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QUANTUM SPACES: NOTES AND COMMENTS ON A LECTURE BY S. L. WORONOWICZ

R. J. BUDZYŃSKI
Department of Physics, Warsaw University
Hoża 69, 00-681 Warszawa, Poland
E-mail: budzynsk@fuw.edu.pl

W. KONDRACKI
Institute of Mathematics of Polish Academy of Sciences
Śniadeckich 8, 00-950 Warszawa, Poland
E-mail: witekkon@impan.gov.pl

Introduction. The concept of quantum spaces has been under study for some time now, and recently it has been attracting a growing amount of attention. This is in part due to the interest in quantum groups, which are considered a very promising concept by many, both physicists and mathematicians. It seems that quantum groups appear quite naturally in low-dimensional quantum field theory and statistical mechanics, they have also found important applications in the theory of knots. Other potential hoped-for applications include a possible role of quantum spaces in a future quantized theory of gravity.

The theory of quantum spaces, and quantum groups in particular, is at present in a phase of rapid growth. One of the consequences of this is that the language used in the field still remains somewhat ambiguous, even concerning the definition of basic concepts. Here, the category of locally compact quantum spaces will be

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understood as the dual to the category of $C^\star$ algebras [2]. The main objective of the present notes is to motivate and describe this basic notion.

Compact quantum spaces. Let us begin by considering a compact Hausdorff space X, and the set $C(X)$ of continuous complex valued functions on X. $C(X)$ is naturally endowed with the structure of a commutative algebra with unit over the complex number field, equipped moreover with the anti-linear involution $\star$ given by

$$(f^\star)(p) = \overline{f(p)}$$

and the norm

$$\|f\| = \sup_{p\in X} |f(p)|.$$

This norm can be seen to obey the condition

$$\|f^\star f\| = \|f\|^2.$$

It is also known that the space $C(X)$ is complete with respect to this norm.

Algebras (not necessarily commutative) with the above properties are known as $C^\star$ algebras. We see therefore that every compact Hausdorff space X is in a natural way associated with a commutative $C^\star$ algebra with unit, namely $C(X)$. What's more, every continuous mapping between compact Hausdorff spaces, $T : X \to Y$, determines a $C^\star$ homomorphism: $T^\star : C(Y) \to C(X)$, given by

$$(T^\star f)(p) = f(T(p)).$$

In particular, when X consists of a single point, $C(X) = \mathbf{C}$ and any mapping of X into Y determines a linear and multiplicative functional on $C(Y)$. In brief: points of Y correspond to linear multiplicative functionals on $C(Y)$. The celebrated Gelfand–Naimark theorem states that this correspondence is one to one. This leads to an important conclusion: Any commutative $C^\star$ algebra with unit is the algebra of continuous functions on a compact topological space. This space may be identified with the set of linear multiplicative functionals (also known as characters) on the algebra.

Stating it in the language of category theory: there exists a contravariant isomorphism between the category of compact topological spaces and that of commutative $C^\star$ algebras with unit. Consequently, it is natural to make the following generalization: A compact *quantum* space corresponds, by an extension of this isomorphism, to a *noncommutative* $C^\star$ algebra with unit. It must be stressed that it is not necessary (nor does it seem useful) to consider the quantum space itself as a set.

It may be therefore said that the theory of quantum spaces is no more than the theory of $C^\star$ algebras. Such a point of view, however, misses much of the point. A similar example is the theory of probability: one could just as well say that it is no more than measure theory. The fact is that although both probability and measure theory may be reduced to a common denominator, they differ in the kind of questions they ask and in their applications.

Returning to the main subject, one of the basic concepts of topology is that of the Cartesian product of topological spaces. In the language of $C^\star$ algebras, the corresponding concept is that of the tensor product of algebras. This, however, requires a definition of the tensor product that would operate within the category of $C^\star$ algebras. The problem boils down to properly defining the norm and completion of the algebraic tensor product. To show how this is done, we quote the following theorem due to Gelfand, Naimark and Segal:

THEOREM. *Every separable $C^\star$ algebra has a faithful continuous representation in $B(H)$, the algebra of bounded linear operators on a Hilbert space H. Moreover, this representation is isometric (norm-preserving).*

It follows that the image of such a representation is a closed subalgebra of $B(H)$.

We say that a closed subalgebra $A \subset B(H)$ is nondegenerate iff for every nonzero $x \in H$, $ax \neq 0$ for some $a \in A$. The family of separable, nondegenerate subalgebras of $B(H)$ will be denoted as $C^\star(H)$.

We are now ready to explain the construction of an adequate tensor product of $C^\star$ algebras: Consider the $C^\star$ algebras A and A'. In virtue of the Theorem quoted above, A and A' may be identified with certain elements of $C^\star(H)$. The algebraic tensor product is contained in $B(H) \otimes B(H) \subset B(H \otimes H)$. Completing the image of $A \otimes A'$ with respect to the norm in $B(H \otimes H)$, we obtain a separable $C^\star$ algebra, which we identify as the tensor product of A with A'. It turns out that the resulting algebra does not depend on the choice of (faithful) representations of A and A'.

An important property of the Cartesian product is the existence of natural projections onto the factors:

$$X \times Y \ni (x, y) \longmapsto x \in X,$$

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Moreover, in the case of the Cartesian product of a space with itself there exists a natural *diagonal* mapping:

$$X \ni x \longmapsto (x, x) \in X \times X.$$

In the language of quantum spaces, the projections correspond to the natural injective homomorphisms:

$$A \ni a \longmapsto a \otimes I \in A \otimes A',$$

$$A' \ni a' \longmapsto I \otimes a' \in A \otimes A'.$$

The diagonal mapping, however, does not have a quantum equivalent: the mapping

$$A \otimes A \ni \sum a_i \otimes b_i \longmapsto \sum a_i b_i$$

is not an algebra homomorphism in general.

Examples. Many examples of quantum spaces may be presented as algebras with a finite number of generators and relations. Consider such a finite set of generators (I, a_i) and relations R_k. We form a $\star$-algebra with unit, A, as the quotient of the free algebra in the generators (I, a_i) by the two-sided ideal generated by the relations R_k. Next, we consider the set $\{\pi\}$ of $\star$-representations of A in $B(H)$. If for every $x \in A$, $\sup_\pi \|\pi(x)\|$ exists, it determines a semi-norm in A. By quotienting A additionally by the ideal generated by elements of vanishing semi-norm and completing, we obtain the so-called universal $C^\star$ algebra generated by (I, a_i) with relations R_k.

As an example, we now present the *quantum disk* [3]. Let μ be a real number, $0 < \mu < 1$, and z a generator subject to the relation

$$zz^\star - z^\star z = \mu(I - z^\star z)(I - zz^\star).$$

It turns out that such an algebra admits only two types of inequivalent representations:

- $\pi(z) \in \mathbf{C}$, $|z| = 1$,
- $\pi(z) f_n = (\frac{n\mu}{1+n\mu})^{1/2} f_{n-1}$,

where $(f_n)_{n=0}^\infty$ form an orthonormal basis in the Hilbert space H. The algebra A_μ described here is well known as the *Toeplitz algebra*.

Observe that the mapping

$$z \longmapsto e^{i\phi} \in C(S^1)$$

extends to a homomorphism of A_μ onto $C(S^1)$. In this sense, the usual circle is a *subset* of the quantum disk.

The following nontrivial fact can be shown: the group $SU(1,1)$ of two by two complex matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

such that $\beta = \overline{\gamma}, \delta = \overline{\alpha}$, and $|\alpha|^2 - |\beta|^2 = 1$, acts on A_μ as an automorphism group:

$$\zeta = (\alpha z + \beta)(\gamma z + \delta)^{-1} \in A_\mu$$

satisfies the same relation as z; this justifies the term quantum disk, being in precise analogy with the action of $SU(1,1)$ on the unit disk in $\mathbf{C}$, as its group of biholomorphisms.

What's more, it is possible to contemplate a quantum extension of a well-known construction of closed Riemann surfaces of genus > 1. Any such surface Σ can be presented as the quotient of the disk by the action of a discrete subgroup of $SU(1,1)$, isomorphic to $\pi_1(\Sigma)$. The quantum version of this would be to consider the subalgebra

$$A_\Sigma = \{x \in A_\mu : \varphi_g(x) = x \text{ for all } g \in \pi_1\Sigma\},$$

where φ denotes the $SU(1,1)$ action on A_μ.

To make this precise, it would however be necessary to take into account that one should be quotienting the *open* disk, while A_μ is a quantum version of the *closed* disk with boundary. Before this can be done, one needs a notion of noncompact quantum space.

Another example which we shall present is the quantum sphere S_q^3 [8]. Let α and γ be the generators, q a real parameter ($0 < q \leq 1$), and take the following set of relations:

$$\alpha\gamma = q\gamma\alpha,$$

$$\alpha\gamma^\star = q\gamma^\star\alpha,$$

$$\gamma\gamma^\star = \gamma^\star\gamma,$$

$$\alpha^\star\alpha + \gamma^\star\gamma = I,$$

$$\alpha\alpha^\star + q^2\gamma^\star\gamma = I.$$

When q is set to 1 we obtain a commutative algebra; by inspecting the last two relations we see that, after suitable completion, it is the algebra $C(S^3)$. In the general case, the universal $C^\star$ algebra obtained from this set of generators and relations by the procedure previously described will be called *the algebra of functions on the quantum sphere*, $C(S_q^3)$.

Just as is the case for the classical sphere (i.e. at $q = 1$), also for $q \neq 1$ we can find an action of S^1 on $C(S_q^3)$ by $C^\star$-algebra automorphisms. This action is determined by the formulas

$$S^1 \ni e^{it} \longmapsto \sigma_t : C(S_q^3) \to C(S_q^3),$$

$$\sigma_t(\alpha) = e^{it}\alpha,\ \sigma_t(\gamma) = e^{-it}\gamma.$$

In the algebra $C(S_q^3)$ we may distinguish the set of elements left invariant by the action σ. It forms a $C^\star$ algebra with unit, and therefore corresponds to a certain compact quantum space. In the case $q = 1$ this algebra is $C(S^2)$, and the construction we have outlined is known as the Hopf fibration: $S^2 \sim S^3/S^1$. What we have obtained is therefore a quantum generalization of the Hopf fibration [4].

It must be mentioned that choosing the particular form of "quantum deformation" of S^3 displayed above has a deep justification. It is known that the classical sphere S^3 is itself endowed with the structure of a (topological) group; this is the group $SU(2)$ of unitary two by two complex matrices of determinant 1. We have claimed that any topological notion referring to compact spaces can be translated into the language of commutative $C^\star$ algebras. In particular, a group multiplication is simply a continuous mapping from $G \times G$ to G, subject to certain additional conditions (namely associativity, existence of a unit and of inverse elements). Speaking in terms of $C(G)$, what corresponds to group multiplication on G is a (unital $C^\star$) homomorphism

$$\Delta : C(G) \to C(G) \otimes C(G),$$

called the coproduct. Explicitly:

$$C(G) \ni f \longmapsto \Delta(f) \in C(G) \otimes C(G) \sim C(G \times G)$$

is given by

$$(\Delta(f))(g_1, g_2) = f(g_1 g_2).$$

Of course, the coproduct obeys certain properties derived from the axioms of group multiplication. We may take these as the basic axioms, and, in the spirit of the theory of quantum spaces, drop the requirement that $C(G)$ be commutative. This leads to the theory of quantum groups [9], which are a very important special class of quantum spaces.

In particular, the algebra $C(S^3_q)$ described above provides an example of a quantum group, denoted $SU_q(2)$. The coproduct is determined by the formulas

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^\star \otimes \gamma,$$

$$\Delta(\gamma) = \gamma \otimes \alpha + \alpha^\star \otimes \gamma.$$

It may be verified that Δ fulfills the axioms dual to those of group multiplication.

The quotient construction of the quantum sphere S^2_q sketched above has therefore also provided us with an example of a *quantum homogeneous space* for the quantum group $SU_q(2)$ [5].

Remarks on noncompact quantum spaces. Up to now, we have been considering the category of *compact* quantum spaces, motivating its definition by the classical version of the Gelfand–Naimark theorem. To proceed to the case of locally compact, but noncompact spaces, we quote a generalization of this theorem:

THEOREM. *Every commutative $C^\star$ algebra may be identified with an algebra of continuous functions on a locally compact topological space. If the algebra has a unit element, this space is moreover compact. In the contrary case, we are dealing with the algebra of continuous functions on a noncompact space, subject to the condition of vanishing at infinity.*

Denote the algebra of continuous functions vanishing at infinity on a noncompact space X by $C_\infty(X)$. It turns out that in the noncompact case there is a problem with properly defining the morphisms of function algebras. One may easily find a continuous mapping $F : X \to Y$ between two topological spaces, such that the corresponding dual mapping $F^\star$ is *not* a homomorphism between $C_\infty(Y)$ and $C_\infty(X)$; that is, $F^\star$ might not preserve the condition of vanishing at infinity. $F^\star$ may indeed be defined, as a homomorphism from $C_b(Y)$ to $C_b(X)$, where C_b denotes the algebra of all continuous *bounded* functions. Trying to describe a noncompact topological space X by its algebra of all bounded continuous functions $C_b(X)$ is not, however, the right idea: the latter is a $C^\star$ algebra with unit, and therefore (by Gelfand–Naimark) corresponds uniquely to a certain *compact* topological space (namely, the Čech–Stone compactification of X).

To describe a solution to the problem of uniquely associating with any locally compact topological space a certain function algebra, *and* an algebra morphism with any continuous mapping, we point out the following fact: knowing $C_\infty(X)$ allows one to determine $C_b(X)$. Indeed,

$$f \in C_b(X) \Rightarrow \text{ for every } g \in C_\infty(X),\ fg \in C_\infty(X).$$

In fact, $C_b(X)$ is the largest algebra which contains $C_\infty(X)$ as a "sufficiently large" ideal. By this we mean the following property: for any $a \in C_b(X)$ which is a strictly positive function, $aC_\infty(X)$ is dense in $C_\infty(X)$. We therefore define the morphisms in the category of commutative $C^\star$ algebras in the following way:

$$F^\star \in \mathrm{Mor}(C_\infty(Y), C_\infty(X)) \text{ iff } F^\star : C_\infty(Y) \to C_b(X) \text{ is a homomorphism,}$$

and $F^\star(C_\infty(Y))C_\infty(X)$ is dense in $C_\infty(X)$.

We are now ready to describe the "quantum extension" of this construction. Let A be a $C^\star$ algebra, not necessarily with unit. By the second Gelfand–Naimark theorem, there exists a faithful representation of A in $B(H)$, i.e. $A \in C^\star(H)$. We treat A as playing the role of $C_\infty(X)$, while for that of $C_b(X)$ we take the so-called algebra of multipliers of A, $M(A)$:

$$M(A) = \{a \in B(H)\ :\ aA, Aa \subset A\}.$$

For a pair of $C^\star$ algebras A, B we will define $\mathrm{Mor}(A, B)$ as the class of $C^\star$-homomorphisms $\varphi : A \to M(B)$ such that $\varphi(A)B$ is dense in B. It turns out that any such homomorphism extends in a canonical way to a homomorphism from $M(A)$ to $M(B)$, and the composition of morphisms can be defined.

It therefore remains to interpret, within the language of algebras, the notion of a continuous, not necessarily bounded function. What motivates the suitable quantum extension in this case is the following property, which holds for continuous functions: if f is continuous on a locally compact space X, then $f(I+f^\star f)^{-1/2}$ is continuous and bounded on X. This leads us to introduce the following definition: given an algebra $A \in C^\star(H)$, an operator T (in general unbounded) on H will be called affiliated with A if

$$T(I+T^\star T)^{-1/2} \in M(A),$$

and $(I+T^\star T)^{-1/2}A$ is a dense set in A. The set of affiliated operators is to be considered as the quantum version of the set of continuous, not necessarily bounded functions on a locally compact space [11]. The operators affiliated with an algebra do not, however, in general themselves form an algebra.

Concluding remarks. What we have described in the present notes is of course only the point of departure for the theory of quantum spaces; at present, this theory is at a rather early stage, and much remains to be developed. It does, however, seem possible to translate into the language of $C^\star$ algebras most of the notions and constructions of classical topology; for instance, some amusing examples of surgery on quantum spaces have been displayed. Another promising direction is the study of quantum spaces equipped with additional structures. Here the most studied example are compact matrix quantum groups, though interesting examples of noncompact quantum groups have also been constructed [7, 12]. Perhaps the most interesting problem is that of finding a satisfactory formulation of the notion of differential structure on a quantum space [1] (some examples can be found in [8, 6, 10]).

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