PANORAMAS OF MATHEMATICS
BANACH CENTER PUBLICATIONS, VOLUME 34
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1995

DENSITY OF HEAT CURVES IN THE MODULI SPACE

A. BOBENKO[1]
Technische Universität Berlin, 10623 Berlin, Germany

N. ERCOLANI[2]
University of Arizona, Tucson, AZ 85721, USA

H. KNÖRRER
E. TRUBOWITZ
ETH-Zentrum, CH-8092 Zürich, Switzerland

1. Introduction. Let

$$\Gamma = (L_1, 0)\mathbf{Z} \oplus (0, L_2)\mathbf{Z} \quad L_1, L_2 \neq 0.$$

For q in $L^2(\mathbf{R}^2/\Gamma)$, let

$$H + q = \frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2} + q(x_1, x_2)$$

be the corresponding heat operator. Set

$$\begin{aligned}\widehat{\mathcal{H}}(q) \quad = \quad & \{(k_1, k_2) \in \mathbf{C}^2 | \quad \text{there is a nontrivial} \quad \psi \in H_k^2(\mathbf{R}^2), \\ & \text{such that} \quad (H + q)\psi = 0\},\end{aligned}$$

1991 *Mathematics Subject Classification*: Primary: 30F10. Secondary: 35Q53.
Lecture given by H. Knörrer at the Banach Center Colloquium on 6th May 1993.
[1]Partially supported by the Alexander von Humboldt Stiftung and the Sonderforschungsbereich 288; on leave of absence from St. Petersburg Branch of Steklov Mathematical Institute, St. Petersburg, Russia.
[2]Supported by the National Science Foundation (DMS-9001897) and the Arizona Center for Mathematical Sciences, sponsored by AFOSR Contract FQ 86711-900589.
The paper is in final form and no version of it will be published elsewhere.

where

$$H^2_k(\mathbf{R}^2) = \{\psi \in H^2_{loc}(\mathbf{R}^2)| \text{ such that } \psi(x+\gamma) = e^{i<k,\gamma>}\psi(x) \text{ for all } \gamma \in \Gamma\}.$$

The dual lattice

$$\Gamma^\sharp = \left(\frac{2\pi}{L_1}, 0\right)\mathbf{Z} \oplus \left(0, \frac{2\pi}{L_2}\right)\mathbf{Z}$$

acts by translation on $\widehat{\mathcal{H}}$. The heat curve associated to q is, by definition,

$$\mathcal{H}(q) = \widehat{\mathcal{H}}(q)/\Gamma^\sharp.$$

In [7] it is shown that $\mathcal{H}$ is a complex analytic subvariety of $\mathbf{C}^2$.

In [10] it is shown that for generic q the heat curve $\mathcal{H}(q)$ is smooth and of infinite genus. In [7] a normalized basis of L^2-holomorphic one forms is constructed, the Riemann operator and associated theta function are analyzed, a vanishing theorem is proved and a Torelli theorem is obtained. Using these results we expect to solve the periodic Kadomtsev-Petviashvili (KP) equation with initial data q and to show that the solution is almost periodic in the time whenever $\mathcal{H}(q)$ is smooth. In [10] it is shown that for any T the solution of KP can be arbitrary well approximated on $0 \le t \le T$ by a finite (but large) gap solution of KP. This is done by approximating $\mathcal{H}(q)$ with curves of finite genus.

The results mentioned above suggest that heat curves are a very natural class of transcendental curves. In this paper we further substantiate this suggestion, by proving that every compact Riemann surface can be arbitrarily well approximated by the normalisations of heat curves.

Recall that the normalisation $\widetilde{\mathcal{H}(q)}$ of $\mathcal{H}(q)$ is obtained by constructing the analytic configuration [12] (in the sense of Weyl) associated to any smooth germ of $\mathcal{H}(q)$. It is a Riemann surface. One can imagine that $\widetilde{\mathcal{H}(q)}$ is the affine part of a finite genus curve when there are enough singularities.

Theorem 1. *For each $g \ge 1$, the set of Riemann surfaces of genus g that are normalisations $\widetilde{\mathcal{H}(q)}$ of heat curves $\mathcal{H}(q)$, where $q \in L^2(\mathbf{R}^2/\Gamma)$ for some rectangular lattice Γ, is dense in the moduli space $\mathcal{M}_g$ of all Riemann surfaces of genus g.*

Let C be any compact Riemann surface, p a point on C and ζ a local coordinate centered at p. Also, let

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log\Theta(xU + yV + tW + D) + 2c \qquad (1)$$

be the solution of the Kadomtsev-Petviashvili equation

$$u_{yy} = \frac{4}{3}\left(u_t - \frac{1}{4}(uu_x + u_{xxx})\right)_x$$

generated by C (see [9]). If this solution is periodic in x and y, then $C - \{p\}$ is isomorphic to $\tilde{\mathcal{H}}(u(x_1, x_2, t))$ for any t, D.

Denote by M the space of triples (C, p, ζ). Here, C is a compact Riemann surface, p a point on C and ζ the 3-jet of a local coordinate centered at p. Theorem 1 is a direct consequence of

THEOREM 2. *Let M_{per} be the set of points (C, p, ζ) for which there are $L_1, L_2 \neq 0$ such that the solution of KP given above satisfies*

$$u(x + L_1, y, t) = u(x, y + L_2, t) = u(x, y, t).$$

Then, M_{per} is dense in $\mathcal{M}_g$.

The technique of Schottky uniformisation used to prove Theorem 2 also yields similar results for curves that generate real solutions of the KP-equation and the KdV-equation. To formulate them, let $M_{\mathbf{R}}$ be the set of all quadruples (C, p, ζ, σ) with $(C, p, \zeta) \in M$ and σ an antiholomorphic involution on C whose fixed point set consists of $g+1$ ovals, contains p and satisfies $\sigma^*(\zeta) = \bar{\zeta}$. For these quadruples, the solution (1) is real whenever D is real. We have

THEOREM 3. *Let $M_{\mathbf{R},per}$ be the set of points (C, p, ζ, σ) in $M_{\mathbf{R}}$ for which there are $L_1, L_2, T \neq 0$ such that the corresponding solution (1) satisfies*

$$u(x + L_1, y, t) = u(x, Y + L_2, t) = u(x, y, t), \quad u(x, y, t + T) = u(x, y, t).$$

Then, $M_{\mathbf{R},per}$ is dense in $M_{\mathbf{R}}$.

Theorem 3 has been proved in [8] by a different method. A stronger version of Theorem 2 has been announced by I. Krichever in [11].

To formulate the last result, let N be the set of points (C, p, ζ, σ) in $M_{\mathbf{R}}$ for which C is hyperelliptic and p is Weierstrass point on C. In this case, (1) is independent of y and is a solution of the Korteweg-de Vries equation

$$4u_t = 6uu_x + u_{xxx}.$$

THEOREM 4. *Let N_{per} be the set of points in N for which there are $L, T \neq 0$ such that the corresponding solution of the Korteweg-de Vries equation satisfies*

$$u(x + L, t) = u(x, t + T) = u(x, t).$$

Then, N_{per} is dense in N.

Again, a stronger version of Theorem 4, namely that the differential of the map (7) below has a maximal rank everywhere, has been proven by I. Krichever (see [3]).

2. Proofs. To begin the proofs we first specify the constants in (1). Let $(C, p, \zeta) \in M$ and $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ be a normalized basis of $H_1(C, \mathbf{Z})$ and $w = (w_1, \ldots, w_g)$ the vector of differentials on C normalized so that

$$\int_{a_n} w_m = 2\pi i \delta_{mn}, \quad m, n = 1, \ldots, g.$$

Furthermore, let

$$R(C)_{nm} = \int_{b_m} w_n$$

be the Riemann period matrix of C. Observe that the theta function of C depends on the choice of the basis of homology, but the second logarithmic derivative in (1) does not and that $u(x, y, t)$ only depends on the point $xU + yU + tW + D$ in $\mathrm{Jac}(C) = \mathbf{C}^{2g}/\{2\pi\mathbf{Z}^g \oplus R(C)\mathbf{Z}^g\}$.

The constants U, V, W of (1) are characterized by

$$w = Ud\zeta + V\zeta d\zeta + W\zeta^2 d\zeta$$

near p. We first prove Theorem 2. Write

$$\begin{aligned} U &= 2\pi i \Delta_1 + R(C)\Delta_2 \\ V &= 2\pi i \Delta_3 + R(C)\Delta_4 \end{aligned} \tag{2}$$

with vectors $\Delta_i \in \mathbf{Q}^g$. Clearly (1) has the required periodicity if $\Delta_i \in \mathbf{Q}^g$. Observe that this property is independent of the choice of the basis of homology. Let $\widetilde{M}$ be the covering of M consisting of points (C, p, ζ) of M together with a choice of a basis of homology. Associating to each element of $\widetilde{M}$ the vector $(\Delta_1, \ldots, \Delta_4) \in \mathbf{R}^{4g}$ as above we get a real analytic mapping

$$\Delta : \widetilde{M} \to \mathbf{R}^{4g}.$$

To prove Theorem 2 it suffices to show that Δ has maximal rank almost everywhere. As this map is analytic it is enough to show that Δ has maximal rank at one point of $\widetilde{M}$. For this purpose and also for the proofs of Theorems 3, 4 we use the technique of Schottky uniformization [2], [4], [5], which we now briefly review.

For $(A, B, \mu) \in \mathbf{C}^g \times \mathbf{C}^g \times \mathbf{C}^{*g}$ let σ_n be the linear transformations defined by

$$\frac{\sigma_z z - B_n}{\sigma_n z - A_n} = \mu_n \frac{z - B_n}{z - A_n} \quad (z \in \mathbf{C}), \quad n = 1, \ldots, g.$$

The map σ_n maps the outside of the circle of radius

$$\frac{|A_n - B_n|}{|\sqrt{\mu_n} - \frac{1}{\sqrt{\mu_n}}|}$$

and centered at

$$\frac{B_n\sqrt{\mu_n} - A_n\frac{1}{\sqrt{\mu_n}}}{\sqrt{\mu_n} - \frac{1}{\sqrt{\mu_n}}}$$

to the inside of the circle of the same radius and center at

$$\frac{A_n\sqrt{\mu_n} - B_n\frac{1}{\sqrt{\mu_n}}}{\sqrt{\mu_n} - \frac{1}{\sqrt{\mu_n}}}.$$

Let S be the set of all $(A, B, \mu) \in \mathbf{C}^g \times \mathbf{C}^g \times \mathbf{C}^{*g}$ for which all the discs bounded by these circles are disjoint. Clearly S is an open subset of $\mathbf{C}^g \times \mathbf{C}^g \times \mathbf{C}^{*g}$ and contains the full-dimensional subset of all points for which all A's, B's are mutually different and μ's are sufficiently small. For $(A, B, \mu) \in S$ the complement of the discs mentioned above is a fundamental domain for the Schottky group G generated by $\sigma_1, \ldots, \sigma_g$. Let Ω be the region of discontinuity for G. Then $C = \Omega/G$ is a compact Riemann surface of genus g. It has a distinguished local coordinate $\zeta = z^{-1}$. Further the images of the circles above define a-cycles on C, so that C is extended with a marking of homology. Thus we get a map $Y : S \to \widetilde{M}$ whose image is an open subset of $\widetilde{M}$.

In order to describe the map $\Delta \circ Y$ we give formulas for the relevant data in terms of A, B, μ. Denote by G_n the cyclic subgroup of G generated by σ_n. The series

$$w_n = \sum_{\sigma \in G/G_n} \left(\frac{1}{z - \sigma B_n} - \frac{1}{z - \sigma A_n} \right) dz, \quad n = 1, \ldots, g,$$

(where G/G_n is the right coset space) define the normalized holomorphic differentials on G. The period matrix is given by

$$\begin{aligned} R(C)_{nm} &= \sum_{\sigma \in G_m \backslash G/G_n} \log\{B_m, A_m, \sigma B_n, \sigma A_n\}, \quad m \neq n, \\ R(C)_{nn} &= \log \mu_n + \sum_{\sigma \in G_n \backslash G/G_n, \sigma \neq I} \log\{B_n, A_n, \sigma B_n, \sigma A_n\}, \end{aligned} \tag{3}$$

where the curly brackets indicate the cross-ratio

$$\{z_1, z_2, z_3, z_4\} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

These are (-2)-dimensional Poincaré theta-series and they converge absolutely, if μ is sufficiently small [1], [6]. Furthermore

$$U_n = \sum_{\sigma \in G/G_n} (\sigma A_n - \sigma B_n),$$

$$\begin{aligned} V_n &= \sum_{\sigma\in G/G_n} \Big((\sigma A_n)^2 - (\sigma B_n)^2\Big), \\ W_n &= \sum_{\sigma\in G/G_n} \Big((\sigma A_n)^3 - (\sigma B_n)^3\Big). \end{aligned} \tag{4}$$

Lemma 1. *The asymptotics of the series* (3), (4) *as* $\mu \to 0$ *are*

$$\begin{aligned} U_n &= A_n - B_n + O(|\sqrt{\mu}|), \quad V_n = A_n^2 - B_n^2 + O(|\sqrt{\mu}|), \\ R_{nm} &= O(1), \qquad n \neq m, \quad R_{nn} = \log \mu_n + O(1). \end{aligned}$$

These O-estimates are uniform in derivatives in A and B.

We postpone the proof of this Lemma and continue with the proof of Theorem 2. From (2) one deduces that

$$\begin{pmatrix} \Delta_1 & \Delta_3 \\ \Delta_2 & \Delta_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi} & -\frac{1}{2\pi}(\operatorname{Im} R)(\operatorname{Re} R)^{-1} \\ 0 & (\operatorname{Re} R)^{-1} \end{pmatrix} \begin{pmatrix} \operatorname{Im} U & \operatorname{Im} V \\ \operatorname{Re} U & \operatorname{Re} V \end{pmatrix}.$$

Using the Lemma this gives

$$\begin{aligned} (\Delta_1)_n &= \frac{1}{2\pi} \operatorname{Im}(A_n - B_n) + O\left(\frac{1}{\log|\mu|}\right), \\ (\Delta_2)_n &= \frac{1}{\log|\mu_n|} \operatorname{Re}(A_n - B_n) + O\left(\frac{1}{(\log|\mu|)^2}\right), \\ (\Delta_3)_n &= \frac{1}{2\pi} \operatorname{Im}(A_n^2 - B_n^2) + O\left(\frac{1}{\log|\mu|}\right), \\ (\Delta_4)_n &= \frac{1}{\log|\mu_n|} \operatorname{Re}(A_n^2 - B_n^2) + O\left(\frac{1}{(\log|\mu|)^2}\right). \end{aligned}$$

So if we choose $(\widehat{A}, \widehat{B}, \widehat{\mu}) \in S$ with $\widehat{A}_i, \widehat{B}_i$ all different and $\widehat{\mu}$ sufficiently small then the derivative of $\Delta \circ Y$ with respect to A and B is invertible at this point, since the map $\mathbf{C}^{2g} \to \mathbf{R}^{4g}$,

$$(A, B) \mapsto \Big(\operatorname{Re}(A_n - B_n); \operatorname{Im}(A_n - B_n); \operatorname{Re}(A_n^2 - B_n^2); \operatorname{Im}(A_n^2 - B_n^2)\Big)$$

is invertible at $\widehat{A}, \widehat{B}$.

For the proof of Theorem 3 we need a more detailed expansion of (U, V) than in Lemma 1.

Lemma 2.

$$\begin{aligned} U_n &= A_n - B_n + \sum_{k\neq n} \mu_k u_{nk} + O(|\mu|^2), \\ V_n &= A_n^2 - B_n^2 + \sum_{k\neq n} \mu_k v_{nk} + O(|\mu|^2), \\ W_n &= A_n^3 - B_n^3 + \sum_{k\neq n} \mu_k w_{nk} + O(|\mu|^2), \end{aligned}$$

where

$$u_{nk} = (A_n - B_n)(A_k - B_k)^2 \left[\frac{1}{(A_k - A_n)(A_k - B_k)} + \frac{1}{(B_k - A_n)(B_k - B_n)} \right],$$
$$v_{nk} = (A_n - B_n)(A_k - B_k)^2 \left[\frac{2B_k}{(A_k - A_n)(A_k - B_k)} + \frac{2A_k}{(B_k - A_n)(B_k - B_n)} \right], \quad (5)$$
$$w_{nk} = (A_n - B_n)(A_k - B_k)^2 \left[\frac{3B_k^2}{(A_k - A_n)(A_k - B_k)} + \frac{3A_k^2}{(B_k - A_n)(B_k - B_n)} \right],$$

Proof of Lemma 2 and Lemma 1: We only discuss the formula for U, the other cases being similar. Every element σ of Schottky group is a loxodromic linear transformation and is represented by a matrix in $PSL(2, \mathbf{C})$ of the form

$$\frac{1}{F_1 - F_2} \begin{pmatrix} rF_1 - r^{-1}F_2 & F_1F_2(r^{-1} - r) \\ r - r^{-1} & r^{-1}F_1 - rF_2 \end{pmatrix},$$

where F_1, F_2 are the fixed points of σ. If ℓ is the length of σ represented as a word in $\sigma_1, \ldots, \sigma_g$ (i.e. $\ell = |m_1| + \ldots + |m_k|$ if $\sigma = \sigma_{n_1}^{m_1} \ldots \sigma_{n_k}^{m_k}$ is a reduced representation), then $r = O(|\mu|^\ell)$. In particular

$$\sigma z_1 - \sigma z_2 = r \frac{(z_1 - z_2)(F_1 - F_2)}{(-z_1 + F_1)(-z_2 + F_1)} + O(|\mu|^{\ell+1}). \quad (6)$$

If we take $\sigma = I$ in the series (4) for U_n we get the term $A_n - B_n$. If we take $\sigma = \sigma_k$ and $\sigma = \sigma_k^{-1}$ we get the terms $\mu_k u_{nk} + O(|\mu|^2)$. Because of (6) all other terms in this series are of order $|\mu|^2$.

Proof of Theorem 3: Let

$$S_{\mathbf{R}} := \left\{ (A, B, \mu) \in S \mid B = \bar{A}, \quad \mu \in \mathbf{R}^g \right\}.$$

The image of $S_{\mathbf{R}}$ under the map from S to M described above is $M_{\mathbf{R}}$ [2], [5]. In this case the vectors U, V, W are purely imaginary. The associated solution of the KP equation is triply periodic if $(\frac{1}{2\pi i}U, \frac{1}{2\pi i}V, \frac{1}{2\pi i}W) \in \mathbf{Q}^{3g}$. Therefore it is enough to show that the map given by

$$(U, V, W) : S_{\mathbf{R}} \to i\mathbf{R}^{3g}$$

has maximal rank at one point. The determinant of the Jacobian of this map with respect to A, B, μ at $\mu = 0$ is

$$\left| \begin{matrix} \frac{\partial U_n}{\partial A_k} & \frac{\partial U_n}{\partial B_k} & \frac{\partial U_n}{\partial \mu_k} \\ \frac{\partial V_n}{\partial A_k} & \frac{\partial V_n}{\partial B_k} & \frac{\partial V_n}{\partial \mu_k} \\ \frac{\partial W_n}{\partial A_k} & \frac{\partial W_n}{\partial B_k} & \frac{\partial W_n}{\partial \mu_k} \end{matrix} \right|_{\mu=0} =$$

$$= \begin{vmatrix} I & -I & u_{nk} \\ 2\,\mathrm{diag}(A_1,\dots,A_g) & -2\,\mathrm{diag}(B_1,\dots,B_g) & v_{nk} \\ 3\,\mathrm{diag}(A_1^2,\dots,A_g^2) & -3\,\mathrm{diag}(B_1^2,\dots,B_g^2) & w_{nk} \end{vmatrix} =$$

$$= \begin{vmatrix} 2\,\mathrm{diag}(A_1-B_1,\dots,A_g-B_g) & v_{nk}-2A_nu_{nk} \\ 3\,\mathrm{diag}(A_1^2-B_1^2,\dots,A_g^2-B_g^2) & w_{nk}-3A_n^2u_{nk} \end{vmatrix} =$$

$$= 2^g \prod_{n=1}^{g} (A_n - B_n) \det \widetilde{F},$$

$$\widetilde{F}_{nk} = w_{nk} - \frac{3}{2}(A_n+B_n)v_{nk} + 3A_nB_nu_{nk},$$

To calculate $\det \widetilde{F}$ we note that

$$\begin{aligned} \det \widetilde{F} &= 3^g \prod_{n=1}^{g} (A_n-B_n)^3 \det F, \\ F_{nk} &= \frac{(B_k-A_n)(B_k-B_n)}{(A_k-A_n)(A_k-B_n)} + \frac{(A_k-A_n)(A_k-B_n)}{(B_k-A_n)(B_k-B_n)}, \quad k \neq n, \\ F_{nn} &= 0. \end{aligned}$$

If we set

$$A_n = n + \alpha, \;\; B_n = \bar{A}_n,$$

then in the limit $\mathrm{Im}\,\alpha \to \infty$ we have

$$F_{nk} \to -2, \;\; k \neq n, \quad F_{nn} = 0$$

and finally $\det F \neq 0$.

Proof of Theorem 4: Let

$$S_{hyp} := \left\{(A,B,\mu) \in S \mid A \in i\mathbf{R}^g, \quad B = \bar{A}, \quad \mu \in \mathbf{R}^g \right\}.$$

The image of $S_{\mathbf{R}}$ under the map from S to M described above is N [2], [4]. As before it suffices to show that the map

$$(U,W) : S_{hyp} \to i\mathbf{R}^{2g} \tag{7}$$

has maximal rank at one point. For U_n and W_n we have the following series

$$U_n = 2A_n + \sum_{k\neq n} \mu_n u_{nk} + O(|\mu|^2),$$
$$W_n = 2A_n^3 + \sum_{k\neq n} \mu_n w_{nk} + O(|\mu|^2),$$

where in u_{nk} and w_{nk} given by (5) we should substitute $B = -A$. After this substitution the determinant of the Jacobian of the map (7) with respect to A and μ at $\mu = 0$ is

$$\left| \begin{matrix} \dfrac{\partial U_n}{\partial A_k} & \dfrac{\partial U_n}{\partial \mu_k} \\[2ex] \dfrac{\partial W_n}{\partial A_k} & \dfrac{\partial W_n}{\partial \mu_k} \end{matrix} \right|_{\mu=0} = \left| \begin{matrix} 2I & u_{nk} \\[2ex] 6\ \operatorname{diag}(A_1^2, \dots, A_g^2) & w_{nk} \end{matrix} \right| = 2^g \det \widetilde{F},$$

$$\begin{aligned} \widetilde{F}_{nk} &= W_{nk} - 3A_n^2 U_{nk}, \\ \det \widetilde{F} &= (24)^g \prod_{k=1}^{g} A_k^3 \det F, \\ F_{nk} &= 2, \quad k \neq n, \quad F_{nn} = 0. \end{aligned}$$

Again, $\det F \neq 0$ proves the nondegeneracy.

References

[1] H. F. Baker, *Abel's Theorem and the Allied Theory Including the Theory of Theta-Functions*, Cambridge Univ. Press, Cambridge 1897.

[2] E. Belokolos, A. Bobenko, V. Enolskii, A. Its, V. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer-Verlag 1994.

[3] R. Bikbaev, S. Kuksin, *On the parametrization of finite-gap solutions by frequency vector and wave-number vector and a theorem of I. Krichever*, Lett. Math. Phys. 28 (1993), 115–122.

[4] A. I. Bobenko, *Schottky uniformization and finite-gap integration*, Soviet Math. Dokl. 36 (1988), 38–42.

[5] A. I. Bobenko, L. A. Bordag: *Periodic multiphase solutions of Kadomtsev-Petviashvili equation*, J. Phys. A 22 (1989), 1259–1274.

[6] W. Burnside, Proc. London Math. Soc. 23 (1892), 49–88.

[7] J. Feldman, H. Knörrer, E. Trubowitz, *Riemann surfaces of infinite genus* I, II, III. Preprints ETH Zürich (1993/1994).

[8] D. Gieseker, *A Lattice Version of the KP Equation*, Acta Math. 168 (1992), 219–248.

[9] I. M. Krichever, *Methods of algebraic geometry in the theory of nonlinear equations*, Russian Math. Surveys 32 (1977), 185–213.

[10] I. M. Krichever: *Spectral theory of two dimensional periodic operators and its applications*, Russian Math. Surveys 44 (1989), 145–225.

[11] I. M. Krichever, *The τ-function of the universal Whitham hierarchy, matrix models and topological field theories*, Comm. Pure Appl. Math. 47 (1994), 437–476.

[12] G. Springer, *Introduction to Riemann surfaces*, Addison-Wesley 1957.