TOPOLOGY IN NONLINEAR ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 35
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1996

UPPER SEMICONTINUOUS PERTURBATIONS OF m-ACCRETIVE OPERATORS AND DIFFERENTIAL INCLUSIONS WITH DISSIPATIVE RIGHT-HAND SIDE

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1. Introduction. Let X be a real Banach space and $A : D(A) \subset X \to 2^X \setminus \{\emptyset\}$ be m-accretive. In applications one often has to deal with operators of the type $A + F$. Therefore it is of interest to have sufficient conditions guaranteeing that this sum is m-accretive again. This problem has attracted many people; see [1], [10], [13] and [14], the references given there and also [2], [4] and [12].

Of particular interest to us is Theorem 1 in [1], saying that $A + F$ is m-accretive if $F : X \to X$ is continuous and accretive. In the first part of this paper, we extend this and related results to the case of multivalued perturbations. More precisely, we prove that if $F : \overline{D(A)} \to 2^X \setminus \{\emptyset\}$ is upper semicontinuous with compact convex values such that $A+F$ is accretive, then $A+F$ is m-accretive. This result proves useful in the second part of this paper where we obtain existence of strong solutions of the initial value problem

$$u' \in F(t,u) \qquad \text{on } J = [0,a],\ u(0) = x_0, \tag{1}$$

if, among other assumptions, the $F(t,\,\cdot\,)$ are usc with compact convex values and satisfy a condition of dissipative type.

2. Preliminaries. In the sequel, X will always be a real Banach space with norm $|\cdot|$. Then $2^X \setminus \emptyset$ denotes the set of all nonempty subsets of X, $B_r(x)$ is the open ball in X with center x and radius r, $\overline{B}_r(x)$ denotes its closure and $\rho(x, B)$ is the distance from x to the set $B \subset X$. Given $J = [0, a] \subset \mathbb{R}$, we let $C_X(J)$ be the Banach space

1991 *Mathematics Subject Classification*: Primary 47H06; Secondary 34A60.

The author gratefully acknowledges financial support by DAAD within the scope of the French-German project PROCOPE.

The paper is in final form and no version of it will be published elsewhere.

of all continuous $u : J \to X$ and $L^1_X(J)$ the Banach space of all strongly measurable, Bochner-integrable $w : J \to X$, both equipped with the usual norms which we denote by $|\cdot|_0$, respectively $|\cdot|_1$. Given an operator $A : X \to 2^X$, we let $D(A) = \{x \in X \mid Ax \neq \emptyset\}$, $R(A) = \bigcup_{x\in D(A)} Ax$ and $\mathrm{gr}(A) = \{(x,y) \mid x \in D(A),\ y \in Ax\}$ denote the domain, range and graph of A, respectively.

(i) Recall that $A : X \to 2^X$ is m-accretive if $R(A + \lambda I) = X$ for all $\lambda > 0$ and A is accretive, i.e.

$$(u - v, x - y)_+ \geq 0 \qquad \text{for all } x, y \in D(A),\ u \in Ax \text{ and } v \in Ay.$$

Here $(\cdot\,,\,\cdot)_+$ denotes one of the semi-inner products $(\cdot\,,\,\cdot)_\pm$ defined by

$$(x,y)_+ = |y| \lim_{t\to 0+} \frac{|y + tx| - |y|}{t} \quad \text{and} \quad (x,y)_- = |y| \lim_{t\to 0+} \frac{|y| - |y - tx|}{t};$$

properties of $(\cdot\,,\,\cdot)_\pm$ can be found e.g. in §4.4 of [7]. If A is m-accretive, the resolvents $J_\lambda := (I + \lambda A)^{-1} : X \to D(A)$ and the Yosida-approximations $A_\lambda := \lambda^{-1}(I - J_\lambda) : X \to X$ are well defined for all $\lambda > 0$. In particular, $A_\lambda x \in A(J_\lambda x)$ on X, all J_λ are nonexpansive mappings and $\lim_{\lambda\to 0+} J_\lambda x = x$ for every $x \in \overline{D(A)}$.

We shall use the following characterization of m-accretivity.

Lemma 1. *Let A be an accretive operator in X. Then A is m-accretive if and only if* $\mathrm{gr}(A)$ *is closed and*

$$\underline{\lim}_{h\to 0+} h^{-1}\rho(x + hz, R(I + hA)) = 0 \qquad \text{for all } x \in \overline{D(A)} \text{ and all } z \in X. \tag{2}$$

This is Theorem 5.2 in [10]. More about m-accretive operators on Banach spaces can be found e.g. in [2] or [4]; in the latter reference one can also find Lemma 1 which is Theorem 16.2 there.

(ii) Let us also recall some facts about u.s.c. multivalued maps; for more details see [7]. A multivalued map $F : D \subset X \to 2^X \setminus \emptyset$ is called upper semicontinuous (u.s.c. for short), if $F^{-1}(B) := \{x \in D \mid F(x) \cap B \neq \emptyset\}$ is closed in D, for all closed $B \subset X$. If F has compact values, u.s.c. is equivalent to: for every $\varepsilon > 0$ and $x_0 \in D$ there is $\delta = \delta(\varepsilon, x_0) > 0$ such that $F(x) \subset F(x_0) + B_\varepsilon(0)$ on $B_\delta(x_0) \cap D$. A multivalued map is said to be continuous if it is continuous w.r. to the Hausdorff metric d_H which is given by

$$d_H(A,B) = \max\{\sup_{x\in A} \rho(x,B), \sup_{x\in B} \rho(x,A)\}$$

for bounded $A, B \subset X$.

In case D is compact and F is u.s.c. with convex values, for every $\varepsilon > 0$, there exists a continuous $f_\varepsilon : D \to X$ such that

$$f_\varepsilon(x) \in F(B_\varepsilon(x) \cap D) + B_\varepsilon(0) \qquad \text{on } D;$$

see Proposition 1.1 in [7]. Finally, the following fixed point theorem is a special case of Theorem 11.5 in [7].

LEMMA 2. *Let X be a real Banach space, $\emptyset \neq D \subset X$ compact convex and $F : D \to 2^D \setminus \emptyset$ be* u.s.c. *with closed convex values. Then F has a fixed point.*

(iii) We also need the following criterion for weak relative compactness in $L^1_X(J)$.

LEMMA 3. *Let X be a Banach space, $J = [0, a] \subset \mathbb{R}$ and $W \subset L^1_X(J)$ be uniformly integrable. Suppose that there exist weakly relatively compact sets $C(t) \subset X$ such that $w(t) \in C(t)$ a.e. on J, for all $w \in W$. Then W is weakly relatively compact in $L^1_X(J)$.*

This is Corollary 2.6 in [8] specialized to Lebesgue measure.

3. Upper semicontinuous perturbations.

THEOREM 1. *Let X be a real Banach space, $A : D(A) \subset X \to 2^X \setminus \emptyset$ be m-accretive and $F : \overline{D(A)} \to 2^X \setminus \emptyset$ be* u.s.c. *with compact convex values such that $A + F$ is accretive. Then $A + F$ is m-accretive.*

Proof. Let $B = A + F$ with $D(B) := D(A)$. Then B has closed graph, since $(x_n, y_n) \in \mathrm{gr}(B)$ means $y_n = u_n + v_n$ with $u_n \in Ax_n$ and $v_n \in F(x_n)$, hence $(x_n, y_n) \to (x, y)$ implies $v_n \in F(x) + B_\varepsilon(0)$ for all $n \geq n_\varepsilon$ and therefore $v_{n_k} \to v \in F(x)$ for some subsequence (v_{n_k}) of (v_n), hence also $u_{n_k} \to u := y - v$ and $u \in Ax$ by closedness of $\mathrm{gr}(A)$.
Next, notice that in order to get (2) we may assume $z = 0$, since for any $z \in X$ the map F_z, defined by $F_z(x) := F(x) - \{z\}$ on $\overline{D(A)}$, has the same properties as F. So we are done by Lemma 1, if

$$\varliminf_{h \to 0+} h^{-1} \rho(x, R(I + hB)) = 0 \qquad \text{on } \overline{D(B)}. \tag{3}$$

Fix $x \in \overline{D(B)}$, let $h > 0$, $C := F(x)$ and $G(z) := F(J_h(x - hz))$ for $z \in X$ where $J_h = (I + hA)^{-1}$. Evidently, G is u.s.c. with compact convex values. Hence, given $\varepsilon > 0$, the approximation result mentioned in 2. (ii) yields a continuous $g_\varepsilon : C \to X$ such that $g_\varepsilon(z) \in G(B_\varepsilon(z) \cap C) + B_\varepsilon(0)$ on C. Let $G_\varepsilon(z) = P_C(g_\varepsilon(z))$ for $z \in C$, where $P_C(\cdot)$ is the metric projection onto C, i.e.

$$P_C(x) = \{y \in C \mid |x - y| = \rho(x, C)\} \qquad \text{on } X.$$

Then $G_\varepsilon : C \to 2^C \setminus \emptyset$ is also u.s.c. with compact convex values, since P_C has this properties. Therefore, G_ε has a fixed point $z_\varepsilon \in C$ by Lemma 2. Given $h_n \searrow 0$ and $\varepsilon_n \searrow 0$ we repeat the previous arguments to obtain fixed points z_n of the corresponding G_{ε_n}, i.e. we get a sequence $(z_n) \subset C$ such that

$$z_n \in P_C(y_n) \qquad \text{and} \qquad y_n \in F(J_{h_n}(x - h_n(B_{\varepsilon_n}(z_n) \cap C))) + B_{\varepsilon_n}(0).$$

In particular, there are $e_n, \widehat{e}_n \in B_{\varepsilon_n}(0)$ such that

$$y_n - e_n \in F(J_{h_n}(x - h_n \widehat{z}_n)) \qquad \text{with } \widehat{z}_n = z_n + \widehat{e}_n \in C. \tag{4}$$

Now $x_n := J_{h_n}(x - h_n \widehat{z}_n)$ satisfies $|x_n - x| \leq h_n |\widehat{z}_n| + |J_{h_n}(x) - x|$, i.e. $x_n \to x$ as $n \to \infty$. We may therefore assume $y_n \to y$ for some $y \in F(x)$. Without loss of generality we also have $z_n \to z$ for some $z \in C$, $z_n \in P_C(y_n)$ implies $z \in P_C(y)$, hence $P_C(y) = \{y\}$ yields $y_n - z_n \to 0$. Together with (4) this means $\widehat{z}_n \in F(x_n) + \widetilde{e}_n$ for some $\widetilde{e}_n \to 0$, hence

$x - h_n\widehat{z}_n = J_{h_n}(x - h_n\widehat{z}_n) + h_n A_{h_n}(x - h_n\widehat{z}_n)$ implies

$$x \in x_n + h_n(Ax_n + F(x_n) + \widetilde{e}_n),$$

i.e. (3) holds. □

Remarks 1. Specialized to the case of single-valued perturbations, the conditions on F become "$F : \overline{D(A)} \to X$ continuous such that $A+F$ is accretive". In this situation the result is known and, using Lemma 1, it was first proved in [10] where it is Theorem 5.3. Independently, the same result was obtained in [13] Theorem II, by means of locally Lipschitz approximations of F. The first result about continuous perturbations of m-accretive operators is Theorem 1 in [1], where the assumptions on F are $F : X \to X$ continuous and accretive. In the proof given there, it is shown that such an F is in fact s-accretive, which means

$$(F(x) - F(y), x - y)_- \geq 0 \qquad \text{for all } x, y \in X,$$

hence $A + F$ is accretive. Let us note that s-accretivity of F follows from the fact that $u' = -F(u)$, $u(0) = x$ has a unique C^1-solution on $\mathbb{R}_+$, for every $x \in X$. Hence $-F$ generates a semigroup of nonexpansive operators $S(t)$, given by $S(t)x := u(t;x)$, and therefore

$$\begin{aligned}(F(x) - F(y), x - y)_- &= \lim_{h\to 0+}\Big(-\frac{S(h)x - x}{h} + \frac{S(h)y - y}{h}, x - y\Big)_- \\ &\geq \underline{\lim}_{h\to 0+} h^{-1}|x - y|(|x - y| - |S(h)x - S(h)y|) \geq 0.\end{aligned}$$

In case $F : \overline{D(A)} \to X$ is continuous, accretive and satisfies the subtangential condition

$$\underline{\lim}_{h\to 0+} h^{-1}\rho(x + hF(x), \overline{D(A)}) = 0 \qquad \text{on } \overline{D(A)},$$

the same argument can be used to show that F is s-accretive, since $u' = -F(u)$, $u(0) = x$ has a unique C^1-solution for every $x \in \overline{D(A)}$; see Remark 3 below. Hence $A + F$ is m-accretive, given that A has this property. This is Theorem 2.8.1' in [12]. Without this additional boundary condition the result is not true; a counterexample is given in [13].

In the case of multivalued F the situation is worse, since accretivity of F is not sufficient then even if F is defined on all of X. This is shown by the following

Example 1. Let $X = \mathbb{R}^2$ with $|x|_0 = \max\{|x_1|, |x_2|\}$ and $A : D(A) \to 2^X \setminus \emptyset$ be given by $Ax = \mathbb{R} \times \{0\}$ on $D(A) = \{(s,s) \mid s \in \mathbb{R}\}$. Obviously, $R(I + \lambda A) = X$ for all $\lambda > 0$. Moreover A is accretive, since $x, y \in D_A$, $u \in Ax$, $v \in Ay$ means $x - y = (s,s)$ and $u - v = (h, 0)$ for some $s, h \in \mathbb{R}$, hence

$$(u - v, x - y)_+ = |s| \lim_{t\to 0+} t^{-1}(\max\{|s + th|, |s|\} - |s|) \geq 0.$$

Let $F : X \to 2^X \setminus \emptyset$ be defined by

$$F(x) = \begin{cases} \{(1,-1)\} & \text{if } x_1 > x_2, \\ \{(s,-s) \mid s \in [-1,1]\} & \text{if } x_1 = x_2, \\ \{(-1,1)\} & \text{if } x_1 < x_2. \end{cases}$$

Evidently, F is u.s.c. with compact convex values. Accretivity of F can also be checked in a straight forward way, but we omit the details, since this F is a special case of an example considered in [6]; see p. 296 there.

Now $A + F \equiv \mathbb{R} \times [-1, 1]$ on $D(A)$ which is not accretive, since e.g. $x = (1, 1)$, $y = (0, 0)$, $u = -x$, $v = y$ yield $(u - v, x - y)_+ = -|x|_0^2 = -1$.

2. For concrete applications, it would be useful to weaken the assumptions on F, since the values will often be only weakly compact and convex. We do not know how to prove a corresponding version of Theorem 1 in this case. If F itself is m-accretive one can of course try to apply results about the sum of m-accretive operators like Theorem 3 in [14], saying that $A + F$ is m-accretive given that A and F have this property, X and X^* are uniformly convex and $D(A) \cap \operatorname{int}(D(F)) \neq \emptyset$.

4. Differential inclusions with dissipative right-hand side. By means of Theorem 1 we are going to obtain strong solutions of (1) if, among other assumptions, the $F(t, \cdot)$ are u.s.c. and satisfy a condition of dissipative type. Here u is called strong solution of (1), if u is absolutely continuous with $u(0) = x_0$ and a.e. differentiable such that $u'(t) \in F(t, u(t))$ a.e. on J.

Let us first consider the special case when F is given by $F(t, x) = -G(x) + w(t)$ on $J \times X$, where $w \in L^1_X(J)$ and $G : X \to 2^X \setminus \emptyset$ is accretive and u.s.c. with compact convex values. By Theorem 1 with $A = 0$ we know that G is m-accretive, hence (1) has a unique mild solution by Theorem 4.6 in [4]; see Remark 4 below for the definition of "mild solution". But in this situation u is in fact a strong solution. This follows from the Proposition in [3], saying that every mild solution of $u' \in -G(u) + w(t)$ is also a strong solution if $w \in L^1_X(J)$ and G is weakly u.s.c. with closed domain and convex, weakly compact values; here weakly u.s.c. means $G^{-1}(A)$ closed for all weakly closed $A \subset X$.

The same conclusion holds if G is only ω-accretive, i.e.

$$(y - \overline{y}, x - \overline{x})_+ \geq -\omega|x - \overline{x}|^2 \qquad \text{for all } x, \overline{x} \in X,\ y \in G(x),\ \overline{y} \in G(\overline{x})$$

with some $\omega \in \mathbb{R}$. Notice that the result mentioned above can be applied with $G + \omega I$ instead of G and therefore the usual fixed point approach yields a strong solution for (1) with $F(t, x) = -G(x) + w(t)$. Let us record this information for later use.

Lemma 4. *Let X be a real Banach space, $G : X \to 2^X \setminus \emptyset$ be ω-accretive for some $\omega \in \mathbb{R}$ and* u.s.c. *with compact convex values, $J = [0, a] \subset \mathbb{R}$ and $w \in L^1_X(J)$. Then the Cauchy problem*

$$u' \in -G(u) + w(t) \qquad \text{on } J,\ u(0) = x_0$$

has a unique strong solution, for every $x_0 \in X$.

We shall use Lemma 4 to prove a more general result which allows the right-hand side F to depend on (t, x) in a more complicated way. But still we need a rather strong assumption concerning the t-dependence. In the subsequent theorem we suppose that for every $\eta > 0$ there exists a closed $J_\eta \subset J$ with $\mu(J \setminus J_\eta) \leq \eta$ such that the family $\{F(\cdot, x)|_{J_\eta} \mid x \in X\}$ is locally equicontinuous, i.e. for every $x_0 \in X$ there is $\delta = \delta(\eta, x_0) >$

0 such that the $F(\cdot,x)|_{J_\eta}$ are equicontinuous for all $x\in B_\delta(x_0)$. If this holds we say that $\{F(\cdot,x)\mid x\in X\}$ is almost locally equicontinuous. Now we can prove

THEOREM 2. *Let X be a real Banach space, $J=[0,a]\subset\mathbb{R}$ and let $F:J\times X\to 2^X\setminus\emptyset$ have compact convex values such that the following conditions hold.*

(a) $\|F(t,x)\|:=\sup\{|y|\mid y\in F(t,x)\}\le c(t)(1+|x|)$ *on* $J\times X$ *with* $c\in L^1(J)$.
(b) $(y-\overline{y},x-\overline{x})_-\le k(t)|x-\overline{x}|^2$ *for all* $t\in J$, $x,\overline{x}\in X$, $y\in F(t,x),\overline{y}\in F(t,\overline{x})$ *with* $k\in L^1(J)$.
(c) $F(t,\cdot)$ *is* u.s.c. *for almost all* $t\in J$.
(d) *The family of maps* $\{F(\cdot,x):x\in X\}$ *is almost locally equicontinuous.*

Then initial value problem (1) *has a unique strong solution on* J.

Proof. (1). Let us first reduce to the case $c(t)\equiv k(t)\equiv 1$. For this purpose define $\varphi\in L^1(J)$ by $\varphi=\max\{1,c,k\}$. The map $t\to\int_0^t\varphi(s)\,ds$ from J to $\widetilde{J}:=[0,|\varphi|_1]$ is continuous and strictly increasing. Let ϕ be its inverse and define $\widetilde{F}:\widetilde{J}\times X\to 2^X\setminus\emptyset$ by

$$\widetilde{F}(t,x)=\frac{1}{\varphi(\phi(t))}F(\phi(t),x)\qquad\text{for } (t,x)\in\widetilde{J}\times X.$$

Evidently, u is a solution of (1) iff $v(t):=u(\phi(t))$ is a solution of (1) with F and J replaced by $\widetilde{F}$ and $\widetilde{J}$, respectively. It is easy to check that $\widetilde{F}$ has properties (a)–(c) with $c(t)\equiv k(t)\equiv 1$ on $\widetilde{J}\times X$. To see that $\widetilde{F}$ also satisfies (d), let $\eta>0$ be given. Then there is $\sigma=\sigma(\eta)>0$ such that $\mu(A)\le\sigma$ implies $\int_A\varphi(t)\,dt\le\eta$ for every Lebesgue measurable $A\subset J$. Exploitation of condition (d) for F yields a closed $J_\sigma\subset J$ with $\mu(J\setminus J_\sigma)\le\sigma$ such that the family $\{F(\cdot,x)|_{J_\sigma}\mid x\in X\}$ is locally equicontinuous and $F(t,\cdot)$ is u.s.c. for all $t\in J_\sigma$. Since φ has the Lusin property, we may also assume that $\varphi|_{J_\sigma}$ is continuous. Let $\widetilde{J}_\eta:=\phi^{-1}(J_\sigma)$. Using the fact that J_σ is closed it is easy to check that $\mu(\widetilde{J}_\eta)=\int_{J_\eta}\varphi(t)\,dt$, hence $\mu(\widetilde{J}\setminus\widetilde{J}_\eta)\le\eta$. Now we are done, since

$$d_H(\widetilde{F}(t,x),\widetilde{F}(s,x))\le\left|\frac{1}{\varphi(\phi(t))}-\frac{1}{\varphi(\phi(s))}\right|(1+|x|)+d_H(F(\phi(t),x),F(\phi(s),x)),$$

$\{F(\phi(\cdot),x)|_{\widetilde{J}_\eta}\mid x\in X\}$ is locally equicontinuous and $(\frac{1}{\varphi(\phi(\cdot))})|_{\widetilde{J}_\eta}$ is uniformly continuous. In the sequel we will denote $\widetilde{F}$ and $\widetilde{J}$ by F and J again.

(2). Given $\eta>0$, let $J_\eta\subset J$ be closed with $\mu(J\setminus J_\eta)\le\eta$ such that the family of maps $\{F(\cdot,x)|_{J_\eta}\mid x\in X\}$ is locally equicontinuous, where we may assume $\{0,a\}\subset J_\eta$. Then $J\setminus J_\eta=\bigcup_{n\ge1}(\alpha_n,\beta_n)$ for disjoint $(\alpha_n,\beta_n)\subset J$, since $J\setminus J_\eta$ is open. Let $F_\eta:J\times X\to 2^X\setminus\emptyset$ be defined by

$$F_\eta(t,x)=\begin{cases}F(t,x) & \text{if } t\in J_\eta,\\ F(\alpha_n,x) & \text{if } t\in(\alpha_n,\beta_n)\text{ for some } n\ge1.\end{cases}$$

Then F_η has compact convex values, satisfies (a), (b) with $c(t)\equiv k(t)\equiv 1$ and $F_\eta(t,\cdot)$ is u.s.c. for all $t\in J$. We want to show that (1) with F_η instead of F has a strong solution. For this purpose let us first prove that (1) with F replaced by F_η has an ε-approximate solution $u = u_\varepsilon$ for every $\varepsilon \in (0,1)$, by which we mean

$$u(t) = x_0 + \int_0^t w(s)\,ds \quad \text{on } J \text{ with } w \in L^1_X(J) \qquad (5)$$
$$\text{such that } \int_0^a \rho(w(t), F_\eta(t,u(t)))\,dt \le \varepsilon a.$$

This will be done by using Zorn's Lemma. But notice first that there is $R > 1$ such that every u satisfying (5) for some $\varepsilon \in (0,1)$ has $|u|_0 \le R-1$. Therefore, we can obtain approximate solutions such that also $|u'(t)| \le R$ a.e. on J. Consider the set

$$M = \{(u,h) \mid h \in (0,a],\ u : [0,h] \to X \text{ satisfies (5) with } J \text{ replaced by } [0,h] \text{ such that } |w(t)| \le R \text{ a.e. on } [0,h]\},$$

equipped with the partial ordering $(u,h) \le (\overline{u},\overline{h})$ if $h \le \overline{h}$ and $u(t) = \overline{u}(t)$ on $[0,h]$. Let us show $M \ne \emptyset$. There is $\delta = \delta(\eta, x_0) > 0$ such that $\{F(\,\cdot\,,x)_{|J_\eta} \mid x \in B_\delta(x_0)\}$ is equicontinuous. Hence there is $h_0 > 0$ such that $d_H(F(0,x), F(t,x)) \le \varepsilon$ for every $t \in [0,h_0] \cap J_\eta$ and every $x \in B_\delta(x_0)$. By the definition of F_η this implies

$$d_H(F_\eta(0,x), F_\eta(t,x)) \le \varepsilon \qquad \text{for every } t \in [0,h_0],\ x \in B_\delta(x_0). \qquad (6)$$

Let u be the strong solution of the initial value problem

$$u' \in F_\eta(0,u) \qquad \text{on } J,\ u(0) = x_0,$$

which exists due to Lemma 4 with $G := -F_\eta(0,\cdot)$ and $w := 0$. Since there is $h \in (0,h_0]$ such that $|u(t) - x_0| \le \delta$ on $[0,h]$, estimate (6) implies

$$\int_0^h \rho(u'(t), F_\eta(t,u(t)))\,dt \le \int_0^h d_H(F_\eta(0,u(t)), F_\eta(t,u(t)))\,dt \le \varepsilon h.$$

Hence $|u(t)| \le R-1$ on $[0,h]$, which implies $|u'(t)| \le ||F_\eta(0,u(t))|| \le R$ a.e. on $[0,h]$, and therefore $(u,h) \in M$. It is obvious that every ordered subset of M has an upper bound, hence M has a maximal element (u^*,h^*) by Zorn's Lemma. Moreover $h^* = a$ since otherwise we may repeat the argument given above with $(h^*, u^*(h^*))$ instead of $(0,x_0)$ to get an ε-approximate solution on $[0,h^*+h]$ which extends u^*, a contradiction.

(3). Now let $(\varepsilon_k) \subset (0,1)$ satisfy $\varepsilon_k \to 0+$ and u_k be ε_k-approximate solutions of (1) for F_η. Then, for fixed m and n, $\psi(t) = |u_n(t) - u_m(t)|$ satisfies $\psi(0) = 0$ and

$$\psi(t)\psi'(t) = (u_n'(t) - u_m'(t), u_n(t) - u_m(t))_- \le (\rho_n(t) + \rho_m(t))\psi(t) + \psi(t)^2 \qquad \text{a.e. on } J,$$

where $\rho_k(t) = \rho(u_k'(t), F_\eta(t,u_k(t)))$ on J. This implies $e^{-a}|\psi|_0 \le |\rho_n|_1 + |\rho_m|_1 \le a(\varepsilon_n + \varepsilon_m)$. Consequently, (u_k) is a Cauchy sequence in $C_X(J)$, hence $|u_k - u|_0 \to 0$ for some $u \in C_X(J)$ with $u(0) = x_0$; notice that (u_k) is equicontinuous. Since $F_\eta(t,\cdot)$ is u.s.c. with compact values for all $t \in J$, the sets $F_\eta(t, \overline{\{u_k(t) \mid k \ge 1\}})$ are compact. By Lemma 3 we may therefore assume $w_k = u_k' \rightharpoonup w$ for some $w \in L^1_X(J)$. Together with $u_k \to u$ in $C_X(J)$ this implies $u(t) = x_0 + \int_0^t w(s)\,ds$ on J. By Mazur's Theorem there are $\overline{w}_k \in \operatorname{conv}\{w_j \mid j \ge k\}$ with $\overline{w}_k \to w$ in $L^1_X(J)$, hence w.l.o.g. $\overline{w}_k(t) \to w(t)$ a.e. on J by passing to a certain subsequence. Let $J_0 = \{t \in J \mid w_k(t) \in F_\eta(t,u_k(t))$ for all $k \ge 1$, $\overline{w}_k(t) \to w(t)\}$ and $t \in J_0$. Then, given $\sigma > 0$, we have $w_k(t) \in F_\eta(t,u(t)) + B_\sigma(0)$ for all large k, hence the same for $\overline{w}_k(t)$. Evidently, this implies $w(t) \in F_\eta(t, u(t))$ on J_0, hence a.e. on J and therefore u is a strong solution of (1) with F_η.

4. Let $\eta_k \searrow 0$ and $J_k := J_{\eta_k}$, $F_k := F_{\eta_k}$ be given as in step 2, where we may assume $J_k \subset J_{k+1}$ for $k \geq 1$. By the previous step, initial value problem (1) with F_k instead of F has a solution u_k for every $k \geq 1$. Moreover, $|u_k|_0 \leq R$ for all $k \geq 1$ with some $R > 0$, since all F_k satisfy (a) with $c(t) \equiv 1$. For fixed $m \geq 1$, we have $F_n = F$ on $J_m \times X$ for all $n \geq m$, hence $\psi(t) := |u_n(t) - u_m(t)|$ has

$$\psi'(t) \leq \psi(t)\chi_{J_m}(t) + 2(1+R)\chi_{J\setminus J_m}(t) \qquad \text{a.e. on } J,\ \psi(0) = 0,$$

for those n. Therefore, application of Gronwall's Lemma shows that (u_k) is Cauchy in $C_X(J)$. Hence $|u_k - u|_0 \to 0$ for some $u \in C_X(J)$ with $u(0) = x_0$, and $u'(t) \in F(t, u(t))$ a.e. on J can be seen as in step 3. So, we have shown that (1) has a strong solution. Evidently we are done, since uniqueness is an obvious consequence of (b). □

Additional information is contained in the following

Remarks 3. If X is a real Hilbert space condition (d) can be replaced by "$F(\,\cdot\,, x)$ has a strongly measurable selection" and the values of F need only be closed convex. This is Theorem 10.5 in [7], and Theorem 2 is a first step to extend this result to general Banach spaces. Therefore this gives a partial answer to Problem 10.6 in [7].
Let us also mention that, specialized to the single-valued case, conditions (a) and (d) hold in case F is almost continuous, which is the same as "F is measurable in t and continuous in x" for separable X. For continuous single-valued F a corresponding version of Theorem 2 holds even if the maps $F(t, \cdot\,)$ are only defined on time-dependent sets $D(t) \subset X$, given that $\mathrm{gr}(D)$ is closed from the left and F also satisfies the subtangential condition

$$\varliminf_{h \to 0+} h^{-1} \rho(x + hF(t, x), D(t+h)) = 0 \qquad \text{for all } t \in [0, a),\ x \in D(t);$$

see Theorem 3 in [9]. For multivalued and almost u.s.c. right-hand sides, such an existence result under time-dependent constraints holds if the condition (b) of dissipative type is replaced by a certain compactness assumption. The details concerning the latter case can be found in [5].

4. A different approach to prove a result like Theorem 2 is to get first the existence of a mild solution and then to show that it is in fact a strong solution; remember the proof of Lemma 4. By a mild solution u of (1) one means $u \in C_X(J)$ being the uniform limit of a sequence of approximate solutions u_m (corresponding to a sequence $\varepsilon_m \to 0+$) which solve an implicit difference scheme. More precisely, v is such an approximate solution corresponding to $\varepsilon > 0$ if there are $x_1, \ldots, x_{n+1} \in X$ and a partition $0 = t_0 < t_1 < \ldots < t_n \leq t_{n+1} = a$ of J such that, for all $k = 0, \ldots, n$, one has:

$$t_{k+1} - t_k \leq \varepsilon, \qquad v(t) = x_k \text{ on } [t_k, t_{k+1}) \text{ and}$$
$$\frac{x_{k+1} - x_k}{t_{k+1} - t_k} \in F(t_{k+1}, x_{k+1}) + z_k \qquad \text{with } |z_k| \leq \varepsilon.$$

Now, under the conditions of Theorem 2 where w.l.o.g. $k(t) \equiv \omega$, it is easy to see that we get such approximate solutions, since almost all $-F(t, \cdot\,)$ are m-ω-accretive. In fact one only needs condition (3) with B replaced by $F(t+h,\,\cdot\,)$; see e.g. Chapter 1.3.5 in [12]. Then the main problem is to obtain the uniform convergence of (v_m), and one may try to apply results about time-dependent ω-accretive operators like Theorem 3.5 in [11]. Specialized to the situation under consideration, this theorem guarantees that $|v_m - u|_0 \to 0$ for some $u \in C_X(J)$, given that

$$(y, x-\overline{x})_- + (-\overline{y}, x-\overline{x})_- \le \omega |x-\overline{x}|^2 + \varphi(t,\overline{t})|x-\overline{x}|$$

for all $t,\overline{t} \in J$, $x,\overline{x} \in X$, $y \in F(t,x)$ and $\overline{y} \in F(\overline{t},\overline{x})$ with some $\omega \ge 0$ and a bounded upper semicontinuous symmetric function $\varphi : J \times J \to \mathbb{R}_+$ satisfying

$$\lim_{r\to 0+} \sup\{\varphi(t,\overline{t}) \mid |t-\overline{t}| \le r\} = 0 \qquad \text{on } J \times J.$$

It is sufficient that this condition holds locally, i.e. for all $x,\overline{x} \in B_\delta(\widehat{x})$ for every $\widehat{x} \in X$ and some $\delta = \delta(\widehat{x}) > 0$, where ω and φ may depend on $B_\delta(\widehat{x})$. In the situation described in Theorem 2 it is not clear if this condition is satisfied, but it holds if $k(t) \equiv \omega$ in (b) and the maps $F(\cdot,x)$ are locally equicontinuous. In this case, once the existence of mild solutions of (1) is established, the proof is easily finished: given $v_m \to u$ in $C_X(J)$, consider functions u_m being linear on each $[t_k^m, t_{k+1}^m]$ with $u_m(t_k^m) := v_m(t_k^m)$. Evidently $|u_m - u|_0 \to 0$. Then $u_m' \rightharpoonup w$ in $L_X^1(J)$ and $u'(t) = w(t) \in F(t,u(t))$ a.e. on J can be proved similar to step 3 of the proof of Theorem 2.

Acknowledgment. The first part of this work was completed while the author was visiting the Université de Franche-Comté (Besançon). It is a pleasure to thank the Equipe de Mathématiques de Besançon for its kind hospitality, in particular Professor Ph. Benilan for helpful discussions and suggestions.

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