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# ROTATION NUMBERS FOR LAGRANGIAN SYSTEMS AND MORSE THEORY

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**Introduction.** Consider the linear oscillator:

$$\ddot{x}(t) + \omega^2 x(t) = 0, \quad x(t) \in \mathbf{R} \tag{0.1}$$

All the solutions are given by the formula $x(t) = a\cos\omega t + b\sin\omega t$, for suitable $a, b \in \mathbf{R}$. Drawing the flow lines $(x(t), \dot{x}(t))$ in the phase space $\mathbf{R}^2$, one easly checks that $(0, 0)$ is a stationary point and that every other vector rotates of an angle $\Omega(t) = \omega t$ in time $t$.

The *rotation number* of this linear Lagrangian system is, by definition:

$$\tau = \lim_{t\to+\infty} \frac{\Omega(t)}{2\pi t} = \frac{\omega}{2\pi}$$

Now consider the equation:

$$\ddot{x}(t) - \lambda^2 x(t) = 0, \quad x(t) \in \mathbf{R} \tag{0.2}$$

Again $(0, 0)$ is the only stationary point, but the rotation angle around the origin of any other vector of the phase space remains bounded by $\pi$: a good reason for defining the rotation number of this system as $\tau = 0$.

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Finally, consider the $N$-dimensional system:

$$\ddot{x}(t) + Ax(t) = 0, \quad x(t) \in \mathbf{R}^N, \quad A = \begin{pmatrix} \omega_1^2 & & & & & \\ & \ddots & & & 0 & \\ & & \omega_l^2 & & & \\ & & & -\lambda_1^2 & & \\ & 0 & & & \ddots & \\ & & & & & -\lambda_{N-l}^2 \end{pmatrix} \tag{0.3}$$

It seems natural to define the rotation number of such system as:

$$\tau(A) = \frac{\omega_1 + \ldots + \omega_l}{2\pi}$$

**1. The rotation number of a linear Lagrangian system.** Consider the $N$-dimensional second order linear system:

$$\ddot{x}(t) + A(t)x(t) = 0, \quad x(t) \in \mathbf{R}^N \tag{1.1}$$

System (1.1) is *Lagrangian* if the $N \times N$ matrix $A(t)$ is symmetric. Systems (0.1), (0.2), (0.3) are examples of linear Lagrangian sytems, with $A(t)$ constant in $t$ and diagonal.

It is useful to transform (1.1) into a $2N$-dimensional first order linear system:

$$z(t) := (x(t), \dot{x}(t)) \in \mathbf{R}^{2N}, \quad \dot{z}(t) + B(t)z(t) = 0, \quad B(t) = \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix} \tag{1.2}$$

Let $W(t)$ be the solution of the matrix differential problem:

$$\begin{cases} \dot{W}(t) + B(t)W(t) = 0 \\ W(0) = I \end{cases} \tag{1.3}$$

$W(t)$ is the Wronskian of system (1.2): for each $z_0 \in \mathbf{R}^{2N}$, $t \mapsto W(t)z_0$ is the solution of (1.2) with initial condition $z_0$.

Because of the special form of the matrix $B(t)$, $W(t)$ is symplectic for each $t \in \mathbf{R}$:

$$W(t)^T J W(t) = J, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{1.4}$$

To see this fact, just notice that (1.4) obviously holds for $t = 0$ and the derivative of $W(t)^T J W(t)$ vanishes, with the aid of (1.3).

The symplecticity of $W(t)$ carries many important consequences: the determinant of $W(t)$ equals 1 and, if $\alpha \in \mathbf{C}$ is an eigenvalue of $W(t)$, so are $\bar{\alpha}$, $\alpha^{-1}$ and $\bar{\alpha}^{-1}$. If 1 or $-1$ are eigenvalues, they have even multiplicity.

The eigenvalues of $W(t)$ are called *Floquet multipliers*; Floquet multipliers of system (0.3) are $e^{\pm\omega_1 i}, \ldots, e^{\pm\omega_l i}, e^{\pm\lambda_1}, \ldots, e^{\pm\lambda_{N-l}}$. In this case $W(t)$ turns out to be diagonalizable, but in general it may be not.

In any case $W(t)$ can be put into polar form $W(t) = P(t)O(t)$, where $P(t) = [W(t)W(t)^T]^{\frac{1}{2}}$ is symmetric, positive and symplectic, $O(t) = P(t)^{-1}W(t)$ is orthogonal and symplectic. The matrix $O(t)$ is made of four $N \times N$-blocks; writing the conditions for orthogonality and symplecticity, one finds:

$$O(t) = \begin{pmatrix} U_1(t) & -U_2(t) \\ U_2(t) & U_1(t) \end{pmatrix} = \begin{pmatrix} U_1(t) & 0 \\ 0 & U_1(t) \end{pmatrix} + J \begin{pmatrix} U_2(t) & 0 \\ 0 & U_2(t) \end{pmatrix}$$

Identifying $\mathbf{R}^{2N}$ to $\mathbf{C}^N$, and considering the action of $J$ as the multiplication by $i$, $O(t)$ can be naturally identified to a unitary matrix $U(t)$ which acts on $\mathbf{C}^N$:

$$U(t) := U_1(t) + iU_2(t)$$

The determinant of $U(t)$ is a complex number with modulus 1 and the *rotation at time* $t$ is defined to be:

$$\Omega(t, A) = \frac{1}{i} \log[\det U(t)] \tag{1.5}$$

Here suitable determinations of the multi-valued function log are chosen, so that $\Omega(0, A) = 0$ and $\Omega(\cdot, A)$ is a continuous function.

If the limit $\tau(A) = \lim_{t\to+\infty} \frac{\Omega(t)}{2\pi t}$ exists, $\tau(A)$ is said to be the *rotation number of the Lagrangian linear system (1.1)*. It is an useful exercise to show that, if $A$ is of the form (0.3), $\tau(A)$ coincides with its previous definition.

In the general case the above limit may not exist. However, if $A(t)$ is periodic in $t$, Bott's iteration formula can be used to prove that the limit exists [6]. If $A(t)$ is not periodic, the existence of this limit can be sometimes obtained using ergodic theorems, as we will see at the end of next section.

Roughly speaking, the rotation number has the following meaning: the matrix $P(t)$ in the decomposition $W(t) = P(t)O(t)$ represents the expansion and it is neglected. $O(t)$ is orthogonal and thus it is a combination of rotations. In general one could only speak about the rotation of $O(t)$ around a certain axis, but the fact that $O(t)$ is also symplectic allows to put all the rotational information into a single number, $\Omega(t, A)$. Finally, one looks at the asymptotic behaviour of $\Omega(t, A)$ and defines $\tau(A)$.

More sophisticated interpretations are also possible: one can study the topology of the symplectic group and see the rotation number as the winding number of the path $W(t)$ in this space, as in [8]. Otherwise one can can speak about the rotation of suitable Lagrangian spaces in the Lagrangian manifold as in [3] and [1].

**2. Non-linear systems: the twist number of an orbit.** Consider the $N$-dimensional Lagrangian system:

$$\ddot{\gamma}(t) + V'(t, \gamma(t)) = 0, \quad \gamma(t) \in \mathbf{R}^N \tag{2.1}$$

where $V : \mathbf{R} \times \mathbf{R}^N \mapsto \mathbf{R}$ is a smooth potential, and $V'$ is the gradient of $V$.

Let $\gamma : \mathbf{R} \mapsto \mathbf{R}^N$ be a solution of (2.1) and consider the linearized system at $\gamma$:

$$\ddot{x}(t) + V''(t, \gamma(t))x(t) = 0, \quad x(t) \in \mathbf{R}^N \tag{2.2}$$

We define the *twist at time* $t$ of the orbit $\gamma$ as the rotation at time $t$ of system (2.2):

$$\Omega(t, \gamma) := \Omega(t, V''(t, \gamma(t)))$$

We define the *twist number* of the orbit $\gamma$ as the rotation number of system (2.2):

$$\tau(\gamma) := \tau(V''(t, \gamma(t))) = \lim_{t \to +\infty} \frac{\Omega(t, \gamma)}{2\pi t}$$

provided that this limit exists.

Notice that the twist number of the solution $x(t) \equiv 0$ of a linear Lagrangian system equals the rotation number of the sytem itself.

If $V$ is $T_0$-periodic in time and $\gamma$ is a $kT_0$-periodic orbit, $k \in \mathbf{N}$, $V''(t, \gamma(t))$ is also $kT_0$-periodic and the twist number $\tau(\gamma)$ is well defined. In this case, $\gamma$ is said to be $hkT_0$-resonant, $h \in \mathbf{N}$, if system (2.2) has a $hkT_0$-periodic non trivial solution.

If the orbit $\gamma$ is not periodic, the existence of the twist number can be proved using ergodic theory.

Assume that our system is autonomous, that is $V$ does not depend on time. We recall that the energy surface $\Sigma_h$, $h \in \mathbf{R}$, is the set:

$$\Sigma_h = \left\{ (x, y) \in \mathbf{R}^{2N} | \frac{1}{2}|y|^2 + V(x) = h \right\}$$

and that every trajectory $(x(t), \dot{x}(t))$ leaving from $\Sigma_h$ stays in $\Sigma_h$ for all times.

Let $\Phi : \mathbf{R} \times \mathbf{R}^{2N} \mapsto \mathbf{R}^{2N}$ be the flow which solves system (2.1). A Borel measure $\mu$ on $\mathbf{R}^{2N}$ is said to be invariant if $\mu(\Phi(t, \Gamma)) = \mu(\Gamma)$, for each $t \in \mathbf{R}$ and $\Gamma$ Borel set in $\mathbf{R}^{2N}$.

Every energy surface $\Sigma_h$ supports a special invariant measure, the Liouville measure, which is induced by Lebesgue measure in $\mathbf{R}^{2N}$ (see, for example [11]). An ergodic theorem can be used to prove the following [1]:

THEOREM 1. *Consider an autonomous Lagrangian system in $\mathbf{R}^N$, given by a potential $V$. Let $\Sigma_h \subset \mathbf{R}^{2N}$ be an energy surface with Liouville measure $\mu_h$. If $\Sigma_h$ is compact, then for $\mu_h$-almost every $(x_0, y_0) \in \Sigma_h$ the unique orbit $x(t)$ such that $x(0) = x_0$ and $\dot{x}(0) = y_0$ has a well defined twist number.*

It is also possible to regard arbitrary invariant probability measures as generalizations of solutions: an invariant KAM tori, for example, can be better described by the invariant probability measure it supports, than by all the trajectories which lie on it. The use of the same ergodic theorem allows also to define the twist number of an invariant probability measure $\mu$: it is the integral, with respect to $\mu$, of the twist number of all the trajectories which lie in the support of $\mu$. See [1] for full details.

**3. Asymptotically linear Lagrangian systems.** The Lagrangian system (2.1) is called *asymptotically linear* if:

$$V(t, x) = \frac{1}{2} < A_\infty(t)x, x > +U(t, x) \tag{3.1}$$

where:

(a) $U(t, x)$ is bounded;

(b) $U'(t, x)$ is bounded;

(c) $\lim_{|x| \to +\infty} U''(t, x) = 0$ uniformly in $t$.

Moreover assume that:
(d) $V(t,0)=0$ for each $t\in\mathbf{R}$;
(e) $V'(t,0)=0$ for each $t\in\mathbf{R}$;
(f) $V''(t,0)=: A_0(t)$.

We can associate two linear systems to an asymptotically linear system: the linearization at zero and the linearization at infinity:

$$\ddot{x}(t)+A_0(t)x(t)=0 \tag{3.2}$$

$$\ddot{x}(t)+A_\infty(t)x(t)=0 \tag{3.3}$$

$\tau_0$ will denote the rotation number of system (3.2), while $\tau_\infty$ will denote the rotation number of system (3.3).

Assume now that $V$ is $T_0$-periodic in $t$; if $T=kT_0$, $k\in\mathbf{N}$, the system is called $T$-resonant at zero [ at infinity] if (3.2) [ (3.3)] has a $T$-periodic non trivial solution.

Theorem A. *Assume that system (2.1) is asymptotically linear and $T_0$-periodic in time. Suppose that it is $kT_0$-non-resonant at zero or at infinity and that*:

$$|\tau_0-\tau_\infty|>\frac{2N}{kT_0} \tag{3.4}$$

*Then the system has at least one non trivial $kT_0$-periodic solution. Moreover, if all the $kT_0$-periodic solutions are $kT_0$-non-resonant, the system has at least two $kT_0$-periodic non trivial solutions.*

This theorem was proved in [5]. A sketch of the proof will be given in section 6. There is also a version for Hamiltonian systems, in the completely non-resonant case [8].

Notice that Theorem A implies that, if $\tau_0\neq\tau_\infty$, there always exists a $kT_0$-periodic non trivial solution, for $k$ large enough. If $\tau_0=\tau_\infty$ it is possible that there exist no non trivial periodic solutions: consider system (0.2). It is linear, $\tau_0=\tau_\infty=0$, but $x(t)\equiv 0$ is the only periodic solution.

With some extra assumptions, Theorem A can be used to show the existence of infinitely many closed orbits. For example, if there are no $T_0$-periodic solutions and $\tau_0\neq\tau_\infty$, we find $kT_0$-periodic solutions for each $k$ prime and large enough, and these are all distinct.

**4. Lagrangian systems on manifolds.** Let $M$ be a Riemannian manifold, equipped with its Levi-Civita covariant derivation $D$. A Lagrangian system on $M$ has the form:

$$D_t\dot{\gamma}(t)+V'(t,\gamma(t))=0 \tag{4.1}$$

In most textbooks in mechanics, Lagrangian systems on manifolds are defined just by giving a Lagrangian $\mathcal{L}:\mathbf{R}\times TM\mapsto\mathbf{R}$. System (4.1) corresponds to the case in which $\mathcal{L}$ has the form $\mathcal{L}(t,(x,v))=T(x,v)-V(t,x)$, where $T(x,\cdot)$ is a positive quadratic form on $T_xM$. Then $T$ can be used to define a Riemannian structure on $M$ such that the Lagrange equations take the form (4.1).

The linearization of (4.1) along a solution $\gamma$ is not a linear system in $\mathbf{R}^{2N}$ anymore, but it is a linear system in $TM$ along $\gamma$. Technical difficulty increases, but one can still define the twist at time $t$, $\Omega(t,\gamma)$ and the twist number $\tau(\gamma) = \lim_{t\mapsto+\infty} \frac{\Omega(t,\gamma)}{2\pi t}$, which have still the same geometrical meaning.

The above limit always exists when $V$ is $T_0$-periodic in time and $\gamma$ is $kT_0$-periodic. Ergodic theorems can be still applied to define the twist number for almost every trajectory and for invariant probability measures, as mentioned in section 2. See [1] for details.

Theorem B. *Assume that $M$ is compact and has a finite fundamental group. Suppose that $V$ is $T_0$-periodic. Set*:

$$\mathcal{T} = \{\tau(\gamma)|\gamma \text{ is a } kT_0 - \text{periodic orbit},\ k \in \mathbf{N}\}$$

*Then $\mathcal{T}$ is dense in $\mathbf{R}^+ = \{\alpha \in \mathbf{R}|\alpha \geq 0\}$.*

This Theorem was proved in [1]. A sketch of the proof will be given in section 7.

It is impossible to remove the condition on the topolgy of $M$: let $M$ be a manifold with negative sectional curvature; by Hadamard Theorem $\pi_1(M,x_0)$ is infinite. Consider system (4.1) with $V \equiv 0$: we are looking at geodesics. It is well known that the geodesic flow on a negatively curved compact manifold is an Anosov system [2]. In particular all the Floquet multipliers of a geodesic $\gamma$ lie away from the unit circle in $\mathbf{C}$ and $\tau(\gamma) = 0$ for every $\gamma$. Therefore $\mathcal{T} = \{0\}$ in this case.

We conclude this section stating a consequence of this Theorem [1]:

Theorem 2. *Suppose that all the assumptions of Theorem B are fulfilled. Assume, moreover, that $M$ has a strictly positive sectional curvature and that all the periodic solutions of (4.1) have period multiple of $T_0$. Let $\alpha \geq 0$. Then one of the following assertions must be true*:

(1) *There exists a $kT_0$-periodic orbit $\gamma$ such that $\tau(\gamma) = \alpha$.*

(2) *There exists an invariant probability measure $\mu$ such that $\tau(\mu) = \alpha$ and a sequence of closed orbits $(\gamma_n)$ with diverging periods which weakly converges to $\mu$.*

**5. Twist number and Morse theory.** We recall some definitions and results of infinite dimensional Morse theory. Let $H$ be a Hilbert space and $f$ be a smooth function on $H$. $x \in H$ is a critical point for $f$ if $df(x) = 0$; $K(f)$ will denote the set of critical points of $f$. If there is no critical point $x$ such that $f(x) = c$, $c$ is a *regular value* for $f$.

The Morse index $m(x,f)$ of a critical point $x$ is the dimension of a maximal subspace of $H$ on which the quadratic form $d^2f(x)$ is negative. The critical point $x$ is called non-degenerate if $d^2f(x)$ is non-degenerate.

$f$ is said to satisfy Palais-Smale condition if each sequence $(x_n)$ such that $f(x_n)$ is bounded and $df(x_n) \to 0$, has a converging subsequence.

Let $\mathbf{F}$ be a fixed field ( usually $\mathbf{F} = \mathbf{Z}_2$) and $(X,A)$ a pair of topological spaces, $A \subset X$. We denote by $\beta_q(X,A)$, $q \in \mathbf{N}$, the $q$-th Betti number of $(X,A)$, that is the dimension on $\mathbf{F}$ of the $\mathbf{F}$-vector space $H_q(X,A;\mathbf{F})$, the $q$-th homology space with coefficients in $\mathbf{F}$.

Poincaré polynomial $P(X,A)$ is defined as $P(X,A)(t) = \sum_{q=0}^{\infty} \beta_q(X,A)t^q$. If $X$ and $A$ have infinite topological dimension, $P(X,A)$ may actually be a formal power series, rather than a polynomial.

We set $f^a = \{x \in H|f(x) \leq a\}$, $f_a^b = \{x \in H|a \leq f(x) \leq b\}$.

The fundamental results of Morse theory can be summarized into the Morse relations:

THEOREM 3. *Assume that $f \in C^2(H)$ satisfies Palais-Smale condition and that all its critical points are non-degenerate. Let $a, b \in \mathbf{R}$ be regular values for $f$. Then there exists a polynomial $Q$ with non negative integer coefficients such that*:

$$\sum_{x \in K(f) \cap f_a^b} t^{m(x,f)} = P(f^b, f^a)(t) + (1+t)Q(t)$$

Notice that the set $K(f) \cap f_a^b$ is finite because of Palais-Smale condition and of the non-degenerance of critical points.

This Theorem can be generalized in several directions: for example one can take $f$ bounded from below and $a = -\infty$, $b = +\infty$. Moreover, one can use functions on Hilbert manifolds, and obtain the same result. Also the condition about the non-degenerance of critical points can be avoided, using Conley index. In the next two sections, Morse relations will be used in some of these more general formulations. See [7] and [12] for the proofs and for a complete bibliography on these topics.

Morse theory is an important tool to prove the existence of periodic solutions of (2.2) and to compute their twist number. In fact there is a close relationship between the Morse index and the twist number, as we are going to show.

Let $V$ be a $T_0$-periodic potential in $\mathbf{R}^N$. $\gamma$ is a $T = kT_0$-periodic solution of (2.2) if and only if $\gamma$ is a critical point of the *action functional*:

$$f_T(\gamma) = \int_0^T \left[ \frac{1}{2} |\dot\gamma(t)|^2 - V(t, \gamma(t)) \right] dt \tag{5.1}$$

on the Hilbert space $\Lambda_T = \left\{ \gamma \in H^1([0,T]; \mathbf{R}^N) | \gamma(0) = \gamma(T) \right\}$. It is easy to see that every critical point of $f_T$ has a finite Morse index $m(x, f_T)$.

Notice that $\gamma$ is $T$-non-resonant if and only if it is a non-degenerate critical point of $f_T$.

The main result of this section is the following:

THEOREM 4. *If $V$ is $T_0$-periodic and $\gamma$ is a $T = kT_0$-periodic solution of (2.2)*:

$$\lim_{h \to +\infty} \frac{m(\gamma, f_{hT})}{hT} = \tau(\gamma)$$

*Moreover, the following estimate holds*:

$$hT\tau(\gamma) - N \le m(\gamma, f_{hT}) \le hT\tau(\gamma) + N$$

Theorem 4 is a consequence of a result of Bott [6]. A similiar statement for convex Hamiltonian systems was proved in [9], [10]. See [4] for a proof of the theorem in the above formulation.

**6. Sketch of the proof of Theorem A.** Set $T = kT_0$. We consider only the completely non-resonant case, refering to [4] for the general case: all $T$-periodic solutions of (2.2) are $T$-non-resonant and the system is $T$-non-resonant at infinity.

The action functional has the form:

$$f_T(\gamma) = \int_0^T \left[ \frac{1}{2} |\dot\gamma(t)|^2 - \frac{1}{2} < A_\infty(t)\gamma(t), \gamma(t) > - U(t, \gamma(t)) \right] dt, \quad \gamma \in \Lambda_T$$

If $c \in \mathbf{R}$, set:

$$f_{T,\infty}(\gamma) = \frac{1}{2}\int_0^T \left[|\dot{\gamma}(t)|^2 - < A_\infty(t)\gamma(t), \gamma(t) >\right] dt$$

$$\Sigma_T^c := f_T^c = \{\gamma \in \Lambda_T | f_T(\gamma) \leq c\}$$

$$\Sigma_{T,\infty}^c := f_{T,\infty}^c = \{\gamma \in \Lambda_T | f_{T,\infty}(\gamma) \leq c\}$$

If $c < 0$ is small enough, $\Sigma_T^c$ is topologically equivalent to $\Sigma_{T,\infty}^c$, therefore:

$$P(\Lambda_T, \Sigma_T^c) = P(\Lambda_T, \Sigma_{T,\infty}^c)$$

Since the system is $T$-non-resonant at infinity, $f_{T,\infty}$ is a non-degenerate quadratic form and, if $c < 0$, $P(\Lambda_T, \Sigma_{T,\infty}^c) = t^{m(0, f_{T,\infty})}$.

Since the system is $T$-non-resonant at infinity, it is possible to prove that $f_T$ satisfies Palais-Smale condition.

Assuming that $c$ is a regular value, write Morse relations for $f_T$:

$$\sum_{\substack{\gamma \in K(f_T) \\ f_T(\gamma) \geq c}} t^{m(\gamma, f_T)} = P(\Lambda_T, \Sigma_T^c)(t) + (1+t)Q(t) \tag{6.1}$$

If $c < 0$ is small enough:

$$t^{m(0,f_T)} + \sum_{\substack{\gamma \in K(f_T)\setminus\{0\} \\ f_T(\gamma) \geq c}} t^{m(\gamma, f_T)} = t^{m(0, f_{T,\infty})} + (1+t)Q(t) \tag{6.2}$$

If we apply Theorem 4 to $\gamma(t) \equiv 0$, considered as a $T$-periodic solution of the systems $\ddot{\gamma}(t) + A_\infty(t)\gamma(t) = 0$ and $\ddot{\gamma}(t) + V(t, \gamma(t)) = 0$, we obtain the following estimates:

$$m(0, f_{T,\infty}) - m(0, f_T) \geq T\tau_\infty - N - T\tau_0 - N = T(\tau_\infty - \tau_0) - 2N$$

$$m(0, f_T) - m(0, f_{T,\infty}) \geq T\tau_0 - N - T\tau_\infty - N = T(\tau_0 - \tau_\infty) - 2N$$

Since $T|\tau_\infty - \tau_0| - 2N > 0$, $m(0, f_T) \neq m(0, f_{T,\infty})$ and in equation (6.2) $Q \neq 0$: therefore there are at least two critical points of $f_T$ different from zero.

If some of the $T$-periodic solutions are $T$-resonant, one has to use Morse theory for degenerate critical points, and can deduce only the exixtence of one non trivial solution.

If the system is $T$-resonant at infinity, the problem is much more complicated, because Palais-Smale condition does not hold.

More details can be found in [5].

**7. Sketch of the proof of Theorem B.** Let $\Lambda_T(M) = \{\gamma \in H^1([0,T]; M) | \gamma(0) = \gamma(T)\}$ and define $f_T$ as in (5.1), but for $\gamma \in \Lambda_T(M)$.

We assume that all the $kT_0$-periodic solutions, $k \in \mathbf{N}$ , are $kT_0$-non-resonant.

Assume first that $M$ is simply connected: then infinitely many Betti numbers of $\Lambda_T(M)$ are different from zero, if one takes $\mathbf{Z}_2$ coefficients [13]. Let $a_n$ be the index of the n-th Betti number different from zero.

It is easy to show that the set $\{\frac{a_n}{hT_0} | n, h \in \mathbf{N},\ h \geq k\}$ is dense in $\mathbf{R}^+$. Therefore there exist $a_{\bar{n}}$ and $h$ such that:

$$\left|\alpha - \frac{a_{\bar{n}}}{hT_0}\right| < \frac{\epsilon}{2} \tag{7.1}$$

$$\frac{N}{hT_0} < \frac{\epsilon}{2} \tag{7.2}$$

Here $N$ is the dimension of $M$. Set $T = hT_0$.

$f_T$ satisfies Palais-Smale condition, because $M$ is compact. The functional $f_T$ is bounded from below, therefore we can write Morse relations for the whole manifold $\Lambda_T(M)$ and we find a critical point $\gamma$ such that $m(\gamma, f_T) = a_{\bar{n}}$.

By Theorem 4:

$$m(\gamma, f_T) - N \leq T\tau(\gamma) \leq m(\gamma, f_T) + N \tag{7.3}$$

From (7.1), (7.2) and (7.3):

$$|\tau(\gamma) - \alpha| < \epsilon$$

Therefore the Theorem is proved for $M$ simply conected. If the fundamental group of $M$ is finite, the universal covering $\tilde{M}$ is compact: we can consider the lifted Lagrangian system on $\tilde{M}$ and prove the density of the twist numbers for it.

Projecting a periodic solution of the lifted system down on $M$, we obtain a periodic solution of the original system with the same twist number. Therefore $\mathcal{T}$ is dense in $\mathbf{R}^+$ also in this case.

The resonant case is not more difficult: one has to use Morse theory for degenerate critical points and the proof remains the same.

More details can be found in [1].

## References


[1] A. Abbondandolo, *A rotation number for invariant measures and Morse theory*, (to appear).

[2] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Inst. Steklov 90 (1967), 1-235.

[3] V. I. Arnol'd, *Characteristic class entering in quantization conditions*,Functional Anal. Appl. 1 (1967), 1-14.

[4] V. Benci, *A new approach to Morse-Conley Theory and some applications*, Ann. Mat. Pura Appl. (4) 158 (1991), 231-305.

[5] V. Benci and D. Fortunato, *Periodic solutions of asymptotically linear dinamical systems*, Nonlinear Diff. Eq. and Appl. (to appear).

[6] R. Bott, *On the iteration of closed geodesics and the Sturm intersection theory*, Comm. Pure Appl. Math. 9(1956), 176-206.

[7] K. C. Chang, *Infinite dimensional Morse theory and multiple solutions problems*, Boston-Basel: Birkhauser 1993.

[8] C. Conley and E. Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations*, Comm. Pure Appl. Math. 37 (1984), 207-253.

[9] I. Ekeland, *Convexity methods in Hamiltonian mechanics*, Berlin Heidelberg New York: Springer-Verlag 1990.

[10] I. Ekeland, *An index theory for periodic solutions of convex Hamiltonian systems*, Proceedings for Symposia in Pure Math. 45, 395-423.

[11] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Berlin Heidelberg New York: Springer-Verlag 1987.

[12] J. L. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, New York: Springer-Verlag 1989.

[13] M. Vigué-Poirrier and D. Sullivan, *The homology theory for the closed geodesic problem*, J. Differential Geom. 11(1976), 633-644.