exact two points. The solutions also satisfy (ii) and (iv) with obvious modifications.

R e m a r k 2.3. In [Wz6] more general symmetric domains including ellipsoid domains have been treated. But here we give more information on the symmetry properties of the solutions by proving the exact symmetry property for the solutions.

R e m a r k 2.4. The problem of studying symmetry properties of solutions for nonlinear Neumann problems is a very interesting one, but yet still widely open. Here we are able to construct some special symmetric solutions for $(2.1)_\lambda$ and more importantly we are able to prove some exact symmetry properties for these solutions.

In order to prove Theorem 2.1, let us first define

$$W_G^{1,2}(B) = \{u \in W^{1,2}(B) \ | \ u(gx) = u(x), \ \ a.e. \text{ in } \ \ B, \ \ \forall g \in G \ \},$$

and

$$V_G(B) = V(B) \cap W_G^{1,2}(B) = \{u \in W_G^{1,2}(B) \ | \ ||u||_{p+1} = 1\}. \tag{2.9}$$

By the symmetric criticality principle ([P]), any critical points of $E_\lambda(u)$ in $V_G(B)$ are critical points of $E_\lambda(u)$ in $V(B)$. Though $E_\lambda(u)$ is bounded from below over $V_G(B)$ the infimum of $E_\lambda(u)$ over $V_G(B)$ may not be achieved or may not give rise to $k-$peaked solutions (see Remark 4.1 in Section 4). The idea in [Wz6] of finding critical points of $E_\lambda(u)$ in $V_G(B)$ is to look for *local minima* in some special subsets of $V_G(B)$ where we believe the desired solutions may live. Let us introduce an auxiliary function here, for any $u \in V_G(B)$,

$$\gamma(u) = \int_B |u|^{p+1}|Px|dx, \tag{2.10}$$

where $P : \mathbf{R}^N \to \mathbf{R}^2$ is the linear projection, i.e. $Px = (x^1, x^2, 0, \cdots, 0)$, and $|Px| = \sqrt{|x^1|^2 + |x^2|^2}$ is the Euclidean norm of Px. Then it is easy to check that $\gamma(u)$ is a continuous function of u in $V_G(B)$ and that $\gamma(u) \in (0, 1)$. Next, we introduce a family of special subsets of $V_G(B)$. We define for any $\delta \in (0, 1)$ the following open sets in $V_G(B)$

$$K_\delta := \{u \in V_G(B) \ | \ \gamma(u) > \delta\}, \tag{2.11}$$

and consider the infimum of $E_\lambda(u)$ in these sets,

$$c_{\lambda,\delta} := \inf_{u \in K_\delta} E_\lambda(u). \tag{2.12}$$

The strategy of proving Theorem 2.1 now is to show that $c_{\lambda,\delta}$ is attained by *an interior point* of K_δ for some suitable δ, and to prove that the local minimizers have the desired properties. Before giving a sketch of the proof of Theorem 2.1, we need a few technical results from [Wz6]. Let

$$\delta_k := \frac{k}{k+1}. \tag{2.13}$$

The following estimates are crucial in our proof ([Wz6]).

LEMMA 2.1. *For any $\delta \in (0,1)$, $c_{\lambda,\delta} < k^{\frac{2}{N}} 2^{-\frac{2}{N}} S$.*

LEMMA 2.2. *Let $\delta \in (\delta_k, 1)$ be fixed. Let $u_n \in K_\delta$ and $\lambda_n \to \infty$ be such that $E_{\lambda_n}(u_n) \le k^{\frac{2}{N}} 2^{-\frac{2}{N}} S$. Then*
(a) $\lim_{n\to\infty} E_{\lambda_n}(u_n) = k^{\frac{2}{N}} 2^{-\frac{2}{N}} S$.
(b) There exist $y_n \in \partial B \cap (\mathbf{R}^2 \times \{0\})$, such that $\forall \epsilon > 0, \exists R > 0$

$$\lim_{n\to\infty} \int_{B_{\frac{R}{\sqrt{\lambda_n}}}(y_n)\cap B} |u_n|^{p+1} dx \ge \frac{1}{k} - \epsilon. \tag{2.14}$$

(c) $\lim_{n\to\infty} \gamma(u_n) = 1$.

LEMMA 2.3. *Let $\lambda > 0$ be fixed. Let $u_n \ge 0$ be a $(P-S)_c$ sequence for E_λ in $V_G(B)$ (i.e. as $n \to \infty$, $E_\lambda(u_n) \to c$ and $E'_\lambda(u_n) \to 0$). Then there exist integers $l_1 \ge 0$, $l_2 \ge 0$, a solution $u_0 \ge 0$ of*

$$\begin{cases} -\Delta u + \lambda u = cu^p, & \text{in } B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B, \end{cases} \tag{2.15}$$

and $x_{n,j} \in B, \epsilon_{n,j} \to 0$, as $n \to \infty$, for $j = 1, 2, \dots, l_1 + l_2$, such that

$$\left\| u_n - u_0 - c^{\frac{1}{1-p}} \sum_{j=1}^{l_1+l_2} U_{\epsilon_{n,j}, x_{n,j}} \right\|_{W^{1,2}(B)} \to 0, \quad \text{as } n \to \infty, \tag{2.16}$$

and

$$c = \int_B (|\nabla u_0|^2 + \lambda u_0^2) dx + l_1 c^{\frac{2}{1-p}} \int_{\mathbf{R}^N} |\nabla U|^2 dx + l_2 c^{\frac{2}{1-p}} \int_{\mathbf{R}^N_+} |\nabla U|^2 dx. \tag{2.17}$$

LEMMA 2.4. *Let $\delta \in (\delta_k, 1)$ be fixed. Then there exist $\lambda_0 > 0$, $\beta_0 > 0$, such that for all $\lambda \ge \lambda_0$, if $\{u_n^\lambda\}$ is a minimizing sequence of $E_\lambda(u)$ in K_δ,*

$$\lim_{n\to\infty} \gamma(u_n^\lambda) \ge \delta + \beta_0. \tag{2.18}$$

The following lemma is essentially from [Wz3].

LEMMA 2.5. *Let $T(Z_k)$ be a representation of Z_k in $\mathbf{O}(N)$ such that for any $x \in \mathbf{R}^N \setminus \mathrm{Fix}_{T(Z_k)}$ the orbit of x contains exactly k points. Let $\lambda_n > 0$ and $u_n \in W^{1,2}(B)$ be such that $u_n(x)$ are solutions of $(2.1)_{\lambda_n}$ and that as $n \to \infty$, $\lambda_n \to \infty$ and*

$$\lim_{n\to\infty} E_{\lambda_n}\left(\frac{u_n}{||u_n||_{p+1}}\right) = k^{\frac{2}{N}} 2^{-\frac{2}{N}} S. \tag{2.19}$$

Assume that u_n are invariant functions with respect to $T(Z_k)$. Assume that there exist $\epsilon_0 > 0$ and $y_n \in \partial B$ such that $\mathrm{dist}(y_n, \mathrm{Fix}_{T(Z_k)}) \ge \epsilon_0 > 0$ as $n \to \infty$ and such that for *any $\epsilon > 0$ there exists $R > 0$ with*

$$\frac{\int_{B_{\frac{R}{\sqrt{\lambda_n}}}(y_n)\cap B} |u_n|^{p+1}dx}{||u_n||_{p+1}^{p+1}} \geq \frac{1}{k} - \epsilon. \tag{2.20}$$

Then for n large, u_n attains its maximum over $\overline{B}$ only at k points on ∂B: P_n, TP_n, ..., $T^{k-1}P_n$ for some $P_n \in \partial B$. Moreover, letting $\epsilon_n = [u_n(P_n)]^{-\frac{p-1}{2}}$, we have

$$\lim_{n\to\infty} ||\nabla u_n - \sum_{j=1}^{k} \nabla U_{\epsilon_n, T^j P_n}||_{L^2(B)} = 0. \tag{2.21}$$

Sketch of the proof Theorem 2.1. We fix $\delta \in (\delta_k, 1)$ throughout the proof. For any $\lambda \geq \lambda_0$ (given in Lemma 2.4), we consider a minimizing sequence $\{u_n^\lambda\}$ for $c_{\lambda,\delta}$, i.e.

$$c_{\lambda,\delta} \leq E_\lambda(u_n^\lambda) \leq c_{\lambda,\delta} + o(1), \text{as } n \to \infty.$$

For simplicity, we omit λ and simply write u_n^λ as u_n. Also we may assume $u_n \geq 0$.

Note first that by Lemma 2.4, u_n does not approach to the boundary of K_δ. By Ekeland's variational principle (e.g. [MW]), we may assume that $\lim_{n\to\infty} E'_\lambda(u_n) \to 0$ as $n \to \infty$, i.e. $\{u_n\}$ is a $(P-S)_{c_{\lambda,\delta}}$ sequence. If u_n has a convergent subsequence (still denoted by u_n) such that $u_n \to u_0 \in K_\delta$ as $n \to \infty$, then u_0 gives rise to a solution of $(2.1)_\lambda$ by rescaling. Thus it suffices to show that u_n has a convergent subsequence. To that end, first by Lemma 2.3, we get integers $l_1 \geq 0, l_2 \geq 0$, a nonnegative solution u_0 of (2.15) with c being replaced by $c_{\lambda,\delta}$, and $x_{n,j} \in B, \epsilon_{n,j} \to 0$, as $n \to \infty$, for $j = 1, 2, ..., l_1 + l_2$, such that

$$\left|\left| u_n - u_0 - c_{\lambda,\delta}^{\frac{1}{1-p}} \sum_{j=1}^{l_1+l_2} U_{\epsilon_{n,j}, x_{n,j}} \right|\right|_{W^{1,2}(B)} \to 0, \quad \text{as } n \to \infty, \tag{2.22}$$

and

$$c_{\lambda,\delta} = \int_B (|\nabla u_0|^2 + \lambda u_0^2)dx + l_1 c_{\lambda,\delta}^{\frac{2}{1-p}} \int_{\mathbf{R}^N} |\nabla U|^2 dx + l_2 c_{\lambda,\delta}^{\frac{2}{1-p}} \int_{\mathbf{R}^N_+} |\nabla U|^2 dx. \tag{2.23}$$

We distinguish two cases:

(a). There exists some j_0, $1 \leq j_0 \leq l_1 + l_2$, such that

$$\limsup_{n\to\infty} \frac{\text{dist}(x_{n,j_0}, \{0\} \times \mathbf{R}^{N-2})}{\epsilon_{n,j_0}} = \infty, \quad \text{as} \quad n \to \infty; \tag{2.24}$$

(b). For all $j = 1, ..., l_1 + l_2$,

$$\limsup_{n\to\infty} \frac{\text{dist}(x_{n,j}, \{0\} \times \mathbf{R}^{N-2})}{\epsilon_{n,j}} < \infty, \quad \text{as} \quad n \to \infty. \tag{2.25}$$

Case (a). Since $u_n \in V_G(B)$, from the symmetry and the proof of Lemma 2.3 (e.g. [S]), we can conclude that Tx_{n,j_0}, $T^2 x_{n,j_0}$,..., $T^{k-1}x_{n,j_0}$ are all among the $x_{n,j}$'s. This implies either $l_1 \geq k$ or $l_2 \geq k$. Then from (2.23)

$$c_{\lambda,\delta} \geq k c_{\lambda,\delta}^{\frac{2}{1-p}} \int_{\mathbf{R}^N_+} |\nabla U|^2 dx,$$

and $c_{\lambda,\delta} \geq k^{\frac{2}{N}} 2^{-\frac{2}{N}} S$, a contradiction to Lemma 2.1. So case (a) is impossible.

Case (b). With a lengthy but straight forward calculation (see [Wz6]), we may conclude that $u_0 \neq 0$ and

$$\lim_{n\to\infty} \gamma(u_n) \le ||u_0||_{p+1}^{p+1} \gamma\left(\frac{u_0}{||u_0||_{p+1}}\right). \tag{2.26}$$

Next, by $u_n \rightharpoonup u_0$ as $n \to \infty$ and a result in [BL],

$$1 = \int_B |u_n|^{p+1} dx = \int_B |u_n - u_0|^{p+1} dx + \int_B |u_0|^{p+1} dx + o(1), \quad \text{as } n \to \infty.$$

This implies $||u_0||_{p+1} \le 1$. Then from (2.26) and Lemma 2.4, $\gamma(\frac{u_0}{||u_0||_{p+1}}) \ge \delta + \beta_0 > \delta$, i.e. $\frac{u_0}{||u_0||_{p+1}} \in K_\delta$.

Finally, we assert $l_1 = l_2 = 0$. By Lemma 2.1, $c_{\lambda,\delta} < k^{\frac{2}{N}} 2^{-\frac{2}{N}} S$. If $l_1 + l_2 \ge 1$, by (2.23) and the definition of $c_{\lambda,\delta}$,

$$\begin{aligned} c_{\lambda,\delta} &= \int_B (|\nabla u_0|^2 + \lambda u_0^2) dx + l_1 c_{\lambda,\delta}^{\frac{2}{1-p}} S^{\frac{N}{2}} + l_2 c_{\lambda,\delta}^{\frac{2}{1-p}} \frac{S^{\frac{N}{2}}}{2} \\ &\ge ||u_0||_{p+1}^2 E_\lambda(\frac{u_0}{||u_0||_{p+1}}) + c_{\lambda,\delta}^{\frac{2}{1-p}} \frac{S^{\frac{N}{2}}}{2} \\ &\ge ||u_0||_{p+1}^2 c_{\lambda,\delta} + \frac{1}{k} k^{\frac{2}{N}} 2^{-\frac{2}{N}} S. \end{aligned}$$

We get

$$||u_0||_{p+1}^2 \le \frac{k-1}{k}.$$

By (2.26) again,

$$\frac{k}{k+1} = \delta_k < \delta \le \lim_{n\to\infty} \gamma(u_n) \le ||u_0||_{p+1}^{p+1} \gamma(\frac{u_0}{||u_0||_{p+1}}) \le \frac{k-1}{k} \gamma(\frac{u_0}{||u_0||_{p+1}})$$

and this implies

$$\gamma(\frac{u_0}{||u_0||_{p+1}}) \ge \frac{k^2}{k^2-1} > 1,$$

a contradiction. So $l_1 = l_2 = 0$, and $u_n \to u_0$ in $W^{1,2}(B)$.

So we conclude that for $\lambda \ge \lambda_0$, $c_{\lambda,\delta}$ is achieved by an interior point w_λ of K_δ, therefore a critical point of $E_\lambda(u)$ in $V_G(B)$. By rescaling w_λ we get a solution u_λ of $(2.1)_\lambda$. By comparing energies with constant solutions, we see that for λ large, u_λ is a nonconstant solution. Therefore we have proved that there is a nonconstant solution of $(2.1)_\lambda$ which is at least $G-$invariant and satisfies (iii) using Lemma 2.1 and the assertion (a) of Lemma 2.2. By Lemma 2.5 we may prove (ii) and (iv), and we refer to [Wz6] for details.

Finally, to prove that u_λ is exactly $G-$invariant, let us observe first that u_λ is at least $G-$invariant, i.e. $\forall g \in G$, $u_\lambda(gx) = u_\lambda(x)$. Because we know that u_λ has exactly k maximum points over $\overline{B}$ which are achieved at a $G-$orbit: $\{T^j P | \ j = 1, ..., k\}$ with P given in the assertion (iii), any $g \in G$ which is such that $u_\lambda(gx) = u_\lambda(x)$ for $\forall x \in B$ must satisfy $g \in \mathbf{O(2)} \times \mathbf{O}(N-2)$, i.e. g has $\mathbf{R}^2 \times \{0\}$ as an invariant subspace. Let us write $g = g_1 \times g_2$ with $g_1 \in \mathbf{O}(2)$ and $g_2 \in G_2 = \mathbf{O}(N-2)$. We want to prove that $g_1 \in G_1$. If k is odd the reflection with respect to the x_1-axis can not be a part of g_1 because otherwise the orbit of P contains $2k$ points. If k is even, the reflection with respect to the x_1-axis belongs to G_1. Concerning the rotation part, if g_1 contains a rotation that does not belong to $G_1^{'}$ we get that the orbit of P would contain more than k points again and that u_λ would have more than k maxima. Thus $g_1 \in G_1$. This finishes the proof of Theorem 2.1. $\square$

3. The subcritical exponent case. In this section, we shall show that the methods used in Section 2 apply also to the subcritical exponent problems though some necessary technical modifications have to be made. Recall that we are interested in the existence of nonradial solutions of

$$(3.1)_\lambda \begin{cases} -\Delta u + \lambda u = u^p, u > 0 & \text{in } B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B \end{cases}$$

where p satisfies $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $1 < p < \infty$ if $N = 2$.

While in Section 2, $U_{\epsilon,y}(x)$ (the solution of (2.8)) plays an important role in the estimates there, the ground state solution of the following problem plays the role for the subcritical exponent problem

$$-\Delta w + w = w^p, w > 0, \lim_{|x|\to\infty} w(x) = 0, \text{ in } \mathbf{R}^N. \tag{3.2}$$

By the results in [CL] and [KZ], up to translations there is a unique solution to this problem and we shall use w to denote the solution satisfying $w(0) = \max_{x\in\mathbf{R}^N} w(x)$. By the result in [GNN], w is radially symmetric and there exist constants $C_0 > 0$ and $\mu > 0$ such that

$$|w(x)| + |Dw(x)| \leq C_0 e^{-\mu|x|} \text{ for all } x \in \mathbf{R}^N. \tag{3.3}$$

The notations E_λ, G_1, G_2, G, and δ_k will be used in this section with the same meanings as in the preceding section. And assuming $1 < p < \frac{N+2}{N-2}$, we may similarly define $V_G(B), \gamma(u), K_\delta$ and $c_{\lambda,\delta}$ as before. Then we have the following lemmas.

Lemma 3.1. *For any $\delta \in (0,1)$,*

$$\lim_{\lambda\to\infty} \lambda^{-\alpha_0} c_{\lambda,\delta} \leq k^{\frac{p-1}{p+1}} 2^{-\frac{p-1}{p+1}} \overline{m}, \tag{3.4}$$

where $\alpha_0 := \frac{2N-(p+1)(N-2)}{2(p+1)} > 0$, and

$$\overline{m} := \frac{\int_{\mathbf{R}^N} (|\nabla w|^2 + w^2)dx}{||w||^2_{L^{p+1}(\mathbf{R}^N)}} = \inf_{u\in W^{1,2}(\mathbf{R}^N)} \frac{\int_{\mathbf{R}^N} (|\nabla u|^2 + u^2)dx}{||u||^2_{L^{p+1}(\mathbf{R}^N)}}. \tag{3.5}$$

Sketch of the proof of Lemma 3.1. Taking $P_0 \in \partial B \cap (\mathbf{R}^2 \times \{0\})$ such that the G-orbit of P_0 contains exactly k points, and defining

$$w_\lambda(x) = \sum_{j=1}^{k} \lambda^{\frac{N}{2(p+1)}} w(\lambda^{\frac{1}{2}}(x - T^j P_0))$$

we get $\frac{w_\lambda}{||w_\lambda||_{L^{p+1}(B)}} \in V_G(B)$. Then direct computation shows that there exists $\sigma_0 > 0$ such that as $\lambda \to \infty$

$$||\nabla w_\lambda||^2_{L^2(B)} = \lambda^{\alpha_0} \left(\int_{\mathbf{R}^N_+} |\nabla w|^2 dx + o(1) \right) + O(\lambda^{\frac{N+(p+1)}{p+1}} e^{-\mu\sigma_0\lambda}) \tag{3.6}$$

$$\lambda||w_\lambda||^2_{L^2(B)} = \lambda^{\alpha_0}\left(\int_{\mathbf{R}^N_+} w^2 dx + o(1)\right) + O(\lambda^{\frac{N+(p+1)}{p+1}} e^{-\mu\sigma_0\lambda}) \tag{3.7}$$

and

$$||w_\lambda||^{p+1}_{L^{p+1}(B)} = \left(\int_{\mathbf{R}^N_+} w^{p+1} dx\right) + o(1) + O(\lambda^{\frac{N}{2}} e^{-\frac{\mu\sigma_0(p+1)\lambda}{2}}). \tag{3.8}$$

Then (3.4) follows from (3.5), (3.6), (3.7) and (3.8). □

LEMMA 3.2. *Let $\delta \in (\delta_k, 1)$ be fixed. Let $u_n \in K_\delta$ and $\lambda_n \to \infty$ be such that $\lim_{n\to\infty} E_{\lambda_n}(u_n) \le k^{\frac{p-1}{p+1}} 2^{-\frac{p-1}{p+1}} \overline{m}$. Then there exists a subsequence (still denoted by u_n) satisfying*

(*a*) $\lim_{n\to\infty} \lambda_n^{-\alpha_0} E_{\lambda_n}(u_n) = k^{\frac{p-1}{p+1}} 2^{-\frac{p-1}{p+1}} \overline{m}$.

(*b*) *There exist $y_n \in \partial B \cap (\mathbf{R}^2 \times \{0\})$, such that $\forall \epsilon > 0, \exists R > 0$*

$$\lim_{n\to\infty} \int_{B_{\frac{R}{\sqrt{\lambda_n}}}(y_n)\cap B} |u_n|^{p+1} dx \ge \frac{1}{k} - \epsilon. \tag{3.9}$$

(*c*) $\lim_{n\to\infty} \gamma(u_n) = 1$.

The proof of Lemma 3.2 is rather similar to the proof of Lemma 2.2 though obvious changes need to be made. We refer to [Wz6] for details.

With the help of Lemma 3.2, we have

LEMMA 3.3. *Let $\delta \in (\delta_k, 1)$ be fixed. Then there exist $\lambda_0 > 0$, $\beta_0 > 0$, such that for all $\lambda \ge \lambda_0$, if $\{u_n^\lambda\}$ is a minimizing sequence of $E_\lambda(u)$ in K_δ,*

$$\lim_{n\to\infty} \gamma(u_n^\lambda) \ge \delta + \beta_0. \tag{3.10}$$

Proof of Lemma 3.3. If the conclusion is not true, there exist $\lambda_n \to \infty$, $\beta_n \to 0$, and minimizing sequences for $E_{\lambda_n}(u)$ in K_δ: $\{u_j^{\lambda_n}\}$, such that

$$\lim_{j\to\infty} E_{\lambda_n}(u_j^{\lambda_n}) = c_{\lambda_n,\delta},$$

and

$$\lim_{j\to\infty} \gamma(u_j^{\lambda_n}) \le \delta + \beta_n.$$

By Lemma 3.1, $\lim_{\lambda\to\infty} \lambda^{-\alpha_0} c_{\lambda,\delta} \le k^{\frac{p-1}{p+1}} 2^{-\frac{p-1}{p+1}} \overline{m}$. Then for each n we can find j_n and a_n with $j_n \to \infty$ and $a_n \to \infty$ such that

$$\lambda_n^{-\alpha_0} E_{\lambda_n}(u_{j_n}^{\lambda_n}) \le k^{\frac{p-1}{p+1}} 2^{-\frac{p-1}{p+1}} \overline{m} + \frac{1}{a_n},$$

$$\gamma(u_{j_n}^{\lambda_n}) \le \delta + 2\beta_n.$$

Calling $w_n = u_{j_n}^{\lambda_n}$ we get a sequence satisfying the condition of Lemma 3.2. Then we get

$$\lim_{n\to\infty} \gamma(w_n) = 1 > \delta,$$

a contradiction. Lemma 3.3 is proved. □

THEOREM 3.1. *Let G be given as above satisfying $(S1)$. Then there exists $\lambda_k > 0$ such that for all $\lambda > \lambda_k$, $(3.1)_\lambda$ possesses a nonconstant solution u_λ satisfying the following.*
(i). u_λ is exactly $G-$invariant, i.e. for any $g \in \mathbf{O}(N)$, $u_\lambda(gx) = u_\lambda(x)$ for any $x \in B$ if and only if $g \in G$.
(ii).

$$\lim_{\lambda\to\infty} \lambda^{-\alpha_0} E_\lambda \left(\frac{u_\lambda}{||u_\lambda||_{L^{p+1}(B)}} \right) = k^{\frac{p-1}{p+1}} 2^{-\frac{p-1}{p+1}} \overline{m}. \tag{3.11}$$

(iii). u_λ is k-peaked on ∂B in the sense that u_λ has exactly k local maxima over $\overline{B}$ which all lie on $\partial B \cap (\mathbf{R}^2 \times \{0\})$ and are given by $T^j P$ for $j = 1, ..., k$ with the G-orbit of P containing exactly k points.

Sketch of the proof of Theorem 3.1. Let $\delta \in (\delta_k, 1)$ and $\lambda > \lambda_k$ be fixed. Let $u_n \in K_\delta$ be a minimizing sequence for $c_{\lambda,\delta}$ in K_δ. By Lemma 3.3, $\lim_{n\to\infty} \gamma(u_n) \geq \delta + \beta_0$, i.e. $\{u_n\}$ does not approach to the boundary of K_δ. By Ekeland's variational principle ([MW]), we may assume $E'_\lambda(u_n) \to 0$ as $n \to \infty$. Since $\{u_n\}$ are uniformly bounded, we may assume that u_n weakly converges to $u \in W^{1,2}(B)$. Then $u_n \to u$ in $L^{p+1}(B)$ by the Sobolev Embedding Theorem. Then $||u||_{L^{p+1}(B)} = 1$, i.e. $u \in V_G(B)$. Since $E_\lambda(u)$ is weakly lower semicontinuous, $c_{\lambda,\delta}$ is achieved at an interior point of K_δ. So we obtain a nonconstant solution u_λ for $(3.1)_\lambda$ for λ large. The assertion (a) of Lemma 3.2 proves (ii) of Theorem 3.1. The last part of the proof for Theorem 2.1 proves that u_λ has exact G-symmetry. To prove (iii) we use the same idea in the proof of the assertion (iii) of Theorem 2.1 and some arguments in [NT1] and [Wz1] (note that here we prove that u_λ possesses exactly k *local maxima* over $\overline{B}$, while in Theorem 2.1 the statement is that u_λ possesses k global maxima over $\overline{B}$). □

4. Further remarks. Note that we have essentially used "local minimization" arguments in Section 2 and Section 3. One would ask what happens with the "global minimizations". We give a few remarks here and complete answer will be reported elsewhere.

For the critical exponent problem we believe that the infimum of $E_\lambda(u)$ over $V_G(B)$ is not achieved for $k \geq 3$. Here we just give a weaker result in this regard, which we proved in [Wz6].

PROPOSITION 4.1. *Let $k = 4$. Define*

$$m_\lambda := \inf_{u \in V_G(B)} E_\lambda(u). \tag{4.1}$$

Assume m_λ is achieved at u_λ. Then for λ large, u_λ is not four-peaked on ∂B (see (iii) in Theorem 2.1).

Remark 4.1. For the subcritical problems, we may state a similar result to the above proposition, i.e. the global minimizer of $E_\lambda(u)$ in $V_G(B)$ does not give a multi-peaked solution for our problem $(3.1)_\lambda$ when λ is large. However, in this case m_λ is always achieved because we do have compactness for the subcritical exponent problems. We tend to believe the minimizers in this case are radial functions. This and the proposition above indicate that in general the global minimization would not yield multi-peaked solutions with boundary spike-layers.

Remark 4.2. In regard to condition $(S1)$, in general a Z_k action may have different orbit types. Modifying our arguments slightly (mainly in the proof of Lemma 2.2 and 3.2), we may get the existence of a k'-peaked solution u_λ for λ large, where $k' = \min\{$ the number of points in $G_1(x) \mid x \in (\mathbf{R}^2 \times \{0\}) \setminus \{0\}\}$.

Remark 4.3. In [Wz6] more general symmetric domains have been treated by putting some geometric conditions on the boundary of the domain.

Remark 4.4. All solutions we obtain so far share one common feature: the orbits of peaks for the solutions are planer. It would be interesting to see solutions with other type of peaks. Some results will be reported in [MSW].

Remark 4.5. We should mention that radial solutions have been studied in [LN], [N], [AY] and [BKP] when the domain is a standard ball domain. Our results imply that for both the subcritical and the critical exponent problems, for each $k \geq 1$ there exists $\lambda_k > 0$, such that for all $\lambda \geq \lambda_k$ the problem has at least k nonradial solutions which are not rotationally equivalent. It should be interesting to study the symmetry properties of all positive solutions for our problem. Here we prove the existence of solutions with prescribed exact symmetry.

Remark 4.6. Problem $(1.1)_\lambda$ may be viewed as a prototype of pattern formation in mathematical biology and is related to the steady state problem for a chemotactic aggregation model by Keller and Segel([KS]). Our results indicate that some solutions tend to be more and more concentrated around a finite number of points on the boundary as the parameter λ tends to infinity.

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