

A SUFFICIENT CONDITION FOR THE EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS OF DIFFERENTIAL INCLUSIONS

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1. Introduction. Let K be a convex cone in \mathbb{R}^n and $\Phi : [0, T] \times K \rightarrow \mathbb{R}^n$ a set-valued map. We shall be concerned with the existence of solutions of the periodic problem

$$(1) \quad \begin{cases} x'(t) \in \Phi(t, x(t)), \\ x(0) = x(T). \end{cases}$$

It is known that under regularity conditions on Φ and a certain Nagumo-type boundary condition a solution of (1) exists (see e.g. [2]).

In this note, following ideas of [4] and using results from [11], we prove the existence of two or more periodic trajectories if additional conditions are satisfied: Our method requires that certain complementary potential functions (comp. [9], [7]) in phase space can be detected.

In our approach we apply the topological fixed point index theory to the Poincaré (also called Poincaré-Andronov) translation operator associated with (1). This well-known technique was developed by M. A. Krasnosel'skiĭ in the single valued case, i.e. when we have unique solvability of the respective initial value problem (see [9]). For applications of the set-valued Poincaré map we recommend [4] as a good survey.

We shall proceed as follows: In section 2 we present the fixed point index theory for the so-called decomposable mappings (see our definition (2.3), comp. [4]) as it is developed in [1] and obtain some additional information necessary for our later work. In the third section we prove some abstract multiple fixed point results for compact decomposable mappings defined on cones of arbitrary ordered normed spaces. We choose this general setting since we believe that the results given there might be of independent interest. Our methods are somewhat connected with the theory given in [10] (for single valued maps), where applications to certain boundary value problems and Hammerstein integral

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operators are given. Finally in section 4 we state our main results on the existence of periodic solutions.

We conclude the introduction with some notational conventions and definitions.

By a *space* we always mean a metric space. If X is a space, $A \subset X$ and $\varepsilon > 0$, then by \overline{A} and ∂A we denote closure and boundary of the set A respectively and by $N_\varepsilon(A)$ we denote an open ε -neighborhood of A . By a *map* we mean a continuous transformation of spaces and by a *set-valued map* – an upper semi-continuous multi-valued map with compact values. For maps we reserve Latin letters: f, g, h, \dots and for set-valued maps, Greek letters: $\varphi, \psi, \chi, \dots$ etc. We say that a set-valued map $\varphi : X \rightarrow Y$ is *compact* if the set $\overline{\varphi(X)}$ is compact in Y and it is *completely continuous* if it maps bounded sets onto relatively compact ones.

2. The fixed point index. The main tool in our consideration stand the topological fixed point index as it is introduced in [5] and developed in [1]. All the proofs which are not included here are given in [1]. We would like to mention that this fixed point index can be constructed by using the technique of single valued approximation on the graph.

We consider maps having the so-called proximally ∞ -connected values. After [3] we recall this notion.

DEFINITION 2.1. A compact subset K of a space X is *proximally ∞ -connected* if, for each $\varepsilon > 0$, there is $0 < \delta \leq \varepsilon$ such that the inclusion $N_\delta(K) \hookrightarrow N_\varepsilon(K)$ induces the trivial homomorphism

$$\pi_n(N_\delta(K)) \rightarrow \pi_n(N_\varepsilon(K))$$

for any $n \geq 0$ (we suppress the base points from the notations since they are not necessary).

Remark 2.2. In [5] a whole list of examples of proximally ∞ -connected sets is given. In particular any R_δ -set (i.e. the intersection of a decreasing sequence of compact AR's, see [8]) lying in an ANR space is proximally ∞ -connected. This example will be important for us in the last section.

DEFINITION 2.3.

- (i) (comp. [5]) If $\varphi : X \rightarrow Y$ is a set-valued map, then we say that φ belongs to the class J if, for any $x \in X$, $\varphi(x)$ is proximally ∞ -connected.
- (ii) A set-valued map $\varphi : X \rightarrow Y$ is said to be *decomposable*, provided there is a factorisation

$$(2) \quad D_\varphi : X = X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} X_n = Y,$$

($n = n(\varphi)$ depends on φ) where φ_i are J -maps, $1 \leq i \leq n$, X_0, \dots, X_n are ANR's such that $\varphi = \varphi_n \circ \dots \circ \varphi_1$. In this case D_φ is a *decomposition* of φ .

Now let $\varphi : X \rightarrow X$ be compact, decomposable with a decomposition

$$(3) \quad D_\varphi : X = X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} X_n = X$$

and W be an open subset of X such that $\text{Fix}(\varphi) \cap \partial W = \emptyset$. There exists a *fixed point*

index

$$\text{Ind}(X, D_\varphi, W) \in \mathbb{Z}$$

for the decomposition D_φ of φ on W .

We have indicated by the notation, that Ind depends not only on φ but also on the decomposition D_φ involved. On the other hand we have the following result:

PROPOSITION 2.4 *Let φ be compact, decomposable with decomposition D_φ (see (3)), W an open subset of X and $\text{Fix}(\varphi) \cap \partial W = \emptyset$. Let*

$$D' : X = X'_0 \xrightarrow{\varphi'_1} X'_1 \xrightarrow{\varphi'_2} \dots \xrightarrow{\varphi'_n} X'_n = X$$

be such that for each $1 \leq i \leq n$, there is $h_i : X_i \rightarrow X'_i$ with $h_0 = h_n = \text{id}_X$ and the diagram

$$\begin{array}{ccc} X_{i-1} & \xrightarrow{\varphi_i} & X_i \\ \downarrow h_{i-1} & & \downarrow h_i \\ X'_{i-1} & \xrightarrow{\varphi'_i} & X'_i \end{array}$$

commutes (i.e. $\varphi'_i \circ h_{i-1} = h_i \circ \varphi_i$) for $1 \leq i \leq n$. Then

$$\text{Ind}(X, D_\varphi, W) = \text{Ind}(X, D', W).$$

□

If decompositions are as in the proposition we say that D' *dominates* over D_φ (written $D' > D_\varphi$).

Below we shall state the properties of the fixed point index, which we use in the sequel. First we need the notion of homotopy for decompositions.

DEFINITION 2.5. Let $\varphi, \psi : X \rightarrow Y$ be decomposable with decompositions D_φ (see (2)) and

$$D_\psi : X = X'_0 \xrightarrow{\psi_1} X'_1 \xrightarrow{\psi_2} \dots \xrightarrow{\psi_m} X'_m = Y.$$

We say that the decompositions D_φ and D_ψ are *homotopic* if $n = m$, $X_i = X'_i$, and there is a map $\chi_i \in J(X_{i-1} \times I, X_i)$ with $\chi_i(\cdot, 0) = \varphi_i$, $\chi_i(\cdot, 1) = \psi_i$, $1 \leq i \leq n$. The set-valued map $\chi : X \times I \rightarrow Y$ given by

$$\chi(x, t) := \chi_n(\bar{\chi}_{n-1}(\dots \bar{\chi}_1(x, t) \dots)),$$

where $\bar{\chi}_i(x, t) = \chi_i(x, t) \times \{t\}$ for $x \in X_{i-1}$, $t \in I$, $1 \leq i \leq n-1$, is called a *homotopy* (observe that $\chi(\cdot, 0) = \varphi$, $\chi(\cdot, 1) = \psi$). We say that the decompositions D_φ , D_ψ are *compactly homotopic*, if χ is compact.

THEOREM 2.6. *Let φ be compact, decomposable with decomposition (3), W an open subset of X and $\text{Fix}(\varphi) \cap \partial W = \emptyset$.*

- (i) (Existence) *If $\text{Ind}(X, D_\varphi, W) \neq 0$ then $\text{Fix}(\varphi) \cap W \neq \emptyset$.*
- (ii) (Additivity) *If $\text{Fix}(\varphi) \cap W \subset \bigcup_{j=1}^k W_j$, where W_j , $1 \leq j \leq k$, are disjoint open*

subsets in W , then

$$\text{Ind}(X, D_\varphi, W) = \sum_{j=1}^k \text{Ind}(X, D_\varphi, W_j).$$

(iii) (Homotopy) If ψ is compact, decomposable with a decomposition

$$(4) \quad D_\psi : X = X'_0 \xrightarrow{\psi_1} X'_1 \xrightarrow{\psi_2} \dots \xrightarrow{\psi_m} X'_m = X$$

compactly homotopic to D_φ and the homotopy $\chi : X \times I \rightarrow X$ is such that $x \notin \chi(x, t)$ for $x \in \partial W$, $t \in I$, then

$$\text{Ind}(X, D_\varphi, W) = \text{Ind}(X, D_\psi, W).$$

(iv) (Units) If φ is constant, i.e. for any $x \in X$, $\varphi(x) = T$ and $T \subset X$ is proximally ∞ -connected, then

$$\text{Ind}(X, D_\varphi, W) = \begin{cases} 1 & \text{if } W \cap T \neq \emptyset, \\ 0 & \text{if } W \cap T = \emptyset. \end{cases}$$

□

COROLLARY 2.7. Let X be a closed convex subset of a normed space (and hence an ANR), W an open subset of X and $\varphi, \psi : X \rightarrow X$ compact, decomposable with decompositions D_φ, D_ψ (see (3), (4)). Let the set-valued map $\lambda : X \times I \rightarrow X$ given by

$$\lambda(x, t) := (1 - t)\varphi(x) + t\psi(x)$$

be such that

$$(5) \quad x \notin \lambda(x, t)$$

for $x \in \partial W$, $t \in I$. Then $\text{Ind}(X, D_\varphi, W) = \text{Ind}(X, D_\psi, W)$.

Observe that λ does not establish a homotopy of the decompositions D_φ, D_ψ and thus (2.6)(iii) can not be applied directly.

Proof. We may assume without loss of generality $m \leq n$. Define $X'_{m+1}, \dots, X'_n := X$, $\psi_{m+1}, \dots, \psi_n := id_X$. Consider the decompositions

$$D_1 : X \xrightarrow{d} X \times X \xrightarrow{\varphi_1 \times \psi_1} X_1 \times X'_1 \xrightarrow{\varphi_2 \times \psi_2} \dots \xrightarrow{\varphi_n \times \psi_n} X \times X \xrightarrow{p_0} X,$$

$$D_2 : X \xrightarrow{d} X \times X \xrightarrow{\varphi_1 \times \psi_1} X_1 \times X'_1 \xrightarrow{\varphi_2 \times \psi_2} \dots \xrightarrow{\varphi_n \times \psi_n} X \times X \xrightarrow{q_0} X,$$

where $d(x) := (x, x)$, $p_0(x, y) := x$, $q_0(x, y) := y$. The J -maps

$$D : X \times I \rightarrow X \times X, \quad D(x, t) := d(x),$$

$$\chi_i : X_{i-1} \times X'_{i-1} \times I \rightarrow X_i \times X'_i, \quad \chi_i(x, y, t) := (\varphi_i \times \psi_i)(x, y), \quad 1 \leq i \leq n,$$

$$\chi_{n+1} : X \times X \times I \rightarrow X, \quad \chi_{n+1}(x, y, t) := (1 - t)x + ty.$$

show that D_1 and D_2 are homotopic in the sense of definition (2.5). But since

$$\lambda(x, t) = \chi_{n+1}(\bar{\chi}_n(\dots \bar{\chi}_1(\bar{D}(x, t) \dots)),$$

and λ clearly is compact, it follows from (5) and (2.6)(iii) that

$$\text{Ind}(X, D_1, W) = \text{Ind}(X, D_2, W).$$

The proof is finished if we show

$$\begin{aligned} \text{Ind}(X, D_\varphi, W) &= \text{Ind}(X, D_1, W), \\ \text{Ind}(X, D_\psi, W) &= \text{Ind}(X, D_2, W). \end{aligned}$$

To this end, define maps $p_i : X_i \times X'_i \rightarrow X_i$, $p_i(x, y) := x$ for $1 \leq i \leq n$. Since $p_0 \circ d = id_X$, $p_i \circ (\varphi_i \times \psi_i) = \varphi_i \circ p_{i-1}$ and $p_0 = id_X \circ p_n$ it follows that $D_\varphi > D_1$ ($D_\psi > D_2$ is shown analogously). Hence, the above equalities follow from (2.4). \square

COROLLARY 2.8. *Let X be a closed convex subset of a normed space, $\varphi : X \rightarrow X$ compact, decomposable with decomposition D_φ . Then $\text{Ind}(X, D_\varphi, X) = 1$. \square*

3. Multiple positive fixed points on ordered normed spaces. The purpose of this section is to prove some of the abstract results which will be applied in the problem of periodic solutions of differential inclusions.

We recall that a closed convex subset K of a normed space E is called a *cone* if $tx \in K$ for each $x \in K$ and $t \geq 0$ and if $x \in K$ and $-x \in K$, then $x = 0$. A cone K induces a partial ordering \leq in E by $x \leq y$ if and only if $y - x \in K$.

By a convex (resp. concave) functional a (resp. b) on K we mean a mapping $a : K \rightarrow \mathbb{R}$ (resp. $b : K \rightarrow \mathbb{R}$) such that

$$\begin{aligned} a(tx + (1-t)y) &\leq ta(x) + (1-t)a(y) && \text{for } x, y \in K, t \in I \\ \text{(resp. } b(tx + (1-t)y) &\geq tb(x) + (1-t)b(y) && \text{for } x, y \in K, t \in I). \end{aligned}$$

LEMMA 3.1. *Let $\varphi : K \rightarrow K$ be a compact, decomposable map with the decomposition*

$$D_\varphi : K = X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} X_n = K.$$

Let a, b be convex, concave functionals on K respectively and r, s numbers such that

- (a) *the set $U := \{x \in K \mid a(x) < r, b(x) > s\}$ is nonempty,*
- (b) *$a(x) = r, b(x) \geq s, y \in \varphi(x) \Rightarrow a(y) < r,$*
- (c) *$b(x) = s, a(x) \leq r, y \in \varphi(x) \Rightarrow b(y) > s.$*

Then $\text{Ind}(K, D_\varphi, U) = 1$.

Proof. The set U is obviously open. Since $x \in \partial U$ implies either $a(x) = r$ and $b(x) \geq s$ or $a(x) \leq r$ and $b(x) = s$, we see, by (b) and (c), that $x \notin \varphi(x)$. It follows that $\text{Ind}(K, D_\varphi, U)$ is defined.

Take a point $x_0 \in U$ and the obvious decomposition $D_{x_0} : K \xrightarrow{x_0} K$ of the constant map sending each point to x_0 . Consider

$$\lambda : K \times I \rightarrow K, \lambda(x, t) := (1-t)\varphi(x) + tx_0.$$

Suppose that $x \in \lambda(x, t)$ for some $x \in \partial U, t \in I$. Then there is $y \in \varphi(x)$ such that $x = (1-t)y + tx_0$. Now, if $a(x) = r$ and $b(x) \geq s$ then

$$r = a(x) = a((1-t)y + tx_0) \leq (1-t)a(y) + ta(x_0) < r,$$

by convexity of a and (b). If on the other hand $a(x) \leq r$ and $b(x) = s$, then

$$s = b(x) = b((1-t)y + tx_0) \geq (1-t)b(y) + tb(x_0) > s.$$

Hence in both cases we obtain a contradiction. Thus we have shown $x \notin \lambda(x, t)$ for each $(x, t) \in \partial U \times I$ and by (2.7) we obtain,

$$\text{Ind}(K, D_\varphi, U) = \text{Ind}(K, D_{x_0}, U).$$

Finally using property (2.6)(iv) we see that $\text{Ind}(K, D_{x_0}, U) = 1$. \square

THEOREM 3.2. *Let φ be as above and $m \geq 2$. Let a_1, \dots, a_m be convex functionals, b_1, \dots, b_m concave functionals on K and numbers $r_1, \dots, r_m, s_1, \dots, s_m$ be given such that:*

- (a) $a_m(x) \leq a_{m-1}(x) \leq \dots \leq a_1(x), b_m(x) \leq b_{m-1}(x) \leq \dots \leq b_1(x)$ for $x \in K$,
- (b) $r_1 \leq r_2 \leq \dots \leq r_m, s_1 \leq s_2 \leq \dots \leq s_m$,
- (c) $b_i(x) \leq a_{i-1}(x)$ for $x \in K, r_{i-1} \leq s_i, 2 \leq i \leq m$,
- (d) the sets $U_i := \{x \in K \mid a_i(x) < r_i, b_i(x) > s_i\}, 1 \leq i \leq m$, are nonempty,
- (e) $a_i(x) = r_i, b_i(x) \geq s_i, y \in \varphi(x) \Rightarrow a_i(y) < r_i, 1 \leq i \leq m$,
- (f) $b_i(x) = s_i, a_i(x) \leq r_i, y \in \varphi(x) \Rightarrow b_i(y) > s_i, 1 \leq i \leq m$.

Then φ has at least $m + 1$ fixed points.

Proof. By lemma (3.1), using (d), (e) and (f) above, we have

$$(6) \quad \text{Ind}(K, D_\varphi, U_i) = 1, \quad 1 \leq i \leq m.$$

Thus, by the existence property (2.6)(i) of Ind the existence of m fixed points is established, if we show that the U_i 's are disjoint. Let U_i be given. Then, if $1 \leq j \leq i - 1$ we see that if $x \in U_j$ then $a_{i-1}(x) \leq a_j(x) < r_j \leq r_{i-1}$ by (a) and (b). But, by (c), we have that $b_i(x) \leq a_{i-1}(x) < r_{i-1} \leq s_i$. It follows that if $x \in U_j$ then $b_i(x) < s_i$ and, hence $x \notin U_i$. If $i + 1 \leq j \leq m$ we conclude the same from $b_{i+1}(x) \geq b_j(x) > s_j \geq s_{i+1}$ and $r_i \leq s_{i+1} < b_{i+1}(x) \leq a_i(x)$.

Now consider $U := K \setminus (\overline{U_1 \cup \dots \cup U_m})$. Then U is an open set and U, U_1, \dots, U_m are disjoint. Moreover, since $K \setminus (U \cup U_1 \cup \dots \cup U_m) = \partial U_1 \cup \dots \cup \partial U_m$ and there are no fixed points on $\partial U_i, 1 \leq i \leq m$, we see that $\text{Fix}(\varphi) \subset U \cup U_1 \cup \dots \cup U_m$. Hence, by additivity (2.6)(ii), and (2.8) we obtain

$$1 = \text{Ind}(K, D_\varphi, K) = \text{Ind}(K, D_\varphi, U) + \sum_{i=1}^m \text{Ind}(K, D_\varphi, U_i).$$

By virtue of (6),

$$1 - m = \text{Ind}(K, D_\varphi, U)$$

and since $m \geq 2$ we see that

$$\text{Ind}(K, D_\varphi, U) \neq 0,$$

so that φ has another fixed point in U . It follows that φ has at least $m + 1$ fixed points. \square

By strengthening the assumptions a bit we obtain additional intermediate fixed points.

THEOREM 3.3. *Let all the conditions of theorem (3.2) be fulfilled, but replace (e) and (f) by*

- (e') $a_i(x) = r_i, y \in \varphi(x) \Rightarrow a_i(y) < r_i, 1 \leq i \leq m$,
- (f') $b_i(x) = s_i, a_m(x) \leq r_m, y \in \varphi(x) \Rightarrow b_i(y) > s_i, 1 \leq i \leq m$.

Then φ has at least $2m - 1$ fixed points.

Proof. Following the first part of the proof of theorem (3.2) one sees that

$$(7) \quad \text{Ind}(K, D_\varphi, U_i) = 1, \quad 1 \leq i \leq m,$$

and the sets U_i , $1 \leq i \leq m$, are disjoint. We obtain m fixed points. Now define open sets

$$\begin{aligned} V_i &:= \{x \in K \mid a_i(x) < r_i, b_{i-1}(x) > s_{i-1}\}, \\ W_i &:= V_i \cap (K \setminus (\overline{U_i \cup U_{i-1}})) \end{aligned} \quad \text{for } 2 \leq i \leq m.$$

By lemma (3.1) it follows that

$$(8) \quad \text{Ind}(K, D_\varphi, V_i) = 1 \quad \text{for } 2 \leq i \leq m.$$

Since $V_i \setminus (U_{i-1} \cup U_i \cup W_i) \subset \partial U_i \cup \partial U_{i-1}$ we have

$$\text{Fix}(\varphi) \cap V_i \subset U_{i-1} \cup U_i \cup W_i.$$

Of course the sets U_{i-1} , U_i , W_i are disjoint, so that we obtain, by (8) and (2.6)(ii),

$$\text{Ind}(K, D_\varphi, U_{i-1}) + \text{Ind}(K, D_\varphi, U_i) + \text{Ind}(K, D_\varphi, W_i) = \text{Ind}(K, D_\varphi, V_i) = 1.$$

Using (7), it follows that

$$\text{Ind}(K, D_\varphi, W_i) = -1.$$

The sets W_i , $2 \leq i \leq m$ are disjoint: $W_{i-1} \cap W_i = \emptyset$ follows since $x \in W_{i-1} \cap W_i$ would imply $x \in U_{i-1}$, which is impossible. It remains to prove $W_i \cap W_j = \emptyset$ if $j = 2, \dots, i-2, i+2, \dots, m$. This follows if $2 \leq j \leq i-2$ from $b_{i-1}(x) \leq a_{i-2}(x) \leq a_j(x) < r_j \leq r_{i-2} \leq s_{i-1}$ and, in case $i+2 \leq j \leq m$ from $a_i(x) \geq b_{i+1}(x) \geq b_{j-1}(x) > s_{j-1} \geq s_{i+1} \geq r_i$. Hence, we see that φ has $m - 1$ fixed points in W_2, \dots, W_m . The proof is finished if we show

$$W_i \cap U_j = \emptyset \quad \text{for } 1 \leq j \leq m.$$

If $j = i, i - 1$ this is obvious, so let $1 \leq j \leq i - 2$: In this case we conclude from $b_{i-1}(x) \leq a_{i-2}(x) \leq a_j(x) < r_j \leq r_{i-2} \leq s_{i-1}$ and if $i + 1 \leq j \leq m$ consider $a_i(x) \geq b_{i+1}(x) \geq b_j(x) > s_j \geq s_{i+1} \geq r_i$. \square

COROLLARY 3.4. *If all conditions of the above theorem are fulfilled, but instead of φ compact, let φ be completely continuous. Moreover let*

$$\lim_{\|x\| \rightarrow \infty} a_m(x) = \infty.$$

Then φ has at least $2m - 1$ fixed points.

Proof. The set $A := \{x \in K \mid a_m(x) \leq r_m\}$ is closed, bounded, convex and therefore an AR. Hence we can define a compact, decomposable map $\varphi' : K \rightarrow K$ such that $\varphi'(x) = \varphi(x)$ for $x \in A$. Applying the theorem gives $2m - 1$ fixed points for φ' and all the fixed points are in A . Thus we have also $2m - 1$ fixed points of φ . \square

4. Multiple periodic solutions of differential inclusions. Let K be a cone in \mathbb{R}^n and $\Phi : [0, T] \times K \rightarrow \mathbb{R}^n$ a set-valued map.

Given a point $x_0 \in K$ we consider the initial value problem

$$(9) \quad \begin{cases} x'(t) \in \Phi(t, x(t)), \\ x(0) = x_0. \end{cases}$$

An absolutely continuous map $x : [0, T] \rightarrow K$ is a solution of (9) if $x'(t) \in \Phi(t, x(t))$ for a.e. $t \in [0, T]$ and $x(0) = x_0$. We denote by

$$S_{\Phi}(x_0) := \{x : [0, T] \rightarrow K \mid x \text{ is a solution of (9)}\}$$

the set of all solutions.

In order to guarantee the existence of solutions of (9) we shall assume the following boundary condition:

$$\Phi(t, x) \cap T_K(x) \neq \emptyset \quad \text{for each } t \in [0, T], x \in K,$$

where $T_K(x)$ is the Bouligand cone to K at the point $x \in K$, i.e.

$$T_K(x) := \left\{ y \in \mathbb{R}^n \mid \liminf_{h \rightarrow +0} \frac{\text{dist}(x + hy, K)}{h} = 0 \right\}.$$

We have the following result (comp. [4]):

THEOREM 4.1. *Let $\Phi : [0, T] \times K \rightarrow \mathbb{R}^n$ be a convex valued, bounded map such that (10) holds. Then*

$$S_{\Phi} : K \rightarrow C([0, T], K)$$

is a J -map (by $C([0, T], K)$ we denote the subspace of $C([0, T], \mathbb{R}^n)$ of mappings $[0, T] \rightarrow K$).

Proof. Let $x_0 \in K$. It follows from the results in [11] (comp. [6]) that $S_{\Phi}(x_0)$ is an R_{δ} -set. Since $C([0, T], K)$ — a closed convex subset of $C([0, T], \mathbb{R}^n)$ — is an ANR, (2.2) implies that $S_{\Phi}(x_0) \subset C([0, T], K)$ is proximally ∞ -connected. Finally, using standard arguments (comp. [2], p.79) we see that S_{Φ} is upper semi-continuous. \square

Consider next the periodic problem

$$(11) \quad \begin{cases} x'(t) \in \Phi(t, x(t)), \\ x(0) = x(T). \end{cases}$$

Let $e_T : C([0, T], K) \rightarrow K$ be the evaluation map $e_T(x) := x(T)$ in T and consider

$$K \xrightarrow{S_{\Phi}} C([0, T], K) \xrightarrow{e_T} K.$$

This decomposition defines a set-valued map $P_{\Phi} := e_T \circ S_{\Phi}$; P_{Φ} is called the *Poincaré translation operator*. It is evident that problem (11) is equivalent to the existence of a fixed point of P_{Φ} .

In order to apply the results of the third section we have to control the flow of the considered dynamical system.

LEMMA 4.2. *Let $\Phi : [0, T] \times K \rightarrow \mathbb{R}^n$ be a convex valued, bounded map, $a : \mathbb{R}^n \rightarrow \mathbb{R}$ continuously differentiable and let r be a number such that*

$$\langle \text{grad } a(x), y \rangle < 0 \quad \text{for } t \in [0, T], a(x) = r \text{ and } y \in \Phi(t, x).$$

If $a(x_0) \leq r$ and $x \in S_{\Phi}(x_0)$ then $a(x(t)) < r$ for $t \in (0, T]$.

Proof. Let $g : [0, T] \rightarrow \mathbb{R}$, $g(t) := a(x(t))$. Then $g'(t) = \langle \text{grad } a(x(t)), x'(t) \rangle$ a.e. Now, assume that $t > 0$ is the smallest real such that $g(t) = r$. But, using (12) and the upper semi-continuity of Φ , we see that $g'(s) < 0$ a.e. on some ε -neighborhood of t in

$[0, T]$. It follows for $t' < t$, $|t - t'| < \varepsilon$, that $g(t') = g(t) - \int_{t'}^t g'(s)ds > g(t) = r$ and thus we obtain a contradiction. \square

COROLLARY 4.3. *Let $b : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable and let s be a number such that*

$$\langle \text{grad } b(x), y \rangle > 0 \quad \text{for } t \in [0, T], \quad b(x) = s \text{ and } y \in \Phi(t, x).$$

If $b(x_0) \geq s$ and $x \in S_\Phi(x_0)$ then $b(x(t)) > s$ for $t \in (0, T]$. \square

We are now able to state our main results. We will give two versions of theorems where multiple periodic solutions can be obtained.

THEOREM 4.4. *Let $\Phi : [0, T] \times K \rightarrow \mathbb{R}^n$ be a convex valued, bounded map such that (10) holds. Let a_1, \dots, a_m be continuously differentiable convex functionals, b_1, \dots, b_m continuously differentiable concave functionals on K and numbers $r_1, \dots, r_m, s_1, \dots, s_m$ be given such that:*

- (a) $a_m(x) \leq a_{m-1}(x) \leq \dots \leq a_1(x)$, $b_m(x) \leq b_{m-1}(x) \leq \dots \leq b_1(x)$ for $x \in K$,
- (b) $r_1 \leq r_2 \leq \dots \leq r_m$, $s_1 \leq s_2 \leq \dots \leq s_m$,
- (c) $b_i(x) \leq a_{i-1}(x)$ for $x \in K$, $r_{i-1} \leq s_i$, $2 \leq i \leq m$,
- (d) the sets $U_i := \{x \in K \mid a_i(x) < r_i, b_i(x) > s_i\}$, $1 \leq i \leq m$, are nonempty,
- (e) $\lim_{\|x\| \rightarrow \infty} a_m(x) = \infty$,
- (f) $\langle \text{grad } a_i(x), y \rangle < 0$ for $t \in [0, T]$, $a_i(x) = r_i$ and $y \in \Phi(t, x)$, $1 \leq i \leq m$,
- (g) $\langle \text{grad } b_i(x), y \rangle > 0$ for $t \in [0, T]$, $b_i(x) = s_i$ and $y \in \Phi(t, x)$, $1 \leq i \leq m$.

Then $x' \in \Phi(t, x)$ has at least $2m - 1$ periodic solutions.

Proof. The Poincaré translation operator P_Φ is decomposable and, by finite-dimensionality, completely continuous. Considering (a)–(g) above, and using (4.2) and (4.3), we see that all the conditions of Corollary (3.4) are satisfied. Hence we have $2m - 1$ fixed points of P_Φ . \square

THEOREM 4.5. *Let $\Phi : [0, T] \times K \rightarrow \mathbb{R}^n$ be a convex valued, bounded map such that (10) holds. Let $a, b : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable, convex, concave functionals on K respectively and numbers r, s such that*

- (a) $\|x\| \leq a(x)$, $b(x) \leq a(x)$ for $x \in K$ and $r \leq s$,
- (b) $\{x \in K \mid a(x) < r\} \neq \emptyset$,
- (c) $\lim_{\|x\| \rightarrow \infty} b(x) = \infty$,
- (d) $\langle \text{grad } a(x), y \rangle < 0$ for $t \in [0, T]$, $a(x) = r$ and $y \in \Phi(t, x)$,
- (e) $\langle \text{grad } b(x), y \rangle > 0$ for $t \in [0, T]$, $b(x) = s$ and $y \in \Phi(t, x)$.

Then $x' \in \Phi(t, x)$ has at least two periodic solutions. Particularly, there exists a nonzero periodic solution.

Proof. We apply theorem (3.3) for $m = 2$. By condition (c) there is $c > r$ such that $\|x\| \geq c$ implies $b(x) > s$. Since Φ is bounded there is r' such that for any x_0 , $\|x_0\| \leq c$ it follows that if $x \in S_\Phi(x_0)$ then $\|x(t)\| < r'$ for $t \in [0, T]$. Define $f : K \rightarrow K$,

$$f(x) := \begin{cases} cx/\|x\| & \text{if } \|x\| \geq c, \\ x & \text{if } \|x\| \leq c \end{cases}$$

and consider the decomposition

$$K \xrightarrow{S_\Phi \circ f} C([0, T], K) \xrightarrow{e_T} K$$

of the map $e_T \circ S_\Phi \circ f = P_\Phi \circ f$. Define $a_1 := a$, $a_2 := \|\cdot\|$, $b_2 := b$, $r_1 := r$, $r_2 := r'$ and $s_2 := s$. One easily sees that the conditions of (3.3) are fulfilled, so that we obtain three fixed points of $P_\Phi \circ f$, where in each one of the disjoint sets (with the notation of (3.3)) U_1 , U_2 , W_2 lies at least one fixed point. But, since $U_1 \cup W_2 \subset \{x \in K \mid \|x\| \leq c\}$, we see that f has no influence on the fixed points lying in U_1 or W_2 so that we indeed obtain at least two fixed points of P_Φ . \square

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References

- [1] R. Bader and W. Kryszewski, *Fixed-point index for compositions of set-valued maps with proximally ∞ -connected values on arbitrary ANR's*, Set-Valued Analysis 2 (1994), 459–480.
- [2] K. Deimling, *Multivalued differential equations*, de Gruyter, 1992.
- [3] J. Dugundji, *Modified Vietoris theorems for homotopy*, Fund. Math. 66 (1970), 223–235.
- [4] L. Górniewicz, *Topological approach to differential inclusions*, in: Topological Methods in Differential Equations and Inclusions, A. Granas, M. Frigon (eds.), NATO ASI Series C 472, Kluwer Academic Publ., 1995, 129–190.
- [5] L. Górniewicz, A. Granas and W. Kryszewski, *On the homotopy method in the fixed point index theory of multivalued mappings of compact ANR's*, J. Math. Anal. Appl. 161 (1991), 457–473.
- [6] L. Górniewicz, P. Nistri and P. Zecca, *Control problems on closed subsets of \mathbb{R}^n via feedback controls*, Topol. Methods Nonlinear Anal., 2 (1993), 163–178.
- [7] L. Górniewicz and S. Plaskacz, *Periodic solutions of differential inclusions in \mathbb{R}^n* , Bollettino U.M.I., (7), 7-A (1993), 409–420.
- [8] D. M. Hyman, *On decreasing sequences of compact absolute retracts*, Fund. Math., 64 (1969), 91–97.
- [9] M. A. Krasnosel'skiĭ and P. P. Zabreĭko, *Geometrical methods of nonlinear analysis*, Springer Verlag, 1984.
- [10] R. W. Leggett and L. R. Williams, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J., 28 (1979), 673–689.
- [11] S. Plaskacz, *On the solution sets for differential inclusions*, Bollettino U.M.I., (7), 6-A (1992), 387–394.