

POISSON LIE GROUPS AND THEIR RELATIONS TO QUANTUM GROUPS

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Abstract. The notion of Poisson Lie group (sometimes called Poisson Drinfel'd group) was first introduced by Drinfel'd [1] and studied by Semenov-Tian-Shansky [7] to understand the Hamiltonian structure of the group of dressing transformations of a completely integrable system. The Poisson Lie groups play an important role in the mathematical theories of quantization and in nonlinear integrable equations.

The aim of our lecture is to point out the naturality of this notion and to present basic facts about Poisson Lie groups together with some relations to the recent work on quantum groups.

1. Poisson structures on manifolds. The basic mathematical structures of classical mechanics are the algebra $C^\infty(N)$ of all smooth functions on the phase space N under ordinary multiplication and the Lie structure on $C^\infty(N)$ induced by the Poisson bracket $\{\cdot, \cdot\}$ defined by the symplectic form on N . The Poisson bracket is a biderivation of the associative algebra, i.e.

$$(1.1) \quad \{f, gh\} = \{f, g\}h + g\{f, h\}$$

and in so called canonical coordinates $(\mathbf{q}, \mathbf{p}) = (q_1, \dots, q_n, p_1, \dots, p_n)$ has the form

$$(1.2) \quad \{f, g\} = \sum_{i=1}^n \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i},$$

so the Hamilton equations

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \quad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$$

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may be written in the “Lax form”:

$$\dot{F} = \{H, F\}.$$

We can forget about the symplectic form and consider a Lie bracket on $C^\infty(N)$ with the property (1.1).

(1.3) DEFINITION. A *Poisson bracket* on an associative algebra is a Lie bracket which is a biderivation. A *Poisson structure* on a smooth manifold N is a bivector field $P \in \Gamma(\Lambda^2(TN))$ such that $\{f, g\} := P(df, dg)$ is a Poisson bracket on $C^\infty(N)$.

(1.4) Remark. There is a one-one correspondence between Poisson brackets on $C^\infty(N)$ and Poisson structures on N (which are those bivector fields P which satisfy the equation $[P, P] = 0$, where $[\cdot, \cdot]$ stands for the Schouten bracket of multivector fields).

For any $f \in C^\infty(N)$ the vector field $X_f := i_{df}P = \{f, \cdot\}$ is called the *Hamiltonian vector field with Hamiltonian f* and the mapping $f \mapsto X_f$ is a homomorphism of the Lie algebra $C^\infty(N)$ with the Poisson bracket into the Lie algebra of vector fields.

(1.5) EXAMPLE. Every symplectic form induces a “classical” Poisson bracket with local form as described in (1.2). The corresponding Poisson structure has in the canonical coordinates the form

$$P = \sum_{i=1}^n \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}.$$

The “symplectic” Poisson brackets are exactly those which are non-degenerate in the sense that the mapping $\omega \mapsto P(\omega, \cdot)$ induces an isomorphism of the cotangent and tangent bundles. In particular, N has to be even dimensional. One can show that every Poisson manifold admits a partition into symplectic submanifolds — the orbits of Hamiltonian vector fields.

(1.6) EXAMPLE. Let V be a Lie algebra over \mathbf{R} with a basis $\{x_1, \dots, x_n\}$ and the bracket $[\cdot, \cdot]$. Set $N = V^*$ to be the dual space, so x_1, \dots, x_n can be regarded as smooth (in fact linear) functions on N which form a coordinate system. It is easy to see that

$$\{f, g\} := \sum_{i,j=1}^n [x_i, x_j] \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} = \sum_{i,j,k=1}^n c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j},$$

where c_{ij}^k are the structure constants, is a Poisson bracket on $C^\infty(V^*)$ called *Kostant-Souriau-Kirillov* bracket. Linear functionals form a Lie subalgebra naturally identified with $(V, [\cdot, \cdot])$. The corresponding *Kostant-Souriau-Kirillov* Poisson structure is given by

$$P = \sum_{i < j} [x_i, x_j] \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

The symplectic leaves of this structure are exactly the orbits of the coadjoint representation of the corresponding Lie group.

Consider in particular the Lie algebra $su(2)$ with the basis (X, Y, H) and commutation relations

$$[X, Y] = 2H, \quad [Y, H] = 2X, \quad [H, X] = 2Y.$$

The Kostant-Souriau-Kirillov Poisson structure on the dual $su(2)^*$ in chosen coordinates reads:

$$P = 2\left(H \frac{\partial}{\partial X} \wedge \frac{\partial}{\partial Y} + X \frac{\partial}{\partial Y} \wedge \frac{\partial}{\partial H} + Y \frac{\partial}{\partial H} \wedge \frac{\partial}{\partial X}\right)$$

and symplectic leaves are 2-dimensional spheres in $su(2)^* \simeq \mathbf{R}^3$ centred at 0.

(1.7) EXAMPLE. Let N be the unit sphere in \mathbf{R}^4 with “global coordinates” (a, b, x, y) , $a^2 + b^2 + x^2 + y^2 = 1$. There is a unique Poisson bracket on N satisfying

$$\begin{aligned} \{x, a\} &= -xb, & \{x, b\} &= xa, & \{x, y\} &= 0, \\ \{y, a\} &= -yb, & \{y, b\} &= ya, & \{b, a\} &= x^2 + y^2. \end{aligned}$$

The corresponding Poisson structure can be written in the form $P = A \wedge B$, where

$$A = aC - \frac{\partial}{\partial a}, \quad B = -bC + \frac{\partial}{\partial b},$$

and

$$C = a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$

Note that the vector fields A, B are really tangent to the sphere and that $[B, A] = aB + bA$.

If we identify this sphere with the group $SU(2)$ by

$$(a, b, x, y) \mapsto \begin{pmatrix} \alpha & -\bar{\nu} \\ \nu & \bar{\alpha} \end{pmatrix},$$

where $\alpha = a + ib$, $\nu = x + iy$, then P can also be written in the form

$$(1.8) \quad P = (x^2 + y^2) \hat{X} \wedge \hat{Y} + (by + ax) \hat{X} \wedge \hat{H} + (ay - bx) \hat{Y} \wedge \hat{H},$$

where $\hat{X}, \hat{Y}, \hat{H}$ are the right-invariant vector fields on $SU(2)$ corresponding to the elements

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad H = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

of the Lie algebra $su(2)$ of $SU(2)$.

(1.9) DEFINITION. Let (M, P_M) and (N, P_N) be Poisson manifolds with Poisson brackets $\{\cdot, \cdot\}_M$ and $\{\cdot, \cdot\}_N$. A smooth mapping $\phi : N \rightarrow M$ is called *the Poisson mapping* if the induced $\phi^* : C^\infty(M) \rightarrow C^\infty(N)$ is a homomorphism of Poisson brackets:

$$\{u \circ \phi, v \circ \phi\}_N = \{u, v\}_N \circ \phi, \quad u, v \in C^\infty(M).$$

The product Poisson bracket $\{\cdot, \cdot\}_{M \times N}$ on $M \times N$ is defined by

$$\{u, v\}_{M \times N}(x, y) = \{u^x, v^x\}_N(y) + \{u^y, v^y\}_M(x),$$

where clearly u^x (resp. u^y) is the smooth function of $y \in N$ (resp. $x \in M$) obtained from $u \in C^\infty(M \times N)$ by keeping x (resp. y) constant.

(1.10) Remark. Note that the product Poisson structure is a bivector field $P_{M \times N}$ on $M \times N$ which under the projections of $M \times N$ onto M and N gives respectively P_M and P_N . If we identify $C^\infty(M \times N)$ with an appropriate tensor product $C^\infty(M) \otimes C^\infty(N)$, we may also write the product Poisson bracket in the form:

$$\{u_1 \otimes u_2, v_1 \otimes v_2\}_{M \times N} = \{u_1, v_1\}_M \otimes u_2 v_2 + u_1 v_1 \otimes \{u_2, v_2\}_N.$$

2. Quantum spaces. Roughly speaking, we obtain quantum spaces deforming corresponding “dual” algebraic objects as for example the commutative algebras of a given class functions on the space (cf. Gel’fand-Najmark functor) can be deformed into noncommutative ones. This class may be arbitrarily chosen; Woronowicz [9] prefers to work with C^* -algebras what is more difficult, but fruitful, while Drinfel’d [2] works with purely algebraic version only.

To make the whole thing transparent, let us consider the very classical example of Woronowicz [9] for the group $SU(2)$ which topologically is a three dimensional sphere. We shall keep the notation of (1.7). The $*$ -algebra \mathcal{A} generated by matrix elements is dense in $C(SU(2))$ and can be characterized as the “maximal” unital commutative $*$ -algebra generated by elements α, ν (cf. (1.7)) and satisfying $\alpha^* \alpha + \nu^* \nu = I$. Dropping the assumption about commutativity, Woronowicz proposed to consider the algebra \mathcal{A}_q as the unital non-commutative $*$ -algebra generated by α, ν satisfying $\alpha^* \alpha + \nu^* \nu = I$ and additionally the commutation relations

$$(2.1) \quad \begin{aligned} \alpha \alpha^* - \alpha^* \alpha &= (2q - q^2) \nu^* \nu, & \nu^* \nu - \nu \nu^* &= 0 \\ \nu \alpha - \alpha \nu &= q \nu \alpha, & \nu^* \alpha - \alpha \nu^* &= q \nu^* \alpha. \end{aligned}$$

It is clear that for $q = 0$ we get the previous commutative algebra, so \mathcal{A}_q is a one-parameter deformation of $\mathcal{A} = \mathcal{A}_0$. The crucial point here is that the algebras \mathcal{A}_q do not “collapse” for $1 > q \geq 0$, i.e. the basis remains the same (for example $\{\nu^{*m} \nu^n \alpha^k, \nu^{*m} \nu^n \alpha^{*l} : k, m, n = 0, 1, 2, \dots \text{ and } l = 1, 2, \dots\}$ is a basis in \mathcal{A}_q for all $1 > q \geq 0$), and we may consider the algebras \mathcal{A}_q not as different objects, but as different multiplications \circ_q on the same object, namely the algebra \mathcal{A} . This way we get so called *formal deformation* of \mathcal{A} , since the \circ_q -product of elements $u, v \in \mathcal{A}$ reads

$$(2.2) \quad u \circ_q v = uv + \sum_{n=1}^{\infty} q^n P_n(u, v).$$

For example, using the identification of \mathcal{A}_q with \mathcal{A} via the basis as above, we get

$$\alpha \circ_q \nu = \nu \circ_q \alpha - (\nu \circ_q \alpha - \alpha \circ_q \nu) = \nu \alpha - q \nu \alpha$$

and $\alpha \circ_q \alpha^* = \alpha^* \circ_q \alpha + (\alpha \circ_q \alpha^* - \alpha^* \circ_q \alpha) = (I - \nu^* \nu) + 2q \nu^* \nu - q^2 \nu^* \nu.$

Since the commutator bracket $[u, v]_q = u \circ_q v - v \circ_q u$, as for any associative algebra, is a biderivation, we get easily that

$$\{u, v\} := P_1(u, v) - P_1(v, u)$$

is a Poisson bracket. Thus the first non-trivial term of a formal deformation gives us, after antisymmetrization, a Poisson bracket. Assuming that the term P_1 is given by a bidifferential operator, we always can, after a reparametrization, obtain P_1 antisymmetric. Indeed, consider a formal automorphism $u \mapsto B(u) = u + qB_1(u) + o(q)$. A new but isomorphic product $\tilde{\circ}_q$ defined by $u\tilde{\circ}_qv := B^{-1}(B(u) \circ_q B(v))$ has the form

$$u\tilde{\circ}_qv = uv + q(P_1(u, v) + \delta B(u, v)) + o(q),$$

where $\delta B(u, v) = uB(v) - B(uv) + B(u)v$ is the Hochschild coboundary of B . It is well known (cf. [8]) that the differentiable Hochschild cohomology of $C^\infty(N)$ is the space of multivector fields, so $P_1(u, v) = \{u, v\} + \delta B$ for some $B \in \text{End}(\mathcal{A})$ and we can obtain P_1 being a Poisson bracket choosing the corresponding formal automorphism.

The idea of deformational quantization is now the following: the commutative associative algebra structure of $C^\infty(N)$ is only a room for the physics, a Poisson bracket (the first non-commutative term) gives us the classical mechanics, but what we really need is the quantum mechanics given by the “true” non-commutative product \circ_q whose infinitesimal part delivers the Poisson bracket, i.e. classical mechanics (cf. [4]).

(2.3) EXAMPLE. From the commutation relations (2.1) defining the Woronowicz’ quantum sphere we get easily the corresponding Poisson bracket which on matrix elements of $SU(2)$ has the form

$$\begin{aligned} \{\alpha, \bar{\alpha}\} &= 2\bar{\nu}\nu, & \{\nu, \bar{\nu}\} &= 0, \\ \{\nu, \alpha\} &= \nu\alpha, & \{\bar{\nu}, \alpha\} &= \bar{\nu}\alpha. \end{aligned}$$

Passing to real functions $\alpha = a + ib$, $\nu = x + iy$, we get a purely imaginary bracket with the imaginary part being exactly the Poisson bracket described in (1.7) (cf. also [3]).

3. Quantum groups. To get a quantum group, we must equip a quantum space, i.e. an associative algebra A , with additional structures: a coproduct $\Delta : A \rightarrow A \otimes A$ and so called counit and antipode. The tensor product (in an appropriate sense) $A \otimes A$ represents clearly functions on the product group $G \times G$. All objects, maps and properties are dual to those of the category of groups. For example, the coproduct is the dual of the group multiplication $G \times G \rightarrow G$, counit to the embedding of the neutral element $e \hookrightarrow G$, etc. The coproduct is not only coassociative, but it is an algebra homomorphism (with the obvious associative algebra structure on $A \otimes A$) as the mapping between quantum spaces. What we get is usually called a *Hopf algebra*, so quantum groups are nothing

but Hopf algebras which are deformations of Hopf algebras associated with group structures (e.g. the C^* -Hopf algebras of continuous functions on compact groups in Woronowicz' and the universal enveloping Hopf algebras of Lie algebras in Drinfel'd's approach). The obvious $*$ -coproduct for the algebra \mathcal{A} generated by matrix elements of the group $SU(2)$ is given by

$$\Delta(\alpha) = \alpha \otimes \alpha - \nu^* \otimes \nu, \quad \Delta(\nu) = \nu \otimes \alpha + \alpha^* \otimes \nu.$$

Woronowicz obtained his famous quantum $SU(2)$ group deforming this coproduct to $\Delta_q = \Delta + qD$, where $D(\alpha) = \nu^* \otimes \nu$, $D(\nu) = 0$. This new coproduct is compatible with the deformed multiplication \circ_q , so we get a quantum group (if we do not care at the moment about counit and antipode). The compatibility means that

$$\Delta_q(u \circ_q v) = \Delta_q(u) \circ_q^{\mathcal{A} \otimes \mathcal{A}} \Delta_q(v)$$

and hence

$$(3.1) \quad \Delta_q([u, v]_q) = [\Delta_q(u), \Delta_q(v)]_q^{\mathcal{A} \otimes \mathcal{A}},$$

where $\circ_q^{\mathcal{A} \otimes \mathcal{A}}$ and $[\cdot, \cdot]_q^{\mathcal{A} \otimes \mathcal{A}}$ are the product and commutator bracket in $\mathcal{A} \otimes \mathcal{A}$. The identity (3.1) implies in particular

$$(3.2) \quad \Delta(\{u, v\}) = \{\Delta(u), \Delta(v)\}^{\mathcal{A} \otimes \mathcal{A}},$$

where $\{\cdot, \cdot\}^{\mathcal{A} \otimes \mathcal{A}}$ is the product Poisson bracket on $\mathcal{A} \otimes \mathcal{A}$ (cf. (1.10)). Of course, all this works not only in the particular $SU(2)$ case, but in the general scheme as well. Writing (3.2) on the level of the group G , we get that the corresponding Poisson structure $P \in \Gamma(\Lambda^2(TG))$ satisfies

$$(3.3) \quad P(g_1 g_2) = (L_{g_1})_* P(g_2) + (R_{g_2})_* P(g_1),$$

where $(L_{g_1})_*$ and $(R_{g_2})_*$ are the left and right actions of G on multivector fields. A multivector field P satisfying (3.3) is called *multiplicative*.

(3.4) DEFINITION. A *Poisson Lie group* is a Lie group G equipped with a multiplicative Poisson structure.

We may say equivalently that a Poisson Lie group is a Lie group G equipped with a Poisson structure such that the group multiplication $G \times G \rightarrow G$ is a Poisson map (with the product Poisson structure on $G \times G$). It can be regarded as “infinitesimal” part of the quantum group being the deformation of G (in the category of Hopf algebras, of course). Note also that Poisson Lie structures are never “symplectic”, since they vanish at the neutral element.

(3.5) ANOTHER EXAMPLE. Consider a Lie algebra V with a basis (x_1, \dots, x_n) (cf. (1.6)) and the universal enveloping algebra U which is a Hopf algebra with the coproduct of the form $x \mapsto I \otimes x + x \otimes I$ for $x \in V$. Due to the Campbell-Baker-Hausdorff theorem, U is isomorphic as a vector space with the symmetric algebra over V , or in other words, with the algebra $A = Pol(V^*)$ of polynomials on the dual space V^* . A is therefore a quantum group being a deformation of the Abelian group V^* (with respect to addition). The corresponding Poisson bracket

$\{\cdot, \cdot\}$ on V^* being the infinitesimal part of this deformation restricted to linear functionals from $V \subset U \simeq Pol(V^*)$ coincides with the original bracket $[\cdot, \cdot]$ on V , so $\{x_i, x_j\} = [x_i, x_j]$ and we get the Poisson structure P as in (1.6). It is clearly multiplicative, since V^* acts trivially on tensor fields and P is linear: $P(x^* + y^*) = P(x^*) + P(y^*)$. The universal enveloping algebra of a Lie algebra V may be therefore regarded as a quantum group being the quantization of the dual space V^* and the Kostant-Souriau-Kirillov bracket on it. The universal enveloping Hopf algebra can be deformed once more. For example the Drinfel'd-Jimbo's quantum $SL(2, \mathbf{C})$ group is the formal deformation of the universal enveloping of the Lie algebra $sl(2, \mathbf{C})$ given by deformed commutators

$$[H, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = \frac{2}{q} sh\left(\frac{qH}{2}\right)$$

and deformed coproduct

$$\Delta(X^\pm) = X^\pm \otimes \exp\left(\frac{qH}{4}\right) + \exp\left(\frac{-qH}{4}\right) \otimes X^\pm, \quad \Delta(H) = H \otimes I + I \otimes H,$$

where (X^+, X^-, H) is the standard basis of $sl(2, \mathbf{C})$. For $q = 0$ we get clearly the non-deformed universal enveloping Hopf algebra.

4. Poisson Lie groups, Lie bialgebras and dressing transformations.

The standard references for this section are papers [1], [2], [5], [6], [7].

Since the tangent bundle of a Lie group G can be canonically trivialized, we can regard any Poisson structure P on G as the mapping $\tilde{P} : G \longrightarrow \mathcal{G} \wedge \mathcal{G}$, for \mathcal{G} being the Lie algebra of G , putting

$$\tilde{P}(g) = (R_{g^{-1}})_* P(g).$$

In other words, every Poisson structure may be written in the form $P = \sum_{i,j} r_{ij} \hat{X}_i \wedge \hat{X}_j$, where \hat{X}_i are the right-invariant vector fields on G associated with $x_i \in \mathcal{G}$ and r_{ij} are smooth functions on G , so $\tilde{P} = \sum_{i,j} r_{ij} x_i \wedge x_j$. The multiplicativity of P means that

$$\tilde{P}(g_1 g_2) = (Ad_{g_1}) \tilde{P}(g_2) + \tilde{P}(g_1),$$

i.e. \tilde{P} is a 1-cocycle of G relative to the adjoint representation of G in $\mathcal{G} \wedge \mathcal{G}$. Passing to the intrinsic derivative $p = d_e(\tilde{P})$, we get $p : \mathcal{G} \longrightarrow \mathcal{G} \wedge \mathcal{G}$, which is a 1-cocycle of the Lie algebra \mathcal{G} with coefficients in the adjoint representation of \mathcal{G} in $\mathcal{G} \wedge \mathcal{G}$, i.e.

$$p([x, y]) = ad_x(p(y)) - ad_y(p(x)).$$

The dual mapping is $p^* : \mathcal{G}^* \wedge \mathcal{G}^* \longrightarrow \mathcal{G}^*$. It is easy to see that for any smooth functions u, v on G , we have

$$p^*(du(e), dv(e)) = (d\{u, v\})(e)$$

(the right hand term does not depend on the choice of u and v , since the Lie Poisson bracket $\{\cdot, \cdot\}$ vanishes at e !) and the Jacobi identity for the Poisson

bracket implies now that p^* is a Lie bracket itself.

(4.1) DEFINITION. A *Lie bialgebra* is a Lie algebra \mathcal{G} equipped with a 1-cocycle $p : \mathcal{G} \rightarrow \mathcal{G} \wedge \mathcal{G}$ relative to the adjoint representation of \mathcal{G} in $\mathcal{G} \wedge \mathcal{G}$ such that the dual mapping $p^* : \mathcal{G}^* \wedge \mathcal{G}^* \rightarrow \mathcal{G}^*$ is a Lie bracket (i.e. satisfies the Jacobi identity).

Note that the notion of Lie bialgebra is self-dual, since if (\mathcal{G}, p) is a Lie bialgebra then (\mathcal{G}^*, q) , where $q : \mathcal{G}^* \rightarrow \mathcal{G}^* \wedge \mathcal{G}^*$ is dual to the Lie bracket $[\cdot, \cdot] : \mathcal{G} \wedge \mathcal{G} \rightarrow \mathcal{G}$, is a Lie bialgebra (called the *dual Lie bialgebra*) as well.

(4.2) THEOREM. *Given a simple connected Lie group G , there is a one-one correspondence between Poisson Lie structures on G and Lie bialgebra structures on \mathcal{G} as described above.*

(4.3) DEFINITION. Let (G, P) be a Poisson Lie group, (\mathcal{G}, p) be the tangent Lie bialgebra, and (\mathcal{G}^*, q) be the dual Lie bialgebra. The simple connected Poisson Lie group (G^*, Q) whose tangent Lie bialgebra is (\mathcal{G}^*, q) is called the *dual Poisson Lie group*.

(4.4) DEFINITION. Let (G, Q) be a Poisson Lie group, (N, P) be a Poisson manifold, and let $\Phi : G \times N \rightarrow N$ be an action of G on N . We call this action a *Poisson action* if Φ is a Poisson map (with respect to the product Poisson structure on the product, of course).

(4.5) Remark. Note that if the Poisson structure on the group is trivial the Poisson action simply preserves the Poisson bracket on N . The major advantage of the above notion is that we allow also actions which do not preserve the Poisson structure, but describe a “hidden symmetry” and admit a reduction method.

(4.6) THEOREM. *Let (G, P) be a Poisson Lie group, (\mathcal{G}, p) be the tangent Lie bialgebra, (\mathcal{G}^*, q) be the dual Lie bialgebra, and (G^*, Q) be the dual Poisson Lie group. Then the mappings*

$$\lambda : \mathcal{G}^* \ni \omega \mapsto i_{\hat{\omega}_\lambda} P \in \Gamma(TG)$$

and

$$\rho : \mathcal{G}^* \ni \omega \mapsto i_{\hat{\omega}_\rho} P \in \Gamma(TG),$$

where $\hat{\omega}_\lambda$ and $\hat{\omega}_\rho$ are respectively the left- and right-invariant 1-forms on G generated by ω , are Lie algebra homomorphisms of \mathcal{G}^* into the Lie algebra of vector fields on G .

(4.7) DEFINITION. Integrating λ (resp. ρ) gives rise to a local (and global if the corresponding vector fields are complete) *left* (resp. *right*) *dressing action* of G^* on G and we say that this actions consist of *left* (resp. *right*) *dressing transformations*.

(4.8) THEOREM. *For a Poisson Lie group (G, P) , both the left and right dressing actions of G^* on G are Poisson actions.*

Note that the dressing actions are in general really only Poisson and not preserving the Poisson structure on G .

(4.9) Remark. There is a way to understand the dressing actions of G^* on G (and vice versa) in the language of so called *double groups* (and *Manin triples* on the level of Lie algebras). One can namely construct (at least locally) a Lie group H with G^* and G as closed subgroups such that the multiplication

$$G \times G^* \ni (g, g^*) \mapsto gg^* \in H$$

is a diffeomorphism. Having the projections $\pi_G : H \rightarrow G$ and $\pi_{G^*} : H \rightarrow G^*$, we get for example the right dressing action of G^* on G in the form

$$G \times G^* \ni (g, g^*) \mapsto \pi_G(gg^*) \in G.$$

Unfortunately, we have no place here to go into details.

(4.10) EXAMPLE. Differentiating (1.8) we get the Lie bialgebra structure on $su(2)$ corresponding to the Poisson Lie structure of the Woronowicz' quantum $SU(2)$ -group:

$$p(H) = 0, \quad p(X) = H \wedge X, \quad p(Y) = H \wedge Y.$$

The dual Lie bialgebra is therefore the Lie algebra spanned by (H^*, X^*, Y^*) , with the commutation relations

$$[X^*, Y^*] = 0, \quad [H^*, X^*] = X^*, \quad [H^*, Y^*] = Y^*,$$

i.e. the Lie algebra $sb(2, \mathcal{C})$ of all 2×2 traceless upper triangular complex matrices with real diagonal elements, and the cobracket defined by:

$$q(H^*) = 2X^* \wedge Y^*, \quad q(X^*) = 2Y^* \wedge H^*, \quad q(Y^*) = 2X^* \wedge H^*.$$

The corresponding double group H is the group $SL(2, \mathcal{C})$ with the decomposition into subgroups $SU(2)$ and $SB(2, \mathcal{C})$ being the group of upper triangular matrices of $SL(2, \mathcal{C})$ with real positive diagonal elements.

(4.11) EXAMPLE. The analogous procedure for the Kostant-Souriau-Kirillov Poisson structure (cf. (1.6)) yields the Lie bialgebra structure on the commutative Lie algebra V^* of the form:

$$p(x_k^*) = \sum_{i < j} c_{ij}^k x_i^* \wedge x_j^*.$$

The dual Lie bialgebra is therefore the Lie algebra V itself with the trivial cobracket and the dual Poisson Lie group is the simple connected Lie group G with the Lie algebra V and the trivial Poisson structure. The double Lie group is in this case the cotangent bundle T^*G with G and V^* as closed subgroups.

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