PANORAMAS OF MATHEMATICS
BANACH CENTER PUBLICATIONS, VOLUME 34
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1995

THE XVI-th HILBERT PROBLEM ABOUT LIMIT CYCLES

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1. Introduction. The XVI-th Hilbert problem consists of two parts. The first part concerns the real algebraic geometry and asks about the topological properties of real algebraic curves and surfaces. The second part deals with polynomial planar vector fields and asks for the number and position of limit cycles.

The progress in the solution of the first part of the problem is significant. The classification of algebraic curves in the projective plane was solved for degrees less than 8. Among general results we notice the inequalities of Harnack and Petrovski and Rohlin's theorem. Other results were obtained by Newton, Klein, Clebsch, Hilbert, Nikulin, Kharlamov, Gudkov, Arnold, Viro, Fidler. There are multidimensional generalizations: the theory of Khovansky, the inequalities of Petrovski and Oleinik, and others.

In contrast to the algebraic part of the problem, the progress in the solution of the second part is small. In the present article we concentrate on the second part of the XVI-th Hilbert problem.

2. Basic definitions. We consider the systems of differential equations of the form

$$\dot{x} = f(x,y),\ \dot{y} = g(x,y), \tag{1}$$

where f, g are polynomials of degree $\leq n$. The qualitative properties of such systems are determined by the properties of their solutions $\phi(t) = (x(t), y(t))$.

1991 *Mathematics Subject Classification*: Primary: 34C05, Secondary: 34C23.
Lecture given at the Banach Center Colloquium on 27th January 1994.
The paper is in final form and no version of it will be published elsewhere.

Some of the solutions are *equilibrium points*, the solutions to $f = g = 0$ (in the number $\le n^2$). Other are periodic solutions $\phi(t+T) = \phi(t)$ giving phase curves diffeomorphic to the circle.

A *limit cycle* of system (1) is a periodic solution which is isolated among the set of all periodic solutions. Therefore the trajectories $\phi(t)$ which start near a limit cycle γ tend to γ as $t \to \infty$ or as $t \to -\infty$. It is possible that the limit cycle is stable on one side and unstable on the other side.

The second part of the XVI-th Hilbert problem asks for the number and position of limit cycles of system (1). In particular, one wants to get the bound $N(n)$ for the number of limit cycles of systems of degree $\le n$.

3. History of errors. Interesting is the history of the second part of the XVI-th Hilbert problem.

The first task is to show the *finiteness* of the number of limit cycles for an individual vector field. In the beginning of 1920-th years Dulac published the memoir [4], which contained the proof of the finiteness theorem. The result had been considered to be true until the end of 70-ties when some people started to read Dulac's book carefully and could not understand some of his arguments. It was Il'iashenko who in 1979 definitively showed that Dulac's proof contains an essential gap.

Since then Il'iashenko started to work on completing the proof of the finiteness theorem. He succeeded. His complete demonstration is contained in the book [9] (300 pages), but the main ideas can be found in his shorter work [8].

Another proof was announced by the group of French mathematicians, J. Ecalle, J. Martinet, R. Moussu, J. P. Ramis, in [5]. In that note they gave a general scheme of the proof. Later Ecalle published more details in [6].

Therefore we have two proofs. However only Il'iashenko's proof is clear and well established; he worked on its presentation since 1986. The quality of the French proof is far below the Russian one.

Now the specialists are working on the local finiteness, i.e. on the bound for the number of limit cycles for a local family $V_\lambda,\ \lambda \in \mathbb{R}^k, 0$ of polynomial systems. This would give the existence of the uniform bound $N(n) < \infty$.

Dramatic efforts were connected with estimating $N(2)$. (The linear case is trivial, $N(1) = 0$.)

Probably the first example of a quadratic system with a limit cycle was given in 1929 by the physicist Sommerfeld in [13], where he got two cycles at once. In 1939 Bautin [1] announced that $N(2) \ge 3$, the full proof appeared in 1952 [2]. In 1955 Petrovski and Landis [10] claimed to have a proof that $N(2) = 3$. Later there was found a mistake in it (Novikov) and in 1967 they retracted from this result.

However, the conjecture $N(2) = 3$ remained until 1979. In that year the Chinese mathematician Shi Songling gave the following example of a quadratic system with four limit cycles

$$\begin{aligned}\dot{x} &= \lambda x - y - 10x^2 + (5+\delta)xy + y^2 \\ \dot{y} &= x + x^2 + (-25 + 8\epsilon - 9\delta)xy,\end{aligned} \tag{2}$$

where $\delta = -10^{-13}$, $\epsilon = -10^{-52}$, $\lambda = -10^{-200}$ (!). Thus, now the conjecture is that $N(2) = 4$.

There is no reasonable conjecture about $N(n)$ for arbitrary n. In fact, Petrovski and Landis in [11] claimed an estimate for $N(n)$ but the fate of that result was the same as that from [10].

In what follows we present some details of Shi's example, the general ideas of Il'iashenko's proof and the connection with Abelian integrals.

4. Shi's example. Consider first the situation when $\lambda = \epsilon = \delta = 0$. Then the phase portrait of system (2) in the plane compactified with a circle, representing the directions at infinity, is presented at Figure 1. (It is known that the phase portrait of a polynomial system can be prolonged to such compactification, called the *Poincaré compactification*.)

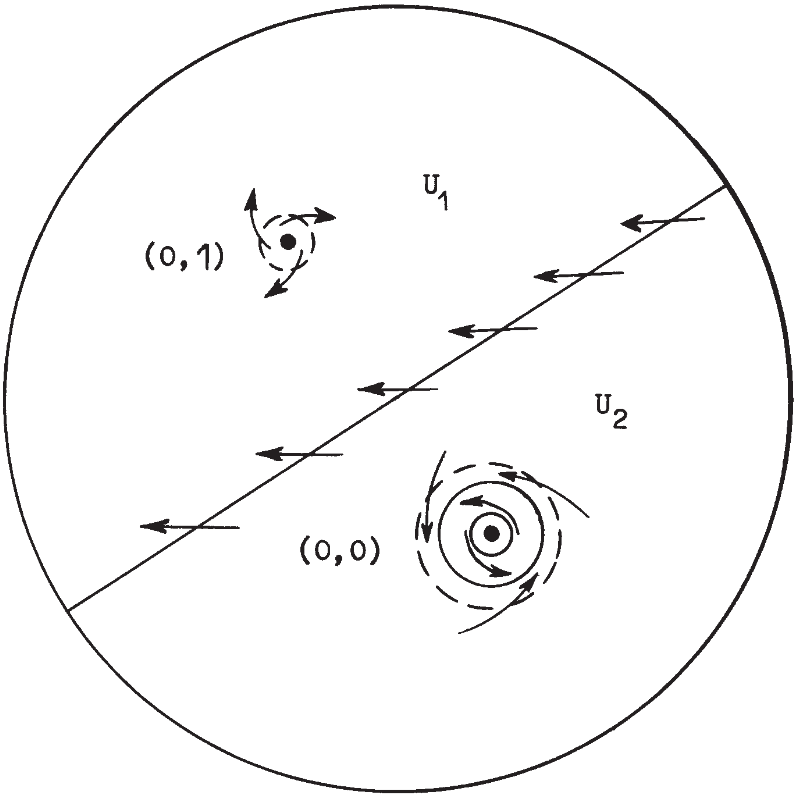

Figure 1

The system has the line $L:\ 1+x-25y=0$ without contact (all trajectories pass through it in one direction). The circle at infinity is invariant for the system. The system has two finite equilibrium points $(0,1)$ and $(0,0)$ separated by L. The point $(0,1)$ is a strong unstable focus and the point $(0,0)$ is a very weak stable focus.

After perturbation near the weak focus $(0,0)$ two limit cycles γ_1,γ_2 appear. The point $(0,1)$ remains an unstable focus. Moreover we have two ring-like domains U_1 and U_2. U_1 is bounded by L, a half-circle at infinity and a small circle around $(0,1)$. U_2 is bounded by L, the other half-circle at infinity and a curve close to the larger of the cycles $\gamma_{1,2}$. U_i do not contain equilibrium points and at their boundaries the trajectories either enter the domain (U_1) or leave the domain (U_2). Such situations lead to the existence of additional limit cycles in U_i (the Bendixson's criterion [3]).

5. Proof of the finiteness theorem. At the beginning we follow the way chosen by Dulac [4], next we shall see his mistake and then we shall show how Il'iashenko overcame the difficulties.

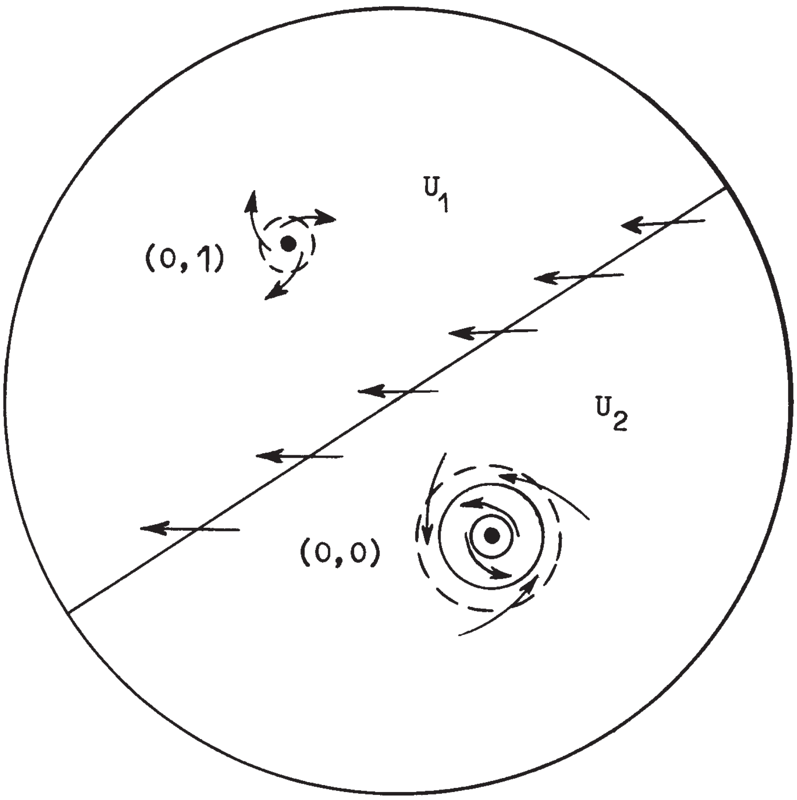

Figure 2

Assume that system (1) has an infinite number of limit cycles. Because any periodic solution γ bounds a region which contains at least one equilibrium point (the index of the vector field along γ is 1), there are infinitely many concentric cycles γ_n. They lie in the compact disc, the Poincaré model.

Consider the limit set of $\{\gamma_n\}$. It cannot contain a periodic orbit δ. (We consider the return map $\Delta_\delta : S \to S$ for a transversal section S to δ which is analytic and cannot have infinitely many accumulating fixed points.) There are two possibilities:

(i) $\lim \gamma_n = \Gamma$, a composed cycle consisting of equilibrium points and separatrices connecting them (see Figure 2);

(ii) $\lim \gamma_n$ is an equilibrium point, which we assume equal to $(0,0)$.

Lemma. *The case (ii) reduces to (i).*

Proof. We have $f(0,0) = g(0,0) = 0$ and in the polar coordinates (r,ϕ) system (1) writes as

$$\dot{r} = r^{l+1}F(r,\phi),\ \dot{\phi} = r^l G(r,\phi),\ G(0,\cdot) \neq 0.$$

We can divide this system by r^l (the phase portrait does not change and only the velocity changes along trajectories). We obtain a vector field in a cylinder with invariant circle $r = 0$. The circle $r = 0$ may contain equilibrium points to which we can apply the above procedure. We get again an invariant circle with possible singularities etc.

The well known theorem of Bendixson–Seidenberg–van den Essen [3] states that this process ends after a finite number of steps. We obtain a contour Γ consisting of elementary critical points and separatrices connecting them.

A singular point $x = y = 0$ is called *elementary* iff in some linear coordinates the system has one of the two forms:

$\dot{x} = \lambda_1 x + \ldots,\qquad \dot{y} = \lambda_2 y + \ldots$ (hyperbolic);

$\dot{x} = ax^{k+1} + \ldots,\qquad \dot{y} = \lambda y + \ldots$ (degenerate).

Of course, after the above resolution $\lim \gamma_n = \Gamma$. ∎

In the case (i) we also resolve all non-elementary singular points appearing in the vertices of Γ. Hence, we can assume that $\lim \gamma_n = \Gamma$ and Γ contains only elementary critical points.

Take a section S transversal to Γ at a point of one of its separatrices. Then the trajectories of the vector field define the return (or *monodromy*) map $\Delta_\Gamma : S \to S$. The fixed points of Δ_Γ correspond to periodic solutions, limit cycles if the points are isolated. We have to show that Δ_Γ does not have infinitely many isolated and accumulating fixed points.

We can represent Δ_Γ as a composition $\Delta_m \circ \ldots \circ \Delta_1$ of maps, where each $\Delta_i : S_{i-1} \to S_i$ (S_i – sections to Γ) is a map defined either by trajectories passing along a separatrix of Γ or by trajectories passing near a single critical point (see

Figure 2). The maps of the first kind are analytic. The maps of the second kind contain essential singularities.

Consider the linear system $\dot{x} = \lambda_1 x,\ \dot{y} = -\lambda_2 y$. Then $\Delta_i(x) = Cx^{\lambda_2/\lambda_1}$. For a non-linear hyperbolic critical point we get the asymptotic *Dulac series*

$$\Delta_i(x) = c_0 x^{\nu_0} + \sum x^{\nu_j} P_j(\ln x),\ \nu_j \to \infty,$$

where P_j are polynomials.

Consider the degenerate critical point $\dot{x} = ax^{k+1},\ \dot{y} = -y$. Then $\Delta_i(x) = Ce^{-a/(kx^k)},\ x \geq 0$.

Therefore each Δ_i is either a Dulac series or a flat map or a map inverse to a flat map. From this we can conclude that Δ_Γ either is a flat map or is inverse to a flat map or expands into the asymptotic Dulac series. To see the latter property we consider the following example:

$$\{e^{-b/x^l}\}^{-1} \circ x^\nu \circ e^{-a/x^k} = (-\frac{1}{b}\ln x)^{-1/l} \circ e^{-a\nu/x^k} = (\frac{a\nu}{bx^k})^{1/l}.$$

The flat or inverse to flat maps do not have accumulating fixed points. If the Dulac series is convergent then it cannot also have accumulating fixed points ($\Delta_\Gamma - \mathrm{id} = dx^\mu(\ln x)^k(1 + \ldots) \neq 0$ near 0).

Here the mistake made by Dulac lies. He assumed that only the Dulac series governs the behaviour of Δ_Γ. But we have the following example:

$$-\ln x \circ x^\nu(1 - x^\mu) \circ e^{-1/x^k} = -\ln x \circ e^{-\nu/x^k}(1 - e^{-\mu/x^k})$$

$$= \nu x^{-k} - \sum \frac{1}{j} e^{-\mu j/x^k}.$$

We see that besides the Dulac series Δ_Γ may contain flat terms which cannot be detected by the Dulac series.

Now we must leave Dulac and follow the way of Il'iashenko. Because this way is very long and complicated, we restrict ourselves to some general ideas.

Il'iashenko goes into the complex domain. Namely, he uses the logarithmic chart $\zeta = -\ln x$, transforming a neighbourhood of the point $x = 0$ in $\mathbb{C}$ into the half-plane $\mathrm{Re}\,\zeta > C_0$.

If all the vertices of Γ are hyperbolic then Δ_Γ prolongs holomorphically to a domain Ω biholomorphically equivalent to $\mathrm{Re}\,\zeta > C_1$. Next he applies the Fragmen-Lindelöf principle to get the property that if $(\Delta_\Gamma - \mathrm{id})(\zeta_n) = 0,\ \zeta_n \to \infty$ then $\Delta_\Gamma = \mathrm{id}$. (The Fragmen-Lindelöf principle allows to estimate a function (with good behaviour at infinity) in a sector in $\mathbb{C}$ by its values at the boundary of the sector.)

If one of the vertices of Γ is a degenerate critical point then the good analytic properties of Δ_i hold only in sectors $\{\arg x \in (\beta_1^{(s)}, \beta^{(s)})\}$ and $\Delta_\Gamma(\zeta)$ is well defined in strips. We have some cocycle corresponding to a covering of the domain Ω by strips. The coboundary of the cocycle is small in intersections of strips. Using this Il'iashenko was able to apply a modification of the Fragmen-Lindelöf principle. Of course, here he had to overcome a lot of technical problems which cannot be included into this article.

Ecalle chose a different way. He applied to Δ_Γ a kind of Borel transform ($\sum a_i x^i \to \sum (a_i/i!) x^i$). The Borel transform of Δ_Γ is analytic with good properties. Next, he applied the integral Laplace transform, which turns out to be inverse to the Borel transform.

6. The linearization of the XVI-th Hilbert problem. Consider the system

$$\dot{x} = \frac{\partial H}{\partial x} + \epsilon P,\ \dot{y} = -\frac{\partial H}{\partial y} + \epsilon Q,$$

where H, P, Q are polynomials of degree $\leq n$ and ϵ is a small parameter. For $\epsilon = 0$ the system is Hamiltonian and its phase curves are given as $H(x,y) = h$; some of them form a continuous family of periodic orbits.

After perturbation this picture becomes spoiled and only few periodic solutions remain. To investigate them we consider the return map $\Delta : S \to S$, where S is a section to the family $H = h$ parametrized by h. We get

$$\Delta(h) - h = \Delta H = \int_0^T \frac{d}{dt} H(x(t), y(t)) dt = \epsilon \int_{\gamma(\epsilon,h)} Qdx - Pdy,$$

where T is the time of return and $\gamma(\epsilon, h)$ is a trajectory of the perturbed system. But $\gamma(\epsilon, h) \approx \{H = h\}$. Thus, $\Delta H = \epsilon I_\omega(h) + O(\epsilon^2)$, where

$$I_\omega(h) = \int_{H=h} \omega$$

is the Abelian integral of the 1-form $\omega = Qdx - Pdy$. Under some generic assumptions about H we conclude that the number of limit cycles is equal to the number of zeroes of I_ω.

The *weakened* (or linearized) *XVI-th Hilbert problem* consists in estimating the number of zeroes of I_ω (Arnold).

The general result was obtained by Varchenko [15]. The number of zeroes is bounded by a constant $C(n)$. He did not give the expression for $C(n)$.

There are only few concrete estimates of the number of zeroes of Abelian integrals. The best was obtained by Petrov [12] for the elliptic Hamiltonian $H = y^2 + x^3 - x$.

Let V_n be the space of functions I_ω, $\deg \omega \leq n$. Then the number of zeroes of $I_\omega \in V_n$, $I_\omega \neq 0$ is not greater than $n - 1$, where $n = \dim V_n$. This property of a space of functions is known as the *Chebyshev property.*

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