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ON A LINEAR HYPERBOLIC EQUATION WITH SMOOTH COEFFICIENTS WITHOUT SOLUTIONS

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Abstract. An example of a locally unsolvable hyperbolic equation of the second order is constructed, which has smooth (C^∞) coefficients, but has no solutions in the class of distributions.

1. Introduction. We consider here an equation of the form

$$Pu = \frac{\partial^2 u}{\partial t^2} - a(t)^2 \frac{\partial^2 u}{\partial x^2} + b(t)\frac{\partial u}{\partial x} = f(t,x), \tag{1}$$

where a, b, f are some real-valued C^∞ functions, and prove that it can have no solutions in the class of distributions in any neighborhood of the origin. Here $a(t) \geq 0$ and the formally adjoint operator P^* is also locally non-solvable.

The equation (1) is (weakly) hyperbolic. The Cauchy problem for weakly hyperbolic equations has been studied by M. Protter [1], M. M. Smirnov [2], V. Ya. Ivriĭ and V. Petkov [3], O. A. Oleĭnik and others.

One of the classical methods to study the equation (1) is factorization. Let

$$v = \frac{\partial u}{\partial t} - a(t)\frac{\partial u}{\partial x}.$$

Then the function v satisfies the relation

$$\frac{\partial v}{\partial t} + a(t)\frac{\partial v}{\partial x} + (b(t) + a'(t))\frac{\partial u}{\partial x} = f.$$

Putting $U = (u, v)$, we obtain the system

$$\frac{\partial U}{\partial t} + A(t)\frac{\partial U}{\partial x} + B(t)U = F(t),$$

where

$$A(t) = \begin{pmatrix} -a(t) & 0 \\ b(t) + a'(t) & a(t) \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

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The principal part of this system can be diagonalized if we put $w = u + k(t)v$, where

$$k(t) = 2a(t)/(a'(t) + b(t)).$$

Then

$$\frac{\partial w}{\partial t} + a(t)\frac{\partial w}{\partial x} = \frac{k'(t)+1}{k(t)}(w-u) + k(t)f(t).$$

This shows that the solvability of the Cauchy problem depends on the properties of the function k (which can also be replaced by $k_1(t) = 2a(t)/(a'(t) - b(t))$, if one takes $v_1 = \partial u/\partial t + a(t)\partial u/\partial x$).

Another approach consists in the consideration of energy estimates. Multiplying (1) by $2\partial u/\partial t$, and integrating over $Q_T = (0,T) \times \mathbf{R}$, we obtain

$$\int_{t=T} \left[\left(\frac{\partial u}{\partial t}\right)^2 + a(t)^2\left(\frac{\partial u}{\partial x}\right)^2\right]dxdt = 2\int_{Q_T}\left[f\frac{\partial u}{\partial t} + ((a(t)^2)' - b(t))\left(\frac{\partial u}{\partial x}\right)^2\right]dxdt.$$

The method works if, for example,

$$(a(t)^2)' - b(t) \le ka(t)^2$$

with a constant k. This inequality is sufficient for the Cauchy problem to be well posed.

O. A. Oleĭnik [4] used a modification of this method and proved that the Cauchy problem is well posed if the following condition holds:

There exist some constants $\alpha, A, T_0, \ldots, T_N$, *such that* $0 = T_0 < T_1 < \ldots < T_N = T$ *and for* $T_j < t < T_{j+1}$, $j = 0, 1, \ldots, N-1$, *the inequality*

$$\alpha(t - T_j)b(t)^2 \le Aa(t)^2 + (a(t)^2)'_t$$

or the inequality

$$\alpha(T_{j+1} - t)b(t)^2 \le Aa(t)^2 + \frac{a(t)^2}{\alpha(T_{j+1}-t)} - (a(t)^2)'_t$$

is true.

On the other hand, the well-known classical example of an equation of the first order which is locally non-solvable is one of H. Lewy [5]:

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + i(x+iy)\frac{\partial u}{\partial t} = f(t,x,y).$$

L. Hörmander [6] has proved that the real equation of the second order

$$(y^2 - z^2)\frac{\partial^2 u}{\partial x^2} + (1+x^2)\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2}\right) - xy\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 (xyu)}{\partial x \partial y} + xz\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 (xzu)}{\partial x \partial z} = f(x,y,z)$$

is also locally non-solvable.

In the works of L. Hörmander [6], L. Nirenberg and F. Trèves [7], and Yu. V. Egorov [8] necessary conditions were given for the local solvability of differential equations of the general form (see [9]). The most general result in this direction is the one of L. Hörmander [10]. The local solvability for the equations with double characteristics has been studied by F. Trèves, V. Ya. Ivriĭ, P. R. Popivanov, Ya. Kannai, Yu. V. Egorov and others. Ya. Kannai [11] proved the local unsolvability of the parabolic operator

$$\frac{\partial u}{\partial t} + t\,\frac{\partial^2 u}{\partial x^2} = f(t,x).$$

This equation is the "inverse heat equation" for all $t \neq 0$. In the work [12] of F. Colombini and S. Spagnolo an example of equation of the form (1) is given with a positive function $a(t)$ (however it is not regular), for which the Cauchy problem is locally non-solvable, and an example of the equation of the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{a(t)}{a(x)}\frac{\partial u}{\partial x}\right) = x,$$

having no solutions of class C^1 in any neighborhood of the origin, where $2^{-1} \le a(t) \le 2$, $a(t) \in C^\alpha$ for $\alpha < 1$, but $a(t) \notin C^1$.

In our example (1) the function $a(t) \in C^\infty(\mathbf{R})$. It is important to note also that the usual technique of the construction of the asymptotic solutions is not applicable in this case. In fact it is impossible to construct in a neighborhood of the origin a smooth phase function $w(t,x)$, with a positive imaginairy part, satisfying the equation

$$w_t^2 - a(t)^2 w_x^2 = 0$$

or the equation

$$w_t^2 - a(t)^2 w_x^2 + ib(t)w_x = 0.$$

2. Main result

Theorem. *There exist real functions*

$$a(t), b(t) \in C^\infty(\mathbf{R}), \quad a(t) \ge 0, \quad f(t,x) \in C^\infty(\mathbf{R}^2),$$

for which equation (1) *has no solution in the class of distributions in any neighborhood of the origin. The formally adjoint operator P^* is also locally non-solvable in any neighborhood of the origin.*

Lemma 1. *If equation* (1) *is locally solvable in a neighborhood ω of the origin, then there exist constants $C_1 \in \mathbf{R}$ and $N \in \mathbf{N}$ such that*

$$\|u\|_0 \le C_1 \|P^* u\|_N, \qquad u \in C_0^\infty(\omega). \tag{2}$$

Proof of Lemma 1. From the local solvability of equation (1) in the domain ω follows the existence of real constants C, s and r such that

$$\|u\|_s \le C\|P^* u\|_r, \qquad u \in C_0^\infty(\omega). \tag{3}$$

The statement of Lemma 1 is evident if $s \ge 0$. If $s < 0$, we choose $n_1 \in \mathbf{N}$ for which $n_1 + s \ge 0$. Since $D_x^i u \in C_0^\infty(\omega)$ for all i, if $u \in C_0^\infty(\omega)$, we have

$$\|D_x^i u\|_s \le C\|D_x^i f\|_r,$$

where $u \in C_0^\infty(\omega)$, $f = P^* u$, $i = 0, 1, \ldots$ Of course, we can assume that $r \ge s$.

Writing down $D_t^2 u$ from the equation $P^* u = f$, we obtain

$$\begin{aligned}\|D_t^2 u\|_{s-1} &\le C(\|D_x^2 u\|_{s-1} + \|D_x u\|_{s-1} + \|f\|_{s-1}) \\ &\le C_1(\|D_x u\|_s + \|u\|_s + \|f\|_{r-1}) \le C_2(\|D_x f\|_r + \|f\|_r).\end{aligned}$$

Therefore,

$$\begin{aligned}\|u\|_{s+1} &\le C_3(\|u\|_s + \|D_x^2 u\|_{s-1} + \|D_t^2 u\|_{s-1})\\ &\le C_4(\|f\|_r + \|D_x f\|_r + \|f\|_{r+1}) \le C_1'\|f\|_{r+1}.\end{aligned}$$

Repeating these arguments, we obtain

$$\|u\|_{s+2} \le C_2'\|f\|_{r+2}, \ \ldots, \ \|u\|_{s+n_1} \le C_{n_1}'\|f\|_{r+n_1}.$$

Since $s + n_1 \ge 0$, we have

$$\|u\|_0 \le C_{n_1}\|f\|_{r+n_1}.$$

The index $r + n_1$ on the right-hand side can always be enlarged. Therefore $\|u\|_0 \le C_{n_1}\|f\|_N$, where $N \in \mathbf{N}$. ∎

3. Proof of the Theorem. Let ω be a neighborhood of the origin, $\lambda \gg 1$ and the function F be of class $C_0^\infty(-1, +1)$. Let $k \in \mathbf{N}$ and $I_k = (1/k\pi, 1/(k-1)\pi)$.

The functions $a(t)$ and $b(t)$ in our example have the following form:

$$a(t) = \exp(-t^{-2} - \sin^{-2}(1/t)), \qquad b(t) = -2a(t)\mu'(t) - a'(t),$$

where $\mu(t) = -\sin^{-4}(1/t) - \ln|t|$ is a function such that $e^{\mu(t)} \to 0$ and $D_t^i e^{\mu(t)} \to 0$, as t tends to the end points of the interval I_k. It is obvious that $b \in C^\infty$ and

$$\int\limits_{I_k} e^{2\mu(t)} dt = \int\limits_0^\pi \exp(-2\sin^{-4} s) ds = c_1.$$

After the substitution $x_1 = x - A(t)$, where A is a function such that $A'(t) = a(t)$, the equation $P^* u = 0$ takes the form

$$P_1 u \equiv \frac{\partial^2 u}{\partial t^2} - 2a(t)\frac{\partial^2 u}{\partial t \partial x} - (b(t) + a'(t))\frac{\partial u}{\partial x} = 0$$

(we drop the subscript 1 for simplicity).

We wish to construct a function $u_\lambda(t, x) \in C_0^\infty(K)$, where $K = I_k \times (-\lambda^{-1}, \lambda^{-1})$, such that

$$\|u_\lambda\|_0 \ge c_0 > 0, \qquad \|P^* u_\lambda\|_N \le C\lambda^{-1}. \tag{4}$$

After substitution of this function in (2), we are led to a contradiction for λ and k sufficiently large. Since for any neighborhood ω of the origin the domain K is inside ω for $\lambda > \lambda_\omega, k > k_\omega$, this proves that the operator P is locally non-solvable at the origin.

Let

$$u(t, x) = F(\lambda x)e^{\mu(t)} v(t, x).$$

Then the function v satisfies the equation

$$\frac{\partial^2 v}{\partial t^2} - 2a(t)\frac{\partial^2 v}{\partial t \partial x} - 2a(t)\lambda\frac{F'(\lambda x)}{F(\lambda x)}\frac{\partial v}{\partial t} + 2\mu'(t)\frac{\partial v}{\partial t} + (\mu'(t)^2 + \mu''(t))v = 0.$$

The change of the variable $x_2 = \lambda x_1$ gives:

$$\frac{\partial^2 v}{\partial t^2} - 2a(t)\lambda\frac{\partial^2 v}{\partial t \partial x} - 2a(t)\lambda\frac{F'(x)}{F(x)}\frac{\partial v}{\partial t} + 2\mu'(t)\frac{\partial v}{\partial t} + (\mu'(t)^2 + \mu''(t))v = 0 \tag{5}$$

(we drop again the subscript 2).

We are looking for an approximate solution of the equation (5) of the form

$$v(t,x) = \sum_{j=0}^{N+1} \lambda^{-j} v_j(t,x),$$

where $v_0(t,x) = 1, v_j(t,x) = \mu_j(t)F_j(x)$, for $j = 1,\dots,N+1$ and

$$\frac{\partial^2(F(x)v_j(t,x))}{\partial t \partial x} = \frac{F(x)}{a(t)}\left[\frac{\partial^2 v_{j-1}(t,x)}{\partial t^2} + 2\mu'(t)\frac{\partial v_{j-1}(t,x)}{\partial t} + [\mu'(t)^2 + \mu''(t)]v_{j-1}\right],$$

$j = 1,2,\dots,N+1$. But then

$$G'_j(x) = G_{j-1}(x), \qquad \mu'_j(t) = [\mu''_{j-1} + 2\mu'\mu'_{j-1} + (\mu'' + \mu'^2)\mu_{j-1}]/a(t),$$

where $G_j(x) = F(x)F_j(x)$, $j = 1,2,\dots,N+1$, $G_0(x) = F(x)$, $\mu_0(t) = 1$. Of course, this means that we have first to choose a smooth function $G_{N+1}(x)$ such that F_{N+1} is flat at the end points of $(-1,+1)$ and next to put $G_{N+1-j}(x) = G^{(j)}_{N+1}(x),\ j = 1,\dots,N+1$. It is easy to see that the function $\mu_j(t)e^{\mu(t)}$ is smooth in I_k and flat at the end points of I_k.

On the other hand,

$$\iint_K e^{2\mu(t)}F(\lambda x)^2 dxdt \geq c_0\lambda^{-1} > 0, \qquad \iint_K e^{2\mu(t)}F(\lambda x)^2 v_j(t,\lambda x)^2 dxdt \leq C_{j,k}\lambda^{-1},$$

$j = 1,2,\dots,N+1$ and thus $\|u\|_0^2 \geq c_0\lambda^{-1}/2$ for $\lambda > \Lambda(\omega,k,N)$. At the same time $\|P^*u(t,x)\|_N^2 \leq C\lambda^{-3}$. Therefore the inequalities (4) are valid for the function

$$u_\lambda = \lambda^{1/2}F(\lambda(x - A(t)))e^{\mu(t)}v(t,\lambda(x - A(t))).$$

This implies the statement of Theorem in the standard way (see [10]).

In order to prove the theorem for the adjoint operator P^* it is sufficient to remark that it can be obtained from P after the substitution $x = -x_1$.

The proof is complete.

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