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EXISTENCE OF PERIODIC SOLUTIONS FOR SEMILINEAR PARABOLIC EQUATIONS

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Abstract. In this paper, we are concerned with the semilinear parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = g(t,x,u) & \text{if } (t,x) \in \mathbf{R}_+ \times \Omega \\ u = 0 & \text{if } (t,x) \in \mathbf{R}_+ \times \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $g : \mathbf{R}_+ \times \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ is T-periodic with respect to the first variable. The existence and the multiplicity of T-periodic solutions for this problem are shown when $\frac{g(t,x,\xi)}{\xi}$ lies between two higher eigenvalues of $-\Delta$ in Ω with the Dirichlet boundary condition as $\xi \to \pm\infty$.

1. Introduction. Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and $g \in C^{1,\alpha}(\mathbf{R}_+ \times \bar{\Omega} \times \mathbf{R})$ with $\alpha > 0$ is T-periodic with respect to the first variable. In this paper, we are concerned with unstable T-periodic solutions for the semilinear parabolic equation

$$(P) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = g(t,x,u), & (t,x) \in \mathbf{R}_+ \times \Omega \\ \quad u(t,x) = 0, & (t,x) \in \mathbf{R}_+ \times \partial\Omega. \end{cases}$$

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Many authors have studied the existence of periodic solutions not only for the problem (P) but also for a more general problem of the form

$$(AP) \qquad \frac{du}{dt} + Au = F(t,u),$$

where A is an m-accretive operator (linear or nonlinear) on a Banach space X and $F : \mathbf{R}_+ \times X \to X$ is a continuous mapping which is T-periodic with respect to the first valiable. The existence and multiplicity of periodic solutions for problem (P) is established by Amann [2] The abstract problem (AP) is studied in [7], [11] and [12].

For the existence of periodic solutions, it is usually assumed that the operator $A - F$ satisfies coercivity conditions. In the case of problem (P), the operator $-\Delta - g(*)$ is coercive if

$$\limsup_{|\xi|\to\infty} \sup\{| \, g(t,x,\xi)/\xi \, | : (t,x) \in [0,T] \times \Omega\} < \lambda_1.$$

Here λ_1 is the first eigenvalue of the Laplacian on Ω with Dirichlet boundary condition.

Our purpose in this paper is to consider the existence and multiplicity of T-periodic solutions for (P) when $\limsup_{|\xi|\to\infty} \frac{g(t,x,\xi)}{\xi}$ lies between two higher eigenvalues of the Laplacian on Ω with Dirichlet boundary condition. We also show the instability of T-periodic solutions for (P). For the stability and instability of periodic solutions for (P), we refer to Alikakos, Hess and Matano [1], Hess [6], Hirano [9] and Hirsch [10].

2. Case of a general nonlinearity $g(t,x,\xi)$. Throughout the rest of this paper, we fix a positive number T. Let $|\cdot|$ and $\|\cdot\|$ be the norms of $L^2(\Omega)$ and $L^2(0,T;L^2(\Omega))$, respectively. The inner products of $L^2(\Omega)$ and $L^2(0,T;L^2(\Omega))$ are denoted by $\langle\cdot,\cdot\rangle$ and $\ll \cdot,\cdot \gg$, respectively. We call $u : \mathbf{R}_+ \to H_0^1(\Omega)$ a T-periodic solution for the problem (P) provided that $u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ satisfies

$$\frac{\partial u}{\partial t} - \Delta u = g(t,x,u)$$

in $L^2(\Omega)$ a.e. in $(0,T)$ and $u(t+T) = u(t)$ for all $t \in \mathbf{R}_+$. A T-periodic solution u is said to be stable if for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for each $v_0 \in L^2(\Omega)$ with $|v_0 - u(0)| < \delta(\epsilon)$, it holds that $|v(t) - u(t)| < \epsilon$ for all $t > 0$, where $v(t) : (0,\infty) \to H^2(\Omega) \cap H_0^1(\Omega)$ is the solution of the initial value problem

$$(I) \qquad \begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = g(t,x,v) & \text{in } (0,\infty) \times \Omega \\ v = 0 & \text{on } (0,\infty) \times \partial\Omega \\ v(0) = v_0 & \text{in } \Omega. \end{cases}$$

A T-periodic solution u is called unstable if u is not stable.

Let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \cdots$ be the sequence of the eigenvalues of the boundary value problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We denote by φ_i an eigenfunction corresponding to λ_i. Throughout this paper, it is supposed that $g \in C^{1,\alpha}(\mathbf{R}_+ \times \bar{\Omega} \times \mathbf{R})$ with $\alpha > 0$ is T-periodic with respect to the first variable. We assume the following conditions on g :

i) There exists $M > 0$ satisfying

$$\lambda_1 \leq \frac{\partial g}{\partial \xi}(t,x,\xi) \leq M \quad \text{for all } (t,x,\xi) \in \mathbf{R}_+ \times \bar{\Omega} \times \mathbf{R}$$

and

$$\frac{\partial g}{\partial \xi}(t,x,0) > \lambda_1 \quad \text{for some } (t,x) \in \mathbf{R}_+ \times \partial\Omega.$$

ii) There are $m \geq 1$ and $\alpha > 0$ such that

$$\lambda_m + \alpha \leq \liminf_{\xi \to \pm\infty} \frac{g(t,x,\xi)}{\xi} \leq \limsup_{\xi \to \pm\infty} \frac{g(t,x,\xi)}{\xi} \leq \lambda_{m+1} - \alpha$$

uniformly for $(t,x) \in \mathbf{R}_+ \times \Omega$.

The purpose of this section is to prove the following results.

THEOREM 1. *Under the hypotheses i) and ii), the problem (P) possesses an unstable T-periodic solution.*

In case that $g(t,x,0) = 0$ for all $(t,x) \in \mathbf{R}_+ \times \Omega$, $u \equiv 0$ is a T-periodic solution for (P). Then $u \equiv 0$ may be unstable. We can prove the existence of a nontrivial unstable T-periodic solution for (P) assuming the following condition :

iii) There are $2 \leq l \leq m$ and $\beta > 0$ such that

$$\lambda_{l-1} + \beta \leq \liminf_{\xi \to 0} \frac{g(t,x,\xi)}{\xi} \leq \limsup_{\xi \to 0} \frac{g(t,x,\xi)}{\xi} < \lambda_1 - \beta$$

uniformly for $(t,x) \in \mathbf{R}_+ \times \Omega$.

THEOREM 2. *Under the assumptions i) - iii), if $m - l + 1$ is an odd integer, then there exists a nontrivial unstable T-periodic solution for the problem (P). Moreover if there exists a nontrivial T-periodic solution u for (P) which is nondegenerate, i.e., 0 is not an eigenvalue of the problem*

$$(L) \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta v - g'(t,x,u)v = \mu v & \text{in } \mathbf{R}_+ \times \Omega \\ v = 0 & \text{on } \mathbf{R}_+ \times \partial\Omega \\ v(T) = v(0) & \text{in } \Omega, \end{cases}$$

then the problem (P) possesses at least two nontrivial unstable T-periodic solutions.

For simplicity, we write $H = L^2(0,T;L^2(\Omega))$ and $\frac{\partial g}{\partial \xi}(t,x,\xi) = g'(t,x,\xi)$. Let

$$L = \frac{\partial}{\partial t} - \Delta$$

with domain

$$D(L) = \{u \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) : u(0) = u(T)\}.$$

It is well known that there is a unique solution u_f for $Lu_f = f$ for any $f \in H$ and the operator K defined by $K = L^{-1}$ is a compact mapping from H into H. It is easy to see that u is a T-periodic solution for (P) if and only if u is a fixed point of $K \circ g$.

LEMMA 1. *Under the assumptions of Theorem 1, there is $R > 0$ such that*

$$\deg\,(I - K \circ g, B_R(0), 0) = (-1)^m,$$

where deg *means the Leray-Schauder degree and $B_R(u)$ is the closed ball in H with radius R centered at u.*

Proof. Let E_1 and E_2 be the closed subspaces of $L^2(\Omega)$ spanned by $\{\varphi_i : i \geq m+1\}$ and $\{\varphi_i : 1 \leq i \leq m+1\}$, respectively. We denote by P_i the projection from $L^2(\Omega)$ onto E_i for $i = 1, 2$. Since $L^2(0, T; E_1)$ and $L^2(0, T; E_2)$ are orthogonal in H and $H = L^2(0, T; E_1) \oplus L^2(0, T; E_2)$, P_i is canonically extended to the projection $\tilde{P}_i$ from H onto $L^2(0, T; E_i)$ for $i = 1, 2$. From the assumption ii), we obtain $C_1, C_2 > 0$ such that

$$\langle -\Delta v - g(t, x, v), P_1 v - P_2 v\rangle \geq C_1 |v|^2 - C_2$$

for each $v \in H^2(\Omega) \cap H_0^1(\Omega)$ and $t \in \mathbf{R}_+$ by the usual argument for semilinear elliptic equations with the Dirichlet boundary condition (see [8]). It follows that

$$\ll Lv - g(t, x, v), \tilde{P}_1 v - \tilde{P}_2 v \gg \geq C_1 \|v\|^2 - C_2 T$$

for all $v \in D(L)$. Therefore there exists $R > 0$ satisfying

$$\ll Lv - g(t, x, v), \tilde{P}_1 v - \tilde{P}_2 v \gg > 0$$

for any $v \in D(L)$ with $\|v\| \geq R$. Take $\lambda_m < a < \lambda_{m+1}$. We consider a homotopy of compact mappings defined by $\{K(sg + (1-s)aI) : 0 \leq s \leq 1\}$. For each $s \in [0, 1]$ and $v \in D(L)$ with $\|v\| = R$, we get

$$\ll Lv - \{sg(t, x, v) + (1-s)av\}, \tilde{P}_1 v - \tilde{P}_2 v \gg > 0.$$

This shows that

$$v - K(sg(t, x, v) + (1-s)av) \neq 0$$

for all $v \in H$ with $\|v\| = R$. By the homotopy invariance of the Leray-Schauder degree, we have

$$\deg\,(I - K \circ g, B_R(0), 0) = \deg\,(I - aK, B_R(0), 0).$$

Now, let $\nu_1, \cdots, \nu_n$ be the eigenvalues of aK with $\nu_i > 1$ for $1 \leq i \leq n$ and ψ_i be an eigenfunction corresponding to ν_i for $1 \leq i \leq n$. Then for $1 \leq i \leq n$ it holds that

$$L\psi_i = \frac{a}{\nu_i}\psi_i \quad \text{ for } 1 \leq i \leq n.$$

From $\nu_i > 1$, it follows that $\frac{a}{\nu_i} = \lambda_j$ for some j with $1 \leq j \leq m$. On the other hand, for each j with $1 \leq j \leq m$, $\frac{a}{\lambda_j}$ is an eigenvalue of aK with $\frac{a}{\lambda_j} > 1$. This implies $n = m$. Consequently, we see

$$\deg\,(I - aK, B_R(0), 0) = (-1)^m.$$

This completes the proof.

LEMMA 2. *Under the hypotheses of Theorem 2, there exists r with $0 < r < R$ satisfying*

$$deg\,(I - K \circ g, B_r(0), 0) = (-1)^{l-1}.$$

Proof. Let F_1 and F_2 be the closed subspaces of $L^2(\Omega)$ spanned by $\{\varphi_i : i \geq l\}$ and $\{\varphi_i : 1 \leq i \leq l-1\}$, respectively. For $i = 1, 2$, we denote by Q_i and $\tilde{Q}_i$ the projections from $L^2(\Omega)$ onto F_i and from H onto $L^2(0,T;F_i)$, respectively. By the assumptions ii) and iii), there are $d, \rho > 0$ such that

$$\langle -\Delta v - g(t,x,v), Q_1 v - Q_2 v\rangle \geq \rho |v|^2 \tag{1}$$

for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$ with $0 < |v| < d$ and $t \in \mathbf{R}_+$(see [8]). Take $\lambda_{l-1} < b < \lambda_l$. Then we can see that there exists $C_1 > 0$ such that for any $s \in [0,1]$, if $v \in D(L)$ satisfies

$$Lv - \{sg(t,x,v) + (1-s)bv\} = 0, \tag{2}$$

then

$$\sup_{t\in[0,T]} |v(t)| \leq C_1 \|v\|.$$

In fact, if v is a solution of (2) for some $0 \leq s \leq 1$, then we multiply (2) by v and integrate over $[s,t]$, where $|v(\tau)|$ attain its minimal at s. Then

$$|v(t)| \leq s\|g\|\|v\| + (1-s)b\|v\| + \|v\|^2/T.$$

for all $t \in [s,T]$.

It then follows from the periodicity of v that the existence of C_1 satisfying the inequality above. Put $r = \dfrac{d}{C_1}$. Suppose that

$$Lv_s - \{sg(t,x,v_s) + (1-s)bv_s\} = 0$$

for some $s \in [0,1]$ and $v_s \in D(L)$ with $0 < \|v_s\| \leq r$. Since

$$\sup_{t\in[0,T]} |v_s(t)| \leq d,$$

it follows from (1) that

$$\ll Lv_s - \{sg(t,x,v_s) + (1-s)bv_s\}, \tilde{Q}_1 v_s - \tilde{Q}_2 v_s \gg > 0.$$

This is a contradiction. Therefore we have

$$v - K(sg(t,x,v) + (1-s)bv) \neq 0$$

for each $v \in H$ with $0 < \|v\| \leq r$. According to the homotopy invariance of the Leray-Schauder degree, it follows that

$$\deg\,(I - K\circ g, B_r(0), 0) = \deg\,(I - bK, B_r(0), 0).$$

By the same method as in the proof of Lemma 1, we obtain

$$\deg\,(I - bK, B_r(0), 0) = (-1)^{l-1}.$$

This completes the proof.

We next consider a sufficient condition for a T-periodic solution of the problem (P) to be unstable. Let u be a T-periodic solution for (P). Denote by $S(t,s)$ the evolution operator for the following problem

$$(LI) \quad \begin{cases} \dfrac{dv}{dt} - \Delta v = g'(t,x,u)v & \text{in } (s,\infty)\times\Omega \\ \qquad v = 0 & \text{on } (s,\infty)\times\partial\Omega \\ \quad v(s) = z & \text{in } \Omega, \end{cases}$$

that is, $S(t,s)z = v(t)$. Then nonzero eigenvalues of $U(t)$ is independent of t (see [5]). It is known that if the periodic map $U(t) = S(t+T,t)$ for the above problem satisfies

$$\sigma(U(t)) \cap \{\mu : |\mu| > 1\} \neq \emptyset,$$

where $\sigma(A)$ means the set of eigenvalues of a linear operator A, then u is unstable (see Theorem 8.1.2 of [5]).

Putting $L_u = L + (M - g'(t,x,u))$ with domain $D(L)$, it was shown that L_u has the real principal eigenvalue with an associated positive eigenfunction in Beltramo and Hess[3].

LEMMA 3. *Under the assumption i), if u is a T-periodic solution for (P), then u is unstable.*

Proof. Suppose that $\sigma(L_u) \cap (-\infty, M) = \emptyset$. Let μ be the principal eigenvalue of L_u and φ_μ be an eigenfunction corresponding to μ. Then we have $\mu - M \geq 0$, $\varphi_\mu > 0$ and

$$L\varphi_\mu - g'(t,x,u)\varphi_\mu = (\mu - M)\varphi_\mu. \tag{3}$$

On the other hand, it holds that

$$L\varphi_1 = \lambda_1\varphi_1. \tag{4}$$

From (3) and (4), it follows that

$$\begin{aligned}
&\int_0^T \int_\Omega (g'(t,x,u) + \mu - M - \lambda_1)\varphi_\mu\varphi_1 dxdt \\
&= \int_0^T \int_\Omega \{(\varphi_\mu)_t\varphi_1 - (\Delta\varphi_\mu)\varphi_1 - (-\Delta\varphi_1)\varphi_\mu\} dxdt \\
&= 0.
\end{aligned}$$

By the assumption i), this is a contradiction. This implies $\sigma(L_u) \cap (-\infty, M) \neq \emptyset$. Let $\mu = M + \gamma$ be an eigenvalue of L_u with $\gamma < 0$ and φ_γ be an eigenfunction corresponding to $M + \gamma$. Then it holds that

$$\frac{d\varphi_\gamma}{dt} - \Delta\varphi_\gamma - g'(t,x,u)\varphi_\gamma = \gamma\varphi_\gamma$$

and hence

$$\frac{d(e^{-\gamma t}\varphi_\gamma)}{dt} - \Delta(e^{-\gamma t}\varphi_\gamma) - g'(t,x,u)(e^{-\gamma t}\varphi_\gamma) = 0.$$

This implies that $e^{-\gamma t}\varphi_\gamma$ is a solution of the initial value problem (LI) with $z = \varphi_\gamma(0)$. Then we get $U(0)\varphi_\gamma(0) = e^{-\gamma T}\varphi_\gamma(0)$, that is, $U(0)$ has an eigenvalue $e^{-\gamma T} > 1$. Therefore u is unstable. This completes the proof.

We can prove Theorems 1,2 using Lemmas 1-3.

Proof of Theorem 1. By Lemma 1, we obtain a T-periodic solution u for the problem (P). Lemma 3 shows that this solution u is unstable.

Proof of Theorem 2. From Lemmas 1 and 2, it follows that

$$\deg\,(I - K \circ g, B_R(0) \backslash B_r(0), 0) \neq 0$$

since $m - l + 1$ is an odd integer. Therefore there exists a nontrivial T-periodic solution u for (P). By Lemma 3, this u is an unstable T-periodic solution of (P). Next assume the existence of nondegenerate nontrivial T-periodic solution u for (P). Since the problem (L) do not have 0 as an eigenvalue, $I - K \circ g'(u)$ is invertible. Let k be the sum of the algebraic multipliers of the eigenvalues of (L) greater than 1. Then we have

$$\deg\,(I - K \circ g, B_\varepsilon(u), 0) = (-1)^k,$$

for sufficiently small $\varepsilon > 0$. Therefore it holds from Lemmas 1 and 2 that

$$\deg\,(I - K \circ g, B_R(0) \backslash (B_r(0) \cup B_\varepsilon(u)), 0) \neq 0.$$

This implies the existence of another nontrivial T-periodic solution of (P).

Remark 1. Under the hypotheses of Theorem 2, $u \equiv 0$ is an unstable T-periodic solution for (P) by Lemma 3.

3. Case of $g(t,x,\xi) = f(\xi) + h(t,x)$. In the present section, we consider the special case that $g(t,x,\xi) = f(\xi) + h(t,x)$ for $(t,x,\xi) \in \mathbf{R}_+ \times \bar{\Omega} \times \mathbf{R}$, where $f \in C^{1,\alpha}(\mathbf{R})$ and $h \in C^{1,\alpha}(\mathbf{R}_+ \times \bar{\Omega})$ which is T-periodic with respect to the first variable.

THEOREM 3. *Under the assumptions i), ii), if $\lambda_{l-1} < f'(0) < \lambda_l$ for some $l \in \mathbf{N}$ with $2 \le l \le m$ and $m - l + 1$ is odd, then the problem (P) with $g(t,x,\xi) = f(\xi) + h(t,x)$ has at least two unstable T-periodic solutions for h with $\|h\|$ sufficiently small. Moreover if all T-periodic solutions for (P) are nondegenerate, then there exist at least three unstable T-periodic solutions for (P).*

Proof. By the same argument as in the proof of Lemma 2, there are positive numbers δ, ω satisfying that

$$\langle Lv - f(v), Q_1 v - Q_2 v \rangle \ge \omega |v|^2 \tag{5}$$

for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$ with $0 < |v| \le \delta$. Take $\lambda_{l-1} < b < \lambda_l$. By the same argument as in the proof of Lemma 2, we obtain $C_2 > 0$ such that for any $s \in [0,1]$, if $v \in D(L)$ satisfies

$$Lv - \{sg(t,x,v) + (1-s)bv\} = 0,$$

then

$$\sup_{t \in [0,T]} |v(t)| \le C_2(\|v\| + \|h\|).$$

Let $r < \frac{\delta}{2C_2}$ and $\|h\| < \min\{\frac{\delta}{2C_2}, \omega r\}$. Suppose that

$$Lv_s - \{sg(t,x,v_s) + (1-s)bv_s\} = 0$$

for some $s \in [0,1]$ and $v_s \in D(L)$ with $\|v_s\| = r$. Since

$$\sup_{t \in [0,T]} |v_s(t)| \le \delta,$$

it follows from (2) that

$$\ll Lv_s - \{sg(t,x,v_s) + (1-s)bv_s\}, \tilde{Q}_1 v_s - \tilde{Q}_2 v_s \gg > 0.$$

This is a contradiction. Therefore we get

$$v - K\{sg(t,x,v_s) + (1-s)bv\} \neq 0$$

for all $v \in H$ with $\|v\| = r$. By the same method as in the proof of Lemma 2, it holds that

$$\deg\,(I - K \circ g, B_r(0), 0) = (-1)^{l-1}.$$

In order to show the rest of the proof, it is sufficient to take the same process as in the proof of Theorem 2.

We next give a sharper result than the above theorem. A solution w of the semilinear elliptic problem

$$(S) \qquad \begin{cases} -\Delta w = f(w) & \text{in } \Omega \\ \quad w = 0 & \text{on } \partial\Omega \end{cases}$$

is said to be nondegenerate if 0 is not an eigenvalue of the linearized problem of (S)

$$(SL) \qquad \begin{cases} -\Delta v - f'(w)v = \lambda v & \text{in } \Omega \\ \qquad\qquad w = 0 & \text{on } \partial\Omega. \end{cases}$$

The stability and instability of solutions for (S) are defined as same as those of T-periodic solutions for (P).

THEOREM 4. *Under the hypotheses of Theorem 3, if $l = m$ and f' is strictly increasing on $[0, +\infty)$ and strictly decreasing on $(-\infty, 0)$, then the problem (P) with $g(t, x, \xi) = f(\xi) + h(t, x)$ possesses at least three unstable T-periodic solutions for h with $\|h\| > 0$ sufficiently small.*

Remark 2. From the proof of Theorem 4, we can see that if $\|h\|$ is sufficiently small, then there are three unstable solutions u_1, u_2, u_3 and they lie in small neighborhoods in $L^2(0, T; L^2(\Omega))$ of unstable solutions $w_1, w_2, 0$ for (S), respectively.

We need the following two lemmas.

LEMMA 4. *Under the assumptions of Theorem 4, if w is a solution for (S), then there are $\delta_1, \rho_1 > 0$ such that for $\delta \le \delta_1$ and $0 < \|h\| \le \rho_1\delta$,*

$$deg\,(I - K \circ g, B_\delta(w), 0\} = (-1)^n,$$

where n is the sum of the multiplicities of the eigenvalues of $K \circ f'(w)$ greater than 1.

Proof. Let X_1 and X_2 be closed subspaces of $L^2(\Omega)$ spanned by eigenfunctions corresponding to the eigenvalues of (SL) greater and less than 0, respectively. Then X_1 and X_2 are orthogonal. Denote by Q_i and $\tilde{Q}_i$ the projections of $L^2(\Omega)$ onto X_i and the canonically extended projection of Q_i on H onto $L^2(0, T; X_i)$ for $i = 1, 2$, respectively. It is easy to see the existence of some positive number γ satisfying

$$\int_\Omega (-\Delta v - f'(w)v)(Q_1 v - Q_2 v)dx \ge \gamma |v|^2$$

for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Since $f : H \to H$ is of class C^1, we get

$$f(u) = f(w) + f'(u - w) + \phi(u - w)$$

for $u \in H$, where $\phi \in o(\|v\|)$ as $\|v\| \to 0$. It follows that

$$Lu - g(t, x, u) = L(u - w) - f'(w)(u - w) - \phi(u - w) - h.$$

Therefore for $s \in [0,1]$ and $u \in D(L)$, we have

$$\begin{aligned}
&\ll s\{Lu - g(t,x,u)\} + (1-s)\{L(u-w) - f'(w)(u-w)\}, \\
&\qquad\qquad \tilde{Q}_1(u-w) - \tilde{Q}_2(u-w) \gg \\
&= \ll L(u-w) - f'(w)(u-w)s\phi(u-w) - sh, \\
&\qquad\qquad \tilde{Q}_1(u-w) - \tilde{Q}_2(u-w) \gg \\
&= \int_0^T \int_\Omega \{u_t - \Delta(u-w) - f'(w)(u-w) - s\phi(u-w) - sh\} \\
&\qquad\qquad \{Q_1(u-w) - Q_2(u-w)\} dxdt \\
&= \int_0^T \int_\Omega \{-\Delta(u-w) - f'(w)(u-w) - s\phi(u-w) - sh\} \\
&\qquad\qquad \{Q_1(u-w) - Q_2(u-w)\} dxdt \\
&\geq \gamma\|u-w\|^2 - (\|\phi(u-w)\| + \|h\|)\|u-w\|.
\end{aligned}$$

By $\phi(v) \in o(\|v\|)$, for $0 < \varepsilon < \gamma$ there is $\delta_\varepsilon > 0$ such that $\|\phi(v)\| \leq \varepsilon\|v\|$ if $\|v\| \leq \delta_\varepsilon$. Taking $\delta_1 < \delta_\varepsilon$ and $\rho_1 = \gamma - \varepsilon$, if $\delta \leq \delta_1$ and $\|h\| \leq \rho_1\delta$, then it holds that

$$\begin{aligned}
&\ll s\{Lu - g(t,x,u)\} + (1-s)\{L(u-w) - f'(w)(u-w)\}, \\
&\qquad\qquad \tilde{Q}_1(u-w) - \tilde{Q}_2(u-w) \gg > 0
\end{aligned}$$

for $s \in [0,1]$ and $u \in \partial B_\delta(w)$. This shows that

$$s\{u - K \circ g(t,x,u)\} + (1-s)\{u - w - K \circ f'(w)(u-w)\} \neq 0$$

for $s \in [0,1]$ and $u \in \partial B_\delta(w)$. According to the homotopy invariance of the Leray-Schauder degree, it follows that

$$\deg\,(I - K \circ g, B_\delta(w), 0) = \deg\,(I - K \circ f'(w), B_\delta(0), 0).$$

Suppose that

$$K \circ f'(w)v = v,$$

i.e.,

$$v_t - \Delta v - f'(w)v = 0$$

for some $v \neq 0$. Multiplying this equality by v_t and integrating on $(0,T) \times \Omega$, we obtain $v_t \equiv 0$ and hence

$$-\Delta v = f'(w)v,$$

which contradicts that w is nondegenerate. This implies that 1 is not an eigenvalue of $K \circ f'(w)$. Consequently, we see

$$\deg\,(I - K \circ f'(w), B_\delta(0), 0) = (-1)^n,$$

where n is the sum of the multiplicities of the eigenvalues of $K \circ f'(w)$ greater than 1. This completes the proof.

We investigate a relation for stability and instability between a solution for (S) and a T-periodic solution for (P). For a solution w of (S) and a T-periodic solution u of (P), denote by λ_w and μ_u the first eigenvalue of (SL) and a real principal eigenvalue of (L), respectively.

LEMMA 5. *Let $w \in C^2(\bar{\Omega})$ be a solution of the problem (S) which is nondegenerate. Then there exist $\delta_2, \rho_2 > 0$ such that if $u \in B_{\delta_2}(w)$ is a T-periodic solution for (P) with $g(t,x,\xi) = f(\xi) + h(t,x)$ with $\|h\| \le \rho_2$, then u is nondegenerate and the sign of μ_u coincides with that of λ_w.*

Proof. Suppose that u is a T-periodic solutions for (P) and w is a solution for (S). Let φ and ψ be positive eigenfunctions corresponding to λ_w and μ_u, respectively. Then it holds that

$$\int_0^T \int_\Omega \{f'(u) - f'(w) - \lambda_w + \mu_u\}\varphi\psi dxdt = 0. \tag{6}$$

By $f \in C^{1,\alpha}(\mathbf{R})$, there is $C_1 > 0$ satisfying that

$$|f'(\xi_1) - f'(\xi_2)| \le C_1|\xi_1 - \xi_2|^\alpha \tag{7}$$

for $\xi_1, \xi_2 \in \mathbf{R}$. Since u is a T-periodic solution for (P) and w is a solution for (S), it follows that

$$\frac{\partial(u-w)}{\partial t} - \Delta(u-w) - \{f(u) - f(w)\} - h = 0.$$

On the other hand we have by the same argument as in the proof of Lemma 2, there are $\delta_2, \rho_2 > 0$ such that

$$\sup_{(t,x)\in[0,T]\times\bar{\Omega}} |u(t,x) - w(x)| < (\frac{|\lambda_w|}{C_1})^{1/\alpha} \tag{8}$$

if $\|h\| \le \rho_2$ and $u \in B_{\delta_2}(w)$ is any T-periodic solution for (P) with $g(t,x,\xi) = f(\xi) + h(t,x)$ since f is Lipschitz continuous. Let $\|h\| \le \rho_2$ and $u \in B_{\delta_2}(w)$ be a solution for (P). In the case of $\lambda_w < 0$, assuming that $\mu_u \ge 0$, we have by (7) and (8),

$$f'(u) - f'(w) - \lambda_w + \mu_u > 0,$$

which contradicts (6). This implies that $\mu_u < 0$. By the same argument as the above, we can prove the case of $\lambda_w > 0$. This completes the proof.

Proof of Theorem 4. Under the hypotheses of Theorem 4, there exist at least two nontrivial solution w_1 and w_2 in $C^2(\bar{\Omega})$ for (S) which are nondegenerate and unstable (see [4]). It is immediate that 0 is nondegenerate unstable solution for (S). Choosing positive numbers δ and ρ sufficiently small, by lemmas 4 and 5, there are at least three unstable T-periodic solutions u_1, u_2, u_3 for (P) with $g(t,x,\xi) = f(\xi) + h(t,x)$ and $0 < \|h\| \le \rho$ such that $u_i \in B_\delta(w_i)$ for $i = 1,2$ and $u_3 \in B_\delta(0)$.

Both stable T-periodic solutions and unstable ones exist in the following cases.

THEOREM 5. *Suppose that*

$$f'(0) < \lambda_1 < \liminf_{|\xi|\to\infty} \frac{g(\xi)}{\xi} \le \limsup_{|\xi|\to\infty} \frac{g(\xi)}{\xi} < \lambda_2$$

and f' is strictly increasing on $[0, \infty)$ and strictly decreasing on $(0, \infty)$. Then the problem (P) with $g(t, x, \xi) = f(\xi) + h(t, x)$ has at least one stable T-periodic solution and two unstable T-periodic solutions if $\|h\| > 0$ is sufficiently small.

Proof. By [4], there are at least two nontrivial solutions of (S) which are nondegenerate and unstable. Obviously, 0 is a stable solution for (S). Using Lemmas 4 and 5, we can obtain the consequence of this theorem.

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