SINGULARITIES AND DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 33
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1996

A COUNTEREXAMPLE TO THE L^p-HODGE DECOMPOSITION

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Abstract. We construct a bounded domain $\Omega \subset \mathbb{R}^2$ with the cone property and a harmonic function on Ω which belongs to $W_0^{1,p}(\Omega)$ for all $1 \le p < 4/3$. As a corollary we deduce that there is no L^p-Hodge decomposition in $L^p(\Omega, \mathbb{R}^2)$ for all $p > 4$ and that the Dirichlet problem for the Laplace equation cannot be in general solved with the boundary data in $W^{1,p}(\Omega)$ for all $p > 4$.

1. Introduction. It is natural to consider the Poisson equation $\Delta u = h$ in the Sobolev space $W^{1,2}(\Omega)$. However, in some problems we have to seek for solutions in $W^{1,p}(\Omega)$, where $p \neq 2$. This appears for example in the context of the L^p-Hodge decomposition which has recently found many applications to Navier–Stokes equations [13], quasiregular mappings [10], [7], [8], harmonic mappings [1], calculus of variations [9], higher integrability of the Jacobian [11], [5], [6], elliptic equations [12], [16], [18] and many other problems. Usually one assumes that the boundary of the domain is sufficiently regular. Then one proves the uniqueness and the existence of the solution to the Dirichlet problem for the Poisson equation and hence that for the L^p-Hodge decomposition. In this note we give a very elementary example which shows that a single cusp on the boundary can imply nonexistence and nonuniqueness results. In a slightly different context the nonuniqueness for Dirichlet problem in a domain with a single cusp on the boundary has been studied in [14].

2. Result. For the sake of simplicity we assume that Ω is a bounded domain. By $W_0^{1,p}(\Omega)$ we denote the closed subspace of $W^{1,p}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ in

1991 *Mathematics Subject Classification*: Primary 35J05, 35J25.

Key words and phrases: Dirichlet problem, Laplace equation, nonuniqueness, Hodge decomposition.

This work is supported by KBN grant no. 2 1057 91 01.

The paper is in final form and no version of it will be published elsewhere.

the $W^{1,p}$ norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. It is a well known and trivial fact that the only harmonic function in $W_0^{1,2}(\Omega)$ is the constant equal to zero. Hence the same holds in $W_0^{1,p}(\Omega)$ for all $p \geq 2$ (because $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,2}(\Omega)$). In the case $p \geq 2$ we do not need any assumptions about the regularity of the boundary. But what about the case $p < 2$? In the case $p = 1$ it suffices to assume that $\partial\Omega \in C^{1,1}$ (cf. [4, Vol. 1, Chapter 2, Prop. 13]) and in the case $1 < p < 2$ that $\partial\Omega \in C^1$ (see Remark 2 below).

We start with a very elementary example which shows that if we loosen the smoothness assumptions on the boundary, then the above result on the uniqueness of Dirichlet problem may fail for $1 \leq p < 2$.

Let Ω be the image of the two dimensional disc $D = \{z \mid |z - i| < 1\}$ under the mapping $z \mapsto z^2$. The boundary of Ω is a smooth Jordan curve except the point $z = 0$, where we have an interior cusp. Moreover, this domain has the cone property, i.e., there exists a cone V with a finite height so that every point $x \in \Omega$ has a cone V_x congruent to V and with vertex at x such that $V_x \subset \Omega$.

THEOREM 1. *If Ω is as above, then the function $u(z) = \operatorname{Im}(z^{-1/2} + i/2)$ is harmonic in Ω and belongs to $W_0^{1,p}(\Omega)$ for all $1 \leq p < 4/3$.*

Proof. The mapping $z \mapsto 1/z + i/2$ maps D onto the halfplane $\{\operatorname{Im} z < 0\}$ and hence the boundary ∂D is mapped onto the x-axis. So $f(z) = \operatorname{Im}(1/z + i/2)$ is harmonic and it vanishes on ∂D, except the discontinuity point $z = 0$. But since Ω and u are obtained from D and f by composition with $\sqrt{z}$, it follows that u is a harmonic function and equal to zero on the boundary of Ω, except the point $z = 0$, where u is not continuous. Elementary estimation shows that $|\nabla u| \in L^p$ for all $p < 4/3$. Now it easily follows that $u \in W_0^{1,p}(\Omega)$.

Remark 1. As was pointed out to the author by Pekka Koskela it follows from Brennan's conjecture that the exponent $4/3$ in Theorem 1 is in some sense critical. Namely, if $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain and $\phi : \Omega \to B^2$ is the Riemann mapping (B^2 is the unit disc) then as was conjectured by Brennan [3], $|\nabla\phi| \in L^p(\Omega)$ for all $p < 4$. Assume now that $u \in W_0^{1,q}(\Omega)$ for some $q > 4/3$. If Brennan's conjecture is true then it easily follows that $u \circ \phi^{-1} \in W_0^{1,1}(B^2)$. Moreover, if u is harmonic then $u \circ \phi^{-1}$ is harmonic as well. The only harmonic function in $W_0^{1,1}(B^2)$ is the constant function equal to zero and hence we deduce that if $u \in W_0^{1,q}(\Omega)$ is a nontrivial harmonic function then $q \leq 4/3$. As far as I know, Brennan's conjectute is still open and the best known result in this direction is due to Pommerenke [15] who proved that $|\nabla\phi| \in L^p(\Omega)$ for all $p < 3.399$. Of course this result also gives a certain estimate for q.

The rest of this paper is devoted to showing how to use Theorem 1 to obtain examples of nonexistence for the L^p-Hodge decomposition and nonexistence of solutions to the Dirichlet problem for Laplace and Poisson equations. We start with an elementary general observation.

THEOREM 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $1 < p, q < \infty$, $1/p + 1/q = 1$. Then*:

(A) *The following two conditions are equivalent*:

(a) *For every* $h \in (W_0^{1,q}(\Omega))^*$ *the Dirichlet problem*

$$\Delta u = h, \qquad u \in W_0^{1,p}(\Omega),$$

has a solution.

(b) *Every vector field* $F \in L^p(\Omega, \mathbb{R}^n)$ *admits an* L^p*-Hodge decomposition, i.e., there exist* $u \in W_0^{1,p}(\Omega)$ *and* $H \in L^p(\Omega, \mathbb{R}^n)$, $\operatorname{div} H = 0$, *such that*

$$F = \nabla u + H.$$

Moreover, *the solution of the Dirichlet problem of* (a) *is unique if and only if the* L^p*-Hodge decomposition of* (b) *is unique.*

(B) *If the above conditions* (a), (b) *are satisfied* (*we do not assume uniqueness*), *then the only harmonic function in* $W_0^{1,q}(\Omega)$ *is the constant function equal to zero.*

Proof. (A) First we prove that (b)⇒(a). Let $h \in (W_0^{1,q}(\Omega))^*$ be taken at will. It follows from the Poincaré inequality that $\nabla : W_0^{1,q}(\Omega) \to L^q(\Omega, \mathbb{R}^n)$ is an isomorphism onto a closed subspace and hence by the Hahn–Banach theorem the functional h can be represented in the form $h[v] = \int \langle F, \nabla v\rangle$ for a certain $F \in L^p(\Omega, \mathbb{R}^n)$. Let $F = \nabla u + H$ be the L^p-Hodge decomposition. Then $h[v] = \int \langle \nabla u, \nabla v\rangle$ and hence $h = -\Delta u$. The proof of the implication (a)⇒(b) is easier and involves similar arguments.

Equivalence of uniqueness of solutions to (a) and (b) is evident.

(B) To the contrary we assume that $v \in W_0^{1,q}(\Omega)$, $v \not\equiv 0$, $\Delta v = 0$. Let $h \in (W_0^{1,q}(\Omega))^*$ be such that $h[v] \neq 0$. It follows from (a) that there exists $u \in W_0^{1,p}(\Omega)$ with $\Delta u = h$. Now $0 \neq \Delta u[v] = u[\Delta v] = 0$. This contradiction completes the proof.

Remark 2. If the boundary of the domain is sufficiently regular, say $\partial\Omega \in C^1$, then it follows from Calderón–Zygmund theory that both conditions (a) and (b) of the above theorem are satisfied (see [17, Thm. 4.6]) and hence by Theorem 2(B) for every $1 < p < \infty$ the only harmonic function in $W_0^{1,p}(\Omega)$ is the constant equal to zero. However, as follows from Theorem 1 the uniqueness does not hold for a domain which appears in Theorem 1. Hence we get the following theorem.

Theorem 3. *If* Ω *is as in Theorem* 1 *and* $p > 4$, $1/p + 1/q = 1$, *then there exists* $h \in (W_0^{1,q}(\Omega))^*$ *such that the Dirichlet problem* $\Delta u = h$, $u \in W_0^{1,p}(\Omega)$, *does not admit any solution. Moreover*, *there exists a vector field* $F \in L^p(\Omega, \mathbb{R}^2)$ *which does not admit an* L^p*-Hodge decomposition.*

Remark 3. The domain Ω which appears in Theorem 3 has the cone property and hence Theorem 3 is slightly surprising when compared with the result of Bogovskiĭ [2] who proved that if $\Omega \subset \mathbb{R}^n$ is a bounded domain with the cone property and $1 < p < \infty$ is taken arbitrarily, then every function $f \in L^p(\Omega)$ admits an L^p-Hodge decomposition, i.e., there exists a vector field $F \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ and a constant C such that $f = \operatorname{div} F + C$.

Theorem 4. *If* Ω *is as in Theorem* 1 *and* $p > 4$, *then there exists* $u_0 \in W^{1,p}(\Omega)$ *such that the Dirichlet problem*

$$\Delta u = 0, \qquad u - u_0 \in W_0^{1,p}(\Omega),$$

does not admit any solution $u \in W^{1,p}(\Omega)$.

Remark 4. Note that in this domain, the classical Dirichlet problem for the Laplace equation, for any continuous boundary data, admits a unique solution.

Proof of Theorem 4. Let $F \in L^p(\Omega, \mathbb{R}^2)$ be a vector field which does not admit an L^p-Hodge decomposition. We extend this vector field onto $\mathbb{R}^2$ putting $F = 0$ outside Ω. Then we can apply to this extended F a variant of Hodge decomposition which holds on the entire $\mathbb{R}^2$ (see [10], [9]):

$$F = \nabla v + H,$$

where $v \in L^{1,p}(\mathbb{R}^2) = \{f \in \mathcal{D}'(\mathbb{R}^2) \mid \nabla f \in L^p(\mathbb{R}^2)\}$, $\operatorname{div} H = 0$. It is well known that $L^{1,p} \subset W^{1,p}_{\mathrm{loc}}$, so we get $u_0 = v|_\Omega \in W^{1,p}(\Omega)$. Now the Dirichlet problem stated in Theorem 4 (with just defined u_0) does not admit any solution. For if not we would have the L^p-Hodge decomposition $F = \nabla(v - u) + (H + \nabla u)$ in Ω.

Acknowledgements. The author wishes to thank Pekka Koskela who pointed out to him the consequences of Brennan's conjecture.

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