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ESTIMATES FOR THE POISSON KERNELS AND A FATOU TYPE THEOREM

APPLICATIONS TO ANALYSIS ON SIEGEL DOMAINS

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Abstract. This is a short description of some results obtained by Ewa Damek, Andrzej Hulanicki, Richard Penney and Jacek Zienkiewicz. They belong to harmonic analysis on a class of solvable Lie groups called NA. We apply our results to analysis on classical Siegel domains.

1. NA **groups.** Let $\mathbf{s}$ be a solvable Lie algebra. We assume that $\mathbf{s}$ as a linear space is the direct sum of two subalgebras

$$\mathbf{s} = \mathbf{n} \oplus \mathbf{a},$$

where $\mathbf{n}$ is nilpotent and $\mathbf{a}$ Abelian. We assume that there exists a basis $E_1, \ldots, E_n$ of $\mathbf{n}$ such that for every H in $\mathbf{a}$

$$[H, E_j] = \langle \lambda_j, H \rangle E_j, \quad \lambda_j \in \mathbf{a}^*, \ j = 1, \ldots, n.$$

We call λ_j's the roots. For $\lambda \in \{\lambda_1, \ldots \lambda_n\} = \Lambda$ let

$$\mathbf{n}^\lambda = \{Y \in \mathbf{n} : \ \mathrm{ad}_H Y = \langle \lambda, H \rangle Y\} \text{ for all } H \in \mathbf{a}.$$

We say that a subspace $\mathbf{n}'$ of $\mathbf{n}$ is homogeneous, if for every H in $\mathbf{a}$

$$\mathrm{ad}_H \mathbf{n}' \subset \mathbf{n}'.$$

Let

$$S = \exp \mathbf{s}, \ N = \exp \mathbf{n} \text{ and } A = \exp \mathbf{a}.$$

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Then

$$S = NA$$

is a semidirect product of the groups N and A, A acting on N by

$$a\exp\{\sum x_j E_j\}a^{-1} = \exp\{\sum x_j e^{\langle\lambda_j,\log a\rangle} E_j\}. \tag{1.1}$$

In general, A is multidimensional, hence for $a \in A$ $\log a$ is a vector. We say that $a \to 0$ with respect to a subset Λ_1 of Λ, if $\langle\lambda, \log a\rangle \to -\infty$ for $\lambda \in \Lambda_1$. Then, of course,

$$a\exp\{\sum x_j E_j\}a^{-1} \to e,$$

if $a \to 0$ with respect to Λ_1.

The name NA comes from the main source of examples of such groups: the NA part of the Iwasawa decomposition of a semisimple (non-compact, finite center) group: NAK. We note that the symmetric space NAK/K admits a simply transitive group of isometries of the form NA acting on the left.

But also every proper homogeneous cone Ω in $\mathbf{R}^n$ admits a simply transitive group of linear transformations which is of the form NA, [V].

Every bounded homogeneous domain $D \subset \mathbf{C}^n$ admits a simply transitive group of biholomorphic transformations of the form NA, [V].

All known examples of non-compact Riemannian harmonic spaces, also the non-symmetric ones produced by E. Damek and F. Ricci, have the form NA, N being a so called group of the Heisenberg type, [DR1], [DR2].

Let us look at the following four examples:

Example 1. *Let*

$$D = \{z \in \mathbf{C} : \Im z > 0\}.$$

We consider the following two groups of biholomorphic maps of D onto itself:

$$N = \mathbf{R} \text{ and } A = \mathbf{R}^+.$$

N acts on D by translations parallel to the real axis:

$$x_0 \cdot z = x_0 + z,$$

A acts by dilations:

$$a \cdot z = az.$$

The group generated by these two groups of transformations is denoted by NA. Of course

$$NA = N \times A \text{ as a manifold}$$

and the group multiplication is

$$xa \cdot yb = (x + ay)ab.$$

We also see that the action of NA on D is simply transitive:

$$xa \cdot i = x + ia.$$

We note that our group NA acts also on the boundary $B = \{z : \Im z = 0\}$ of D by affine transformations.

EXAMPLE 2. *Let*
$$D = \{(w', w) \in \mathbf{C} \times \mathbf{C}^n : \Im w' - \sum |w_j|^2 > 0\}.$$

As the upper half-plane is biholomorphic with the unit disc, the corresponding Cayley transformation maps the unit ball $B \subset \mathbf{C} \times \mathbf{C}^n$ biholomorphically onto D:
$$\mathbf{C} \times \mathbf{C}^n \ni (w', w) \to \left(\frac{w'+i}{1+iw'}, \frac{w}{1+iw'}\right) \in \mathbf{C} \times \mathbf{C}^n.$$

Again there is a group N of transformations acting on D "parallel to the boundary"
$$B = \{(w', w) \in \mathbf{C} \times \mathbf{C}^n : \Im w' - \sum |w_j|^2 = 0\}.$$
Here N is $\mathbf{H}^n$ the Heisenberg group. As a manifold $N = \mathbf{R} \times \mathbf{C}^n$, the multiplication is defined by $(x, z)(y, w) = (x + y + 2\Im \sum w'_j \bar{w}_j, w' + w)$. The action of N on D is given by the formula
$$N \times D \ni ((x, u), (w', w)) \to (w' + x + 2i\Phi(w, u) + i\Phi(u, u), w + u) \in D,$$
where $\Phi(u, w) = \sum u_j \bar{w}_j$. A simple calculation shows that in fact the action of N on $\mathbf{C}^{n+1}$ preserves the form
$$\rho(w', w) = \Im w' - \sum |w_j|^2,$$
thus it maps D onto D and also B onto B.

The group A is again equal to $\mathbf{R}^+$. A acts on D by non-isotropic dilations:
$$a \cdot (w', w) \to (aw', a^{\frac{1}{2}} w).$$
Under this action the form ρ transforms as follows:
$$\rho(aw', a^{\frac{1}{2}} w) = a\rho(w', w),$$
hence A also preserves both D and B.

We see that the group NA of transformations of D generated by N and A is the group which contains N as a normal subgroup of codimension 1 and the group A acts as a group of automorphisms of N by non-isotropic dilations. Moreover, the group NA acts simply transitively on the domain D as a group of biholomorphic transformations.

EXAMPLE 3. *Let M be the space of symmetric real $n \times n$ matrices and let Ω be the open cone in M consisting of positive definite matrices. Let*
$$D = M + i\Omega \subset \mathbf{C}^d,\ d = \frac{(n+1)n}{2}.$$

To relate a NA group to this example we proceed as follows. Let E_{ij} be an elementary $n \times n$ matrix. We put
$$\mathbf{a} = \mathrm{lin}\{E_{11}, ..., E_{nn}\}$$

$$\mathbf{n}_0 = \mathrm{lin}\{E_{ij} : i < j\}$$
$$\mathbf{n}_1 = \mathrm{lin}\{E_{kl} + E_{lk} : l \le k\} = M.$$

Then

$$\mathbf{s} = \mathbf{n}_1 \oplus \mathbf{n}_0 \oplus \mathbf{a} = \mathbf{n}_1 + \mathbf{s}_0$$

is a Lie algebra, the Lie bracket being defined as follows:

$$[E,F] = EF - FE \text{ for } E,F \in \mathbf{s}_0$$
$$[X,Y] = 0 \text{ for } X,Y \in \mathbf{n}_1$$
$$[E,X] = E^t X + XE \text{ for } E \in \mathbf{s}_0, X \in \mathbf{n}_1.$$

We see that the group

$$S = \exp \mathbf{s} = N_1 N_0 A = N_1 S_0$$

acts on D simply transitively :

$g(Y+iH) = g^t Y g + i g^t H g$ for $g \in S_0$ and $Y_0(Y+iH) = Y_0 + Y + iH$ for $Y_0 \in N_1$.

We notice that $N = N_1 N_0$ is a nilpotent group on which A acts diagonally, $\mathbf{n}_0$ and $\mathbf{n}_1$ are homogeneous subalgebras, and the basis $\{E_{ij} : i < j\} \cup \{E_{kl} + E_{lk} : k \le l\}$ defines roots on $\mathbf{a}$.

EXAMPLE 4. *Let $\mathcal{C}_2$ be the space of complex $s \times t$ matrices, $\mathcal{C}_1$ — of complex $s \times s$ matrices. We have*

$$\mathcal{C}_1 = \mathcal{H} + i\mathcal{H},$$

where $\mathcal{H}$ is the real space of Hermitian $s \times s$ matrices and for a matrix $E \in \mathcal{C}_1$ we write

$$E = \Re E + i \Im E,$$

where $\Re E, \Im E \in \mathcal{H}$.

Let Ω be the open cone in $\mathcal{H}$ consisting of positive definite matrices, the non-negative definite matrices being its closure.

If we define a Hermitian bilinear form

$$\Phi : \mathcal{C}_2 \times \mathcal{C}_2 \ni E, F \to EF^\star \in \mathcal{C}_1,$$

then we see that $\Phi(E,E) \in \bar{\Omega}$ for all $E \in \mathcal{C}_2$ and $\Phi(E,E) = 0$ implies $E = 0$.

Let

$$D = \{(E,F) \in \mathcal{C}_1 \times \mathcal{C}_2 : \ \Im E - \Phi(F,F) \in \Omega\},$$
$$B = \{(E,F) \in \mathcal{C}_1 \times \mathcal{C}_2 : \ \Im E - \Phi(F,F) = 0\}.$$

Let A be the group of $s \times s$ diagonal matrices with non-zero entries, N_0 the group of unipotent upper triangular $s \times s$ matrices, $S_0 = N_0 A$. For $g \in S_0$ we define the action on D by

$$g(E,F) = (g^* E g, gF).$$

It is clear that

$$\Phi(gF_1, gF_2) = g \cdot \Phi(F_1, F_2).$$

The group $N_1 = \mathcal{H} \times \mathcal{C}_2$ is defined as $N(\Phi)$ below.

The group $N_1 N_0 A$ acts simply transitively on D.

Example 4 shows all the main features of the so called homogeneous Siegel domains. The importance of the homogeneous Siegel domains is stressed by the fact, proved by Piateckij-Shapiro in 1960 [P], that these are precisely all the domains which are biholomorphic with the bounded homogenous domains.

As we see in Examples 3 and 4 above, our boundary B of D is much smaller than the topological boundary of D as embedded in $\mathbf{C}^n$. However, it has the property that for every holomorphic bounded function F on D which extends continuously to B we have

$$\sup\{|F(z)| : z \in D\} = \sup\{|F(x)| : x \in B\}.$$

Abusing slightly, the terminology we call B the *Shilov boundary* of D while in fact it is only a dense subset of the compact Shilov boundary in the bounded realization of D.

A *Siegel domain* (not necessarily homogeneous) is defined as follows:

Let $\Omega \subset \mathbf{R}^{n_1}$ be a regular cone, i.e. a nonempty open convex cone Ω with vertex at 0 and containing no entire straight line.

Given a regular cone in Ω in $\mathbf{R}^{n_1}$, we say that a Hermitian bilinear map

$$\Phi \ : \ \mathbf{C}^{n_2} \times \mathbf{C}^{n_2} \to \ \mathbf{C}^{n_1}$$

is Ω-positive if $\Phi(z_2, z_2) \in \bar{\Omega}$ for all $z_2 \in \mathbf{C}^{n_2}$ and $\Phi(z_2, z_2) = 0$ implies $z_2 = 0$. The domain

$$D = \{(z_1, z_2) \in \mathbf{C}^{n_1} \times \mathbf{C}^{n_2} : \ \Im z_1 - \Phi(z_2, z_2) \in \Omega\}$$

is called a *generalized half-plane* or a *Siegel domain* determined by Φ, Ω. This definition includes also the case where $n_2 = 0$. Then D is *a tube domain* over Ω, as in Examples 1 and 3.

What we call the *Shilov boundary* of D is the set

$$B = \{(z_1, z_2) \in \mathbf{C}^{n_1} \times \mathbf{C}^{n_2} : \Im z_1 - \Phi(z_2, z_2) = 0\}.$$

As in Example 2, the group

$$N(\Phi) = \mathbf{R}^{n_1} \times \mathbf{C}^{n_2}$$

with multiplication

$$(h_1, w_1)(h_2, w_2) = (h_1 + h_2 + 2\Im\Phi(w_1, w_2), w_1 + w_2)$$

acts on D:

$$(x, u)(z_1, z_2) = (z_1 + x + 2i\Phi(z_2, u) + i\Phi(u, u), z_2 + u).$$

Therefore $D = \{(x, u)(it, 0) : \ (x, u) \in N(\Phi), \ t \in \Omega\}$ and the decomposition $z = (x, u)(it, 0)$ of a point $z \in D$ is unique. The orbits of $N(\Phi)$ are parametrized by the elements $it, \ t \in \Omega$. $N(\Phi)$ acts also simply transitively on B.

The *Hardy spaces* $H^p(D)$ are defined as follows.

$H^p(D)$ is the set of all holomorphic $F : D \to \mathbf{C}$ such that

$$\|F\|^p_{H^p(D)} = \sup_{t\in\Omega} \int_{N(\Phi)} |F((x,z)(it,0))|^p \; dxdz \; < \infty.$$

Let

$$F((x,z)(it,0)) = F_t(x,z).$$

Since for $F \in H^p(D)$ the norms $\|F_t\|_{L^p(N(\Phi))}$ are bounded, for $p > 1$ there exists a function F_0 such that

$$\lim_{t\to 0} \|F_t - F_0\|_{L^p(N(\Phi))} = 0.$$

It follows that the map $F \to F_0$ is one-to-one from $H^2(D)$ onto a closed subspace of $L^2(F)$. Thus $H^2(D)$ is a Hilbert space, the inner product being defined by

$$\langle F, G\rangle = \lim_{t\to 0} \int_F F_t(\xi)\bar{G}_t(\xi)d\xi = \langle F_0, G_0\rangle_{L^2(B)}.$$

$H^2(D)$ has a reproducing kernel $S(z,w)$ defined on $D \times D$ called the Szegö kernel [H]. $S(z,w)$ is holomorphic with respect to $z = (z_1, z_2)$ and for fixed w, $S(\cdot,w) \in H^2(D)$. A formula for $S(z,w)$ proved in [KS2] gives $S(z,w)$ in terms of a "Fourier transform" over the dual cone Ω^*.

$$S^w(z) = S(z,w) = \int_{\Omega^*} e^{-2\pi\langle\lambda,\rho(z,w)\rangle} \; \det \Phi_\lambda^{\frac{1}{2}} \; d\lambda, \tag{1.2}$$

where

$$\rho(z,w) = i(\bar{w}_1 - z_1) - 2\Phi(z_2, w_2).$$

If $n_2 = 0$ then

$$S^w(z) = \int_{\Omega^*} e^{-2\pi\langle\lambda,\rho(z,w)\rangle} \; d\lambda.$$

For given w, let S_0^w be the limit of S_t^w in $L^2(B)$. Then $S^w(x) = S_0(x,w)$ and S_0 is a function on $D \times B$, holomorphic with respect to z and square integrable with respect to x, such that for every F in $H^2(D)$

$$F(z) = \int_B S_0(z,x)f(x)dx$$

for some $f \in L^2(B)$. The Poisson kernel for D, introduced by Hua, is the function $P(x,z)$ on $B \times D$ defined by

$$P(x,z) = \frac{|S_0(x,z)|^2}{S(z,z)}, \quad x \in B, \; z \in D.$$

For every z

$$\int_B P(x,z)dx = 1.$$

Every $F \in H^p,\ 1 \leq p \leq \infty$, is a Hua-Poisson integral of a function $f \in L^p$, i.e.

$$F(z) = \int_B f(x)P(x,z)dx.$$

Also for every $f \in L^p(B)$ the function F above is a function on D. It is called the *Hua-Poisson integral* of F.

We see that the definitions given above are modeled on the case of one complex variable (Example 1). We have

$$S(z,x) = \frac{1}{2\pi i}\frac{1}{x-z} = \frac{1}{2\pi i}\frac{1}{x-y-ia}, \text{ if } z = y + ia$$

$$P(z,x) = \frac{1}{\pi}\frac{a}{(y-x)^2+a^2},$$

in which we recognize the Cauchy kernel and the ordinary Poisson kernel.

We also note that both are convolution kernels on the group $N(\Phi)$ and rewrite the formula above as

$$\int_N f(x)P(y\cdot t,x)dx = \int_N f(y^{-1}x)P_t(y)dy = f \star P_t(x), \quad t \in \Omega.$$

This is true in general, the Hua-Poisson kernel can be rewritten as a family of convolution kernels $P_t, t \in \Omega$ on $N(\Phi)$.

Except for the very classical cases, there are no explicite formulas neither for the Szegö nor for the Hua-Poisson kernel. However much is known about both of them. The main tool here is formula (1.2) for the Szegö kernel.

Here are some estimates of the Hua-Poisson kernels which we were able to obtain for general Siegel domains, [DHP1].

Let $\|\cdot\|$ be a norm in $N(\Phi)$. For a fixed $t \in \Omega$ we have

(1.3) $$\text{There exists } \eta > 0 \text{ such that } \int_{N(\Phi)} \|y\|^\eta P_t(y)dy < \infty.$$

For every multiindex I there are constants $c,\ M$ such that

(1.4) $$|\partial^I P_t(y)| \leq c(1+\|y\|)^M.$$

(1.5) $$\text{There exist } c,\epsilon > 0 \text{ such that } P_t(y) \leq c(1+\|y\|)^{-\epsilon}.$$

Properties (1.3)—(1.5) are the only pointwise estimates for P which we can prove in the case of the general Siegel domains. However we have proved that they are sufficient to obtain a satisfactory Fatou type theorem about almost everywhere convergence of the Hua-Poisson integrals of functions in $L^p,\ 1 < p \leq \infty$, to their boundary values, [DHP1].

Another thing which is of importance here is that due to a version of the Harnack inequality for the Hua-Poisson integrals, a consequence of the mean value theorem for holomorphic functions, the maximal function

$$Mf(x) = \sup_{t\in AK} P_t \star f(x), \text{ where } K \text{ is a compact subset in } \Omega$$

is dominated by

$$M'f(x) = \sup_{t\in \Lambda K} P_y \star f(x),$$

where $\Lambda = \{(e^{m_1}\mathbf{e}_1, ..., e^{m_n}\mathbf{e}_n) : m_j \in \mathbf{Z}\}$, $\mathbf{e}_1, ..., \mathbf{e}_n$ form an appropriate basis in the vector group A.

THEOREM 1 (E. Damek, A. Hulanicki, R. Penney [DHP1]). *For $f \in L^p(B)$, $1 < p \le \infty$, the Hua-Poisson integral*

$$\int P(x \cdot (it, 0), y) f(y) dy \to f(x) \quad \textit{almost everywhere}$$

as $t \to 0$ in an appropriate way inside the cone Ω.

The Hua-Poisson integrals are functions on D which are called Hua-harmonic, Hua-harmonic functions are annihilated by the Laplace-Beltrami operator Δ (with respect to the Bergman metric on D) in the case of the *symmetric* domains. However, the class of Δ-harmonic functions on D is much larger, in general. In 1975 A. Korányi and P. Malliavin [KM] presented two elliptic operators L_1 and L_2 on the Siegel domain of 2×2 symmetric matrices as in Example 2, $n = 2$, such that the set of bounded functions annihilated by both L_1 and L_2 are the Hua-Poisson integrals of L^∞ functions on B. In 1980 K. Johnson and A. Korányi [JK] found for symmetric tube domains an elliptic system whose zeros are precisely the Hua-harmonic functions.

THEOREM 2 (E. Damek, A. Hulanicki, R. Penney [DHP2]). *For the domain of our Example 2 (general $n \times n$ matrices) there exists an elliptic real second order differential operator L on D such that bounded Hua-harmonic functions are precisely the bounded functions F such that $LF = 0$.*

2. Harmonic functions. Let L be a second order left-invariant degenerate elliptic operator without a constant term on S:

$$L = X_1^2 + \ldots + X_m^2 + X_0. \tag{2.1}$$

We shall assume that $X_0, X_1, \ldots, X_m$ satisfy the Hörmander condition, i.e. the smallest Lie subalgebra which contains $X_0, \ldots, X_m$ is equal to $\mathbf{s}$. We write

$$X_0 = Y_0 + Z_0, \quad Y_0 \in \mathbf{n},\ Z_0 \in \mathbf{a}.$$

Now let

$$\Lambda_0 = \{\lambda \in \Lambda : \langle \lambda, Z_0 \rangle \ge 0\},$$

We define a subalgebra

$$\mathbf{n}_0(L) = \oplus_{\lambda\in\Lambda_0} \mathbf{n}^\lambda$$

and the corresponding subgroup

$$N_0(L) = \exp \mathbf{n}_0(L).$$

Let $\mathbf{n}_0$ be an arbitrary homogeneous subalgebra of $\mathbf{n}$ containing $\mathbf{n}_0(L)$ and let

$$N_0 = \exp \mathbf{n}_0.$$

Let $\{\mu_t\}_{t>0}$ be a semigroup of probability measures generated by L. Let $\{S_t(\omega)\}$ be the diffusion process defined by the semigroup $\{\mu_t\}_{t>0}$ on S.

Definition of the Furstenberg boundary. Let X be a locally compact space with a probability measure σ. Assume that S acts transitively on X. X is called a *boundary* for the pair S, L if

$$\text{for every } t > 0 \quad \check{\mu}_t \star \sigma = \sigma \tag{2.2}$$

and

$$\lim_{t\to\infty} \int f(S_t(\omega)x) d\sigma(x) = f(x(\omega))$$

for almost every trajectory $S_t(\omega)$, where $x(\omega)$ is a point of X depending only on ω. A Poisson boundary for L is the *maximal Furstenberg boundary* X.

Formula (2.2) implies that for every $f \in L^\infty(\sigma)$ the function F on S defined by

$$F(s) = \int_X f(sx) d\sigma(x) \tag{2.3}$$

is L-harmonic.

If X is the Poisson boundary, then every bounded harmonic function F on S is of the form (2.3).

In [DH1] we have shown that the boundaries of the pair S, L are precisely the S-spaces

$$X = S/N_0A = N/N_0.$$

Let us elaborate this fact. We write $S \times X \ni (s, u) \mapsto su \in X$ for the natural action of S on the quotient space X. We select a point e in X and define the map

$$\mathbf{p} : S \ni s \mapsto se \in X.$$

For a measure ν on X and a bounded measure or a distribution with compact support μ on S we write $\mu * \nu$ for the natural convolution corresponding to this action. We have, cf. [DH1]:

$$\sigma \text{ is a *-weak limit of } \mathbf{p}(\check{\mu}_t) \text{ as } t \to \infty.$$

We see that in our case $X = \mathbf{R}^\chi$ as a manifold. Moreover,

$$\sigma \text{ has a smooth density } d\sigma(x) = P(x)dx, \tag{2.4}$$

where dx is the Lebesgue measure. Let f be a function on X and suppose $f \in L^p(\mathbf{R}^X)$ for some p, $1 \le p \le \infty$. Then (2.3) becomes

$$F(s) = \int_X f(s \cdot x) P(x) dx, \tag{2.5}$$

we call it *the Poisson integral* of f and

$$P(s,x) = \frac{ds^{-1} \cdot x}{dx} P(s^{-1} \cdot x)$$

the Poisson kernel.

In our Example 3 the domain D is identified with NA where $N = N_0 N_1$, $N_0 \subset \Omega$ consists of matrices with ones on the diagonal and A are the diagonal matrices in Ω.

For the proof of Theorem 1 we exhibit an operator of the form (2.1) such that the roots defined by the homogenous basis of M are *negative* on Z_0, cf. (2.2), and the ones defined by the homogeneous basis of $\mathbf{n}_0$ are *non-negative.* Of course the operator has also to annihilate Hua-harmonic functions on D.

3. Estimates for the Poisson kernel. The examples discussed above point to the importance of the Poisson kernels on the Furstenberg boundaries which in the case of non-symmetric Siegel domains *do not coincide with the Hua-Poisson kernel* (i.e. normalized square of the modulus of the Szegö kernel).

The first thing we can tell about them for general NA groups is that they satisfy estimates (1.3)—(1.5).

Properties (1.3)—(1.5) are the only estimates for P we can prove in the case of general NA groups. However, we have proved that they suffices to prove a satisfactory Fatou type theorem on almost everywhere convergence of the Poisson integrals of functions in $L^p,\ 1 < p \le \infty$, to their boundary values.

We note that in general a boundary is only a homogeneous space N/N_0, not necessarily a subgroup of N which is the case for the Poisson boundary. This is the reason why in the general case the proof of the Fatou theorem is much more complicated, cf. [DH3]. In fact, to estimate the maximal functions involved we have to consider maximal functions along surfaces. Here the ideas and techniques of M. Christ [Ch1], [Ch2], are extremely useful.

On the other hand, for the case when NA is a harmonic space and L is the Laplace-Beltrami operator, E. Damek and F. Ricci have proved a formula for P very similar to the corresponding one for symmetric spaces of rank one.

Their result is

$$P(x,z) = c \frac{1}{(1 + \frac{1}{4}|x|^2)^2 + |z|^2)^Q} \simeq c\|x,z\|^{-2Q}, \tag{3.1}$$

where $|\cdot|$ is a specific homogenous gauge on N and Q is the homogenous dimension of N.

Formula (3.1) has been a starting point for a search of better estimates on P and their derivatives first in the case of one-dimensional A.

In this case we identify A with $\mathbf{R}^+$, N becomes a homogeneous group on which A acts by dilations, i.e. there is a

$$\text{homogeneous basis } Y_1, ..., Y_n \text{ of } \mathbf{n}$$

and positive numbers $1 = d_1 \leq \cdots \leq d_n$ such that

$$\delta_a(\exp \sum_{j=1}^{n} y_j Y_j) = \exp(\sum_{j=1}^{n} y_j a^{d_j} Y_j), \quad a \in A$$

are automorphisms of N. Let $Q = d_1 + ... + d_n$ be the homogeneous dimension of N.

In this case operator (2.1) can be written in the form

$$Lf(xa) = \big((a\partial_a)^2 - \alpha a\partial_a + \sum_{i,j=1}^{n} \alpha_{i,j} a^{d_i+d_j} Y_i Y_j + \sum_{i=1}^{n} \alpha_i a^{d_i} Y_i\big) f(xa), \tag{3.2}$$

where the matrix $[\alpha_{ij}]$ is non-negative definite.

(3.3) If L is the Laplace-Beltrami operator on NA then $\alpha = Q$.

Recently, using the method by which Alano Ancona [A] described the minimal positive harmonic functions on Riemannian spaces of negative curvature, E. Damek has proved the following

THEOREM 3 (E. Damek [D2]). *If an operator L is of the form* (3.2), *then*

$$c^{-1}(1 + |x|)^{-\alpha-Q} \leq P(x) \leq c(1 + |x|)^{-\alpha-Q} \tag{3.4}$$

for some constant c.

The proof of the theorem is based on a boundary Harnack inequality for positive harmonic functions on NA and cannot be used to derive estimates for the derivatives. Therefore Ewa Damek, Andrzej Hulanicki and Jacek Zienkiewicz revised a probabilistic approach to the Poisson kernel used earlier by E. Damek and A. Hulanicki [DH2].

Let

$$L_t = \sum_{i,j=1}^{n} \alpha_{i,j} a(t)^{d_i+d_j} Y_i Y_j + \sum_{i=1}^{n} \alpha_i a(t)^{d_i} Y_i),$$

where $a(t)$ is a trajectory with $a(0) = 1$ of the Brownian motion on $\mathbf{R}^+$ generated by

$$(a\partial_a)^2 - \alpha a\partial_a.$$

Then the following holds:

Let $p(\cdot;s,t)$, $s<t$, be a non-negative function on N such that for every $f\in C_c^\infty(N)$ the function $u(x;s,t)=f*p(x;s,t)$ solves

$$L_t u(x;s,t)=\partial_t u(x;s,t),\quad \lim_{t\nearrow s} u(x;s,t)=f(x).$$

Of course $p(\cdot;s,t)$ depends on the trajectory $a(\cdot)$. Then taking the expected value with respect to the Wiener measure we obtain

$$Ep(x;0,\infty)=P(x).$$

We have been able to obtain good estimates for the kernels $p(\cdot;0,\infty)$ in terms of the numbers

$$A_j=\int_0^\infty a(t)^{d_j}\,dt$$

and the explicit formula for the distribution of the Brownian random variables A_j. By this method we have proved the upper bound of (3.4) and

THEOREM 4 (E. Damek, A. Hulanicki, J. Zienkiewicz). *For every multiindex I there is a constant C_I such that*

$$|Y^I P(x)|\le C_I(1+|x|)^{-\alpha-Q-|I|}.$$

The problem of finding precise pointwise estimates for the Poisson kernel in the case when A is multidimensional is still wide open, cf. also [St].

References

[A] A. Ancona, *Negatively curved manifolds, elliptic operators, and the Martin boundary*, Ann. of Math. (2) 125 (1987), 495–536.

[Ch1] M. Christ, *Hilbert transforms along curves I. Nilpotent groups*, Ann. of Math. (2) 122 (1985), 575–596.

[Ch2] M. Christ, *The strong maximal function on a nilpotent group*, Trans. Amer. Math. Soc. 331(1) (1992), 1–13.

[CW] R. R. Coifman and G. Weiss, *Operators associated with representations of amenable groups, singular integrals induced by ergodic flows, the rotations method and multipliers*, Studia Math. 47 (1973), 285-303.

[D1] E. Damek, *Left-invariant degenerate elliptic operators on semidirect extensions of homogeneous groups*, Studia Math. 89 (1988), 169–196.

[D2] E. Damek, *Pointwise estimates on the Poisson kernel on NA groups by the Ancona method*, to appear.

[DH1] E. Damek and A. Hulanicki, *Boundaries for left-invariant subelliptic operators on semi-direct products of nilpotent and Abelian groups*, J. Reine Angew. Math. 411 (1990), 1–38.

[DH2] E. Damek and A. Hulanicki, *Maximal functions related to subelliptic operators invariant under an action of a solvable Lie group*, Studia Math. 101 (1991), 33–68.

[DH3] E. Damek and A. Hulanicki, *Boundaries and the Fatou theorem for subelliptic second order operators on solvable Lie groups*, Colloq. Math. 68 (1995), 121–140.

[DHP1] E. Damek, A. Hulanicki, R. Penney, *Admissible convergence for the Poisson-Szegö integrals*, J. Geom. Anal. (to appear)

[DHP2] E. Damek, A. Hulanicki, R. Penney, *Hua operators on bounded homogeneous domains in* $\mathbf{C}^n$, preprint.

[DR1] E. Damek and F. Ricci, *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. 27 (1992), 139–142.

[DR2] E. Damek and F. Ricci, *Harmonic analysis on solvable extensions of H-type groups*, J. Geom. Anal. 2 (1992), 213–248.

[H] L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in Classical Domains*, Vol. 6, Translations of Math. Monographs, Amer. Math. Soc., Providence, 1963.

[JK] K. Johnson, A. Korányi, *The Hua operators on bounded symmetric domains of tube type*, Ann. of Math. (2) 111 (1980), 589–608.

[K1] A. Korányi, *The Poisson integral for generalized halfplanes and bounded symmetric domains*, Ann. of Math. (2) 82 (1965), 332–350.

[K2] A. Korányi, *Boundary behavior of Poisson integrals on symmetric spaces*, Trans. Amer. Math. Soc. 140 (1969), 393–409.

[K3] A. Korányi, *Harmonic functions on symmetric spaces*, in: Symmetric Spaces, Basel-New York 1972.

[KM] A. Korányi and P. Malliavin, *Poisson formula and compound diffusion associated to overdetermined elliptic system on the Siegel half-plane of rank two*, Acta Math. 134 (1975), 185–209.

[KS1] A. Korányi, E. M. Stein, *Fatou's theorem for generalized half-planes*, Ann. Scuola Norm. Sup. Pisa 22 (1968), 107–112.

[KS2] A. Korányi, E. M. Stein, *H^2-spaces of generalized half-planes*, Studia Math. 44 (1972), 379–388.

[NS] A. Nagel and E. M. Stein, *On certain maximal functions and approach regions*, Adv. Math. 54 (1984), 83–106.

[P] I. I. Pjatecki-Shapiro, *Geometry and classification of homogeneous bounded domains in* $\mathbf{C}^n$, Uspekhi Mat. Nauk 2 (1965), 3–51; Russian Math. Surv. 20 (1966), 1–48.

[R] F. Ricci, *Singular integrals on* $\mathbf{R}^n$, Tempus lectures held at the Institute of Mathematics of Wrocław University, 1991.

[Sj] P. Sjögren, *Admissible convergence of Poisson integrals in symmetric spaces*, Ann. of Math. (2) 124 (1986), 313–335.

[S] J. Sołowiej, *The Fatou theorem for NA groups — a negative result*, Colloq. Math. 67 (1994), 131–145.

[St] E. M. Stein, *Boundary behavior of harmonic functions on symmetric spaces: Maximal estimates for Poisson integrals*, Invent. Math. 74 (1983), 63–83.

[V] E. B. Vinberg, *The theory of convex homogeneous cones*, English translation, Trans. Moscow Math. Soc. 12 (1963), 340–403.