ON MARTINET’S SINGULAR SYMPLECTIC STRUCTURES

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Introduction. Let \( V \) be a stratified subspace of \( \mathbb{R}^N \). We call it symplectic if there exists a differential 2-form \( \omega \) on \( \mathbb{R}^N \) such that the restriction of \( \omega \) to each stratum is a symplectic form. In the Marsden-Weinstein singular reduction theory these spaces were studied by several authors [5, 4, 9, 1]. In this paper we classify the symplectic spaces modelled on the so-called symplectic flag \( S \). First we prove the corresponding Darboux theorem and then we show that the only reasonable symplectic structures on \( S \) are those with underlying Martinet’s singular symplectic structure of type \( \Sigma_{2,0} \). Finally we find the normal form for this structure and show the similar result for an example of a stratified symplectic space with singular boundary of the maximal stratum.

1. Singular symplectic spaces. A stratified differential space with each stratum being a symplectic manifold is called a stratified symplectic space. This notion was introduced in [9] (see also [4]) in the context of standard symplectic reduction. For our purpose, in the first step we need embedded symplectic spaces.

Definition 1.1. Let \( S \) be a stratified subset of \( \mathbb{R}^N \) with each stratum \( S_i \) (even dimensional) endowed with a symplectic structure \( \omega_{S_i} \). We assume that there exists a closed two-form \( \omega \) on \( \mathbb{R}^N \) such that \( \omega|_{S_i} = \omega_{S_i} \). Then the pair \((S, \omega)\) is called a singular symplectic space.

A representative model of a singular symplectic space is a disjoint union of semialgebraic sets. We consider the following elementary symplectic flag:

\[
S = S_{2n} \cup S_{2n-2} \subset \mathbb{R}^{2n},
\]

where

\[
S_{2n} = \{(x,y) \in \mathbb{R}^{2n} : x_1 > 0\}, \quad S_{2n-2} = \{(x,y) \in \mathbb{R}^{2n} : x_1 = 0, y_1 = 0\}
\]

endowed with a symplectic structure \( \omega \). By \( t_k : S_i \to \mathbb{R}^N \) we denote the canonical inclusions of \( S_{2n-k} \). Here \( S_{2n-1} = \{x \in \mathbb{R}^{2n} : x_1 = 0\} \).


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Example 1.1. Let $V \subset (M, \omega)$ be an algebraic hypersurface. Let $X_V$ be its Whitney stratification. By $V^d$ we denote an element of $X_V$, $V^d \in X_V$, of dimension $d$. We say $V$ is a coisotropic hypersurface if and only if each stratum of $X_V$ is a coisotropic or an isotropic submanifold of $(M, \omega)$. We easily see that a typical hypersurface $V$ defined by the polynomial equation $F(p) = 0$ is not coisotropic. As an example let us consider the cusp-edge surface in $R^{2n}$ endowed with a symplectic form $\omega$ in general position with respect to $V$. In this case $\omega|_{\text{Sing} V}$ is a symplectic form. It is shown in [2] that $(V, \omega)$ is diffeomorphic to $(\{ x_1^3 - y_1^2 = 0 \}, \sum_{i=1}^n dx_i \wedge dy_i)$ and the reduced symplectic space $V - \text{Sing} V$ is isomorphic to the singular edge of $V$ (cf. [4]).

We conjecture that if $\text{Sing} V$ is a coisotropic submanifold of $(R^{2n}, \omega)$, then $(V, \omega)$ is diffeomorphic to $(\{ x_1^3 - x_2^2 = 0 \}, \sum_{i=1}^n dx_i \wedge dy_i)$. Let $\Phi : R^{2n-1} \to R^{2n}$ be the parameterization of $\{ x_1^3 - x_2^2 = 0 \}$,

$$\Phi(s, y_1, y_2, x_3, y_3, \ldots, x_n, y_n) = (s^3, y_1^3, y_2, x_3, y_3, \ldots, x_n, y_n).$$

Then

$$\Phi^* \omega = ds \wedge d(3s^2 y_2 + 2sy_1) + \sum_{i=3}^n dx_i \wedge dy_i.$$

Let $\pi : R^{2n-1} \to R^{2n-2}$ be the mapping

$$\pi(s, y_1, y_2, x_3, y_3, \ldots, x_n, y_n) = (s, 3s^2 y_2 + 2sy_1, x_3, y_3, \ldots, x_n, y_n).$$

Let $S$ be the image of $\pi$. Then

$$S = \{(x, y) \in R^{2n-2} : x_1 \neq 0\} \cup \{(x, y) \in R^{2n-2} : x_1 = 0, x_2 = 0\}$$

and

$$\pi^* \left( \sum_{i=1}^{n-1} dx_i \wedge dy_i \right) = \Phi^* \omega.$$ 

The reduced space $S$ endowed with the Darboux form on $R^{2n-2}$ is a singular symplectic space.

Now we have a natural extension problem: let $\tilde{\omega}$ be a symplectic form on $S_{2n-2}$, we ask for the existence of the closed two-form on $N^S$ such that $\omega|_{S_{2n-2}} = \tilde{\omega}$ and $\omega|_{S_2}$ is symplectic. The first step in approaching this problem is to classify singular symplectic spaces $(S, \omega)$, where $\omega$ provides a symplectic structure on $R^{2n}$.

By $G_S$ we denote the group of germs of diffeomorphisms $(R^{2n}, 0) \to (R^{2n}, 0)$ preserving $S$, i.e. if $\Phi \in G_S$ then $\Phi(S_{2n}) \subset S_{2n}$, and $\Phi(S_{2n-2}) \subset S_{2n-2}$.

Let $\Phi \in G_S$. Then using the standard setting of singularity theory (cf. [7]) we have

$$\Phi(x_1, y_1, \ldots, x_n, y_n) = (x_1 \phi_1(x, y), x_1 \phi_2(x, y) + y_1 \phi_22(x, y), \phi_3(x, y), \ldots, \phi_{2n}(x, y)),$$

where $\phi_1, \phi_2, \phi_22, \phi_3, \ldots, \phi_{2n}$ are smooth germs of functions on $(R^{2n}, 0)$.

Let $\omega_1, \omega_2$ be two symplectic structures on $S$ (closed two-forms on $(R^{2n}, 0)$).

Definition 1.2. We say that $\omega_1$ and $\omega_2$ are $S$-equivalent ($\omega_1 \sim_S \omega_2$) if and only if there exists $\Phi \in G_S$ such that $\Phi^* \omega_1 = \omega_2$. 


Theorem 1.1 (Darboux form). Let $\omega$ be a symplectic structure on $S$. Assume $\omega$ is a symplectic form on $\mathbb{R}^{2n}$. Then $\omega$ is $S$-equivalent to the Darboux form:

$$\omega \sim S \sum_{i=1}^{n} dx_i \wedge dy_i.$$ 

Proof. We take the homotopy (cf. [8]) $\omega_t = t\omega_1 + (1 - t)\omega_0$, $t \in [0, 1]$. One can check that $\omega_t$ is a nondegenerate form for every $t \in [0, 1]$. We seek for a smooth family $t \mapsto \Phi_t$ such that

$$\Phi_t^* \omega_t = \omega_0, \quad \Phi_0 = id_{\mathbb{R}^{2n}}.$$ 

Differentiating (1) we have

$$L_{V_t} \omega_t + \omega_1 - \omega_0 = 0,$$

where $L_{V_t}$ is the Lie derivative along the vector field $V_t$ generated by the flow $\Phi_t$. But

$$L_{V_t} \omega_t = d(V_t \rfloor \omega_t) + \iota_{V_t} d\omega_t = d(V_t \rfloor \omega_t).$$

We have $d(\omega_0 - \omega_1) = 0$ and $\iota_{S^{2n-1}}(\omega_0 - \omega_1) = 0$. So by the relative Poincaré Lemma (see e.g. [11]) there exists a one-form $\alpha$ such that $d\alpha = \omega_0 - \omega_1$ and $\alpha$ vanishes on $S^{2n-1}$. Thus we have

$$V_t \rfloor \omega_t = \alpha \quad \text{and} \quad \alpha |_{(x,y)} = 0 \quad \text{for every } (x,y) \in S^{2n-1}.$$

Because $\omega_t$ is a nondegenerate form, (2) is always solvable with respect to $V_t$ and moreover $V_t(x,y) = 0$ for every $(x,y) \in S^{2n-1}$. We deduce $\Phi_t$ exists, $\Phi_t \in G_S$ and by compactness of the interval $[0,1]$ we have $\Phi^* \omega_1 = \omega_0$. ■

2. Martinet’s singular symplectic spaces. Before we pass to the more detailed analysis of the degenerate case we recall the basic results on the standard classification of singularities of differential forms [6].

Let $\omega$ be a germ of a closed two-form on $\mathbb{R}^{2n}$ at zero. We denote

$$\Sigma_k(\omega) = \{ x \in \mathbb{R}^{2n} : \operatorname{rank} \omega(x) = 2n-k \}, \quad k \text{ is even}. $$

Let $\omega^n = f \Omega$, where $\Omega$ is the volume form on $\mathbb{R}^{2n}$.

(i) If $f(0) \neq 0$ then $\omega$ is a symplectic form (according to the standard notation denoted by $\Sigma_0$) and by the Darboux theorem we obtain

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

in local coordinates around zero.

(ii) Next we assume $f(0) = 0$ while $(df)(0) \neq 0$. We have $\Sigma_2(\omega) = \{ f = 0 \}$ and let $\iota : \Sigma_2(\omega) \to \mathbb{R}^{2n}$ be the inclusion. If $\iota^* \omega^{2n-1}(0) \neq 0$ then in local coordinates

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$$

and this type of singular form $\omega$ is denoted by $\Sigma_{2,0}$ (and called Martinet’s singular form).

Both types of forms $\Sigma_0$, $\Sigma_{2,0}$ are locally stable (see [6]) and this is why we use them in what follows.
Proposition 2.1. Let $\omega$ be a symplectic structure on $S$. Assume $f(0) = 0$ and $df_0 \neq 0$ (stability conditions), then $\omega$ is a singular form of type $\Sigma_{2,0}$ at zero, i.e. $\omega$ belongs to the standard orbit of (ii) (4).

Remark 2.1. We see that the symplectic form $\omega$ on $S$ may be very singular in general. The singular set of $\omega$ is not visible from $S$ (see Fig. 1). The above proposition says that the typical symplectic forms on $S$ can only have $\Sigma_{2,0}$ or $\Sigma_0$ type singularities in the ambient space. Thus the two remaining stable cases $\Sigma_{2,2,0}$ are naturally excluded from our approach (cf. [3]).

Proof of Proposition 2.1. We see that $\omega$ is a symplectic form on $S_{2n-2}$. Let
\[
\tilde{S} = \Sigma_2(\omega) = \{ f = 0 \},
\]
where $\omega^n = f \Omega$ and $\Omega$ is the standard volume form on $\mathbb{R}^{2n}$. We have $T_0 \tilde{S} = T_0 S_{2n-1}$, because $\omega$ is symplectic on $S_{2n}$. $S_{2n-2} \subset S_{2n-1}$ so $T_0 S_{2n-2} \subset T_0 S_{2n-1}$ and $T_0 S_{2n-2} \subset T_0 \tilde{S}$. By assumption $\iota_{2n-2} \omega$ is symplectic. Thus $(\iota_{2n-2} \omega)^{n-1} \neq 0$ and this implies $(\iota^* \omega)^{n-1} \neq 0$, where $\iota: \tilde{S} \to \mathbb{R}^{2n}$ is the embedding of $\tilde{S}$. $
$
Lemma 2.1. By means of a diffeomorphism $\Phi \in G_S$ of the form
\[
\Phi(x,y) = (\phi(x,y),x_2,\ldots,x_n,y_1,\ldots,y_n)
\]
one can reduce $f$ to the following normal form:
\[
f(x_1,y_1,\ldots,x_n,y_n) = \pm (x_1 - \psi(y_1,x_2,\ldots,x_n,y_n)).
\]

Definition 2.1. We say that $\psi_1$, $\psi_2$ are contact equivalent if and only if there exists a diffeomorphism $\Phi: (\mathbb{R}^{2n-1},0) \to (\mathbb{R}^{2n-1},0)$ and a smooth function-germ $g: (\mathbb{R}^{2n-1},0) \to \mathbb{R}$, $g(0) \neq 0$, such that
\[
\psi_1 = g \cdot (\psi_2 \circ \Phi).
\]

Let $\omega_1, \omega_2$ be two symplectic forms on $S$. Let $f_1, f_2$ define their corresponding singular hypersurfaces, $\omega_1^n = f_1 \Omega$ and $\omega_2^n = f_2 \Omega$ and $\psi_1, \psi_2$ are as in Lemma 2.1. By straightforward check we obtain the following

Proposition 2.2. If $\omega_1$ and $\omega_2$ are $S$-equivalent then $\psi_1$ and $\psi_2$ are contact equivalent.

Let $\omega$ be a symplectic form on $S$, $\omega^n = f \Omega$, $f(0) = 0$ and $df_0 \neq 0$. We see that $\frac{\partial f}{\partial x_i}(0) = 0$ and $\frac{\partial f}{\partial y_j}(0) = 0$ for $i = 2,\ldots,n, j = 1,\ldots,n$, so $\frac{\partial f}{\partial x_1}(0) \neq 0$. Thus
\[
df \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \ldots \wedge dx_n \wedge dy_n(0) \neq 0,
\]
so $\{y_1,x_2,\ldots,x_n,y_n\}$ defines a coordinate system on $\tilde{S} = \{ f = 0 \}$.

Before we formulate the main theorem concerning the normal form of $\omega$ we need some necessary facts ([6]).

Lemma 2.2. Let $\tau$ be a $k$-form on $\mathbb{R}^n$ satisfying
\[
\frac{\partial}{\partial x_1}|\tau = 0, \quad \frac{\partial}{\partial x_1}|\tau = 0.
\]
Then \( \tau = \pi^* \iota^* \tau \), where
\[
\pi : R^n \to \{x_1 = 0\}, \quad \pi(x_1, x_2, \ldots, x_n) = (0, x_2, \ldots, x_n),
\]
\[
\iota : \{x_1 = 0\} \to R^n, \quad \iota(x_2, \ldots, x_n) = (0, x_2, \ldots, x_n).
\]

**Lemma 2.3.** Let \( \tau \) be a \( k \)-form on \( R^n \) satisfying
\[
(6) \quad \frac{\partial}{\partial x_1} |_{\tau} = 0, \quad \frac{\partial}{\partial x_1} |_{d\tau} = \varphi \tau,
\]
where \( \varphi \) is a smooth function on \( R^n \). Then
\[
\tau = \zeta \pi^* \iota^* \tau,
\]
where \( \zeta \) is a smooth function on \( R^n \), and \( \zeta |_{\{x_1 = 0\}} = 1 \).

It is easy to prove the following lemmas.

**Lemma 2.4.** Let \( \alpha \) be a germ of a closed \((n-1)\)-form on \( R^n \) at \( 0 \) satisfying the following conditions:
1. \( \alpha_0 \neq 0 \),
2. a germ of a vector field \( X \) at \( 0 \) such that \( X |_{\alpha} = 0 \) and \( X(0) \neq 0 \) meets \( \{x_1 = 0\} \) transversally at \( 0 \).

Then there exists a germ of diffeomorphism \( \Phi : (R^n, 0) \to (R^n, 0) \), which preserves \( \{x_1 = 0\} \) and
\[
\Phi^* \alpha = dx_2 \wedge \ldots \wedge dx_n,
\]
where \((x_1, \ldots, x_n)\) is a coordinate system on \( R^n \).

**Lemma 2.5.** Let \( \alpha \) be a germ of a 1-form on \( R^{2k+1} \) at \( 0 \) satisfying the following conditions:
1. \( \alpha \wedge (d\alpha)^k \neq 0 \),
2. a germ of a vector field \( X \) at \( 0 \) such that
\[
X |_{\alpha} \wedge (d\alpha)^k = (d\alpha)^k
\]
meets \( \{ z = 0 \} \) transversally at 0.
3. \( \iota^* \alpha_0 \neq 0 \), where \( \iota : \{ z = 0 \} \hookrightarrow \mathbb{R}^{2k+1} \) is the canonical inclusion.

Then there exists a germ of diffeomorphism \( \Phi : (\mathbb{R}^{2k+1}, 0) \to (\mathbb{R}^{2k+1}, 0) \), which preserves \( \{ z = 0 \} \) and

\[
\Phi^* \alpha = dz + dy_1 + \sum_{i=1}^{k} x_i dy_i,
\]

where \((z, x_1, \ldots, x_n, y_1, \ldots, y_n)\) is a coordinate system on \( \mathbb{R}^n \).

Now we prove the main theorem obtaining the normal form (with moduli) of the symplectic structure on \( S \). The geometrical contents of this theorem is illustrated in Fig. 1.

**Theorem 2.1.** Let \( \omega \) be a symplectic structure on \( S \). Assume \( f(0) = 0 \) and \( df_0 \neq 0 \). Then \( \omega \) is \( S \)-equivalent to the form

\[
(7) \quad (x_1 - \psi(x_2, \ldots, x_n, y_1, \ldots, y_n))d(x_1 - \psi(x_2, \ldots, x_n, y_1, \ldots, y_n)) \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i,
\]

where \( \psi \) is a germ at 0 of a smooth function, \( \psi(0) = 0 \), \( \frac{\partial \psi}{\partial x_i}(0) = 0 \), \( i = 2, \ldots, n \), \( \frac{\partial \psi}{\partial y_i}(0) = 0 \), \( i = 1, \ldots, n \).

**Proof.** By Lemma 2.1 we have \( f = \pm(x_1 - q) \), where \( q \) does not depend on \( x_1 \). We are searching for a 1-form \( \alpha \) satisfying the following conditions:

1. \( d\alpha = \omega \),
2. \( \iota^* \alpha \wedge (d\iota^* \alpha)^{n-1}_0 \neq 0 \), where \( \iota : \tilde{S} \hookrightarrow \mathbb{R}^{2n} \) is the canonical inclusion,
3. \( \iota^* \alpha_0 \neq 0 \), where \( \iota : \tilde{S} \cap \{ y_1 = 0 \} \hookrightarrow \mathbb{R}^{2n} \) is the canonical inclusion.

\( \omega \) is closed, then there exists a 1-form \( \alpha \) such that \( d\alpha = \omega \). If \( \alpha \) fails to satisfy condition 3 then we replace it by the 1-form \( \alpha + dy_2 \), which satisfies conditions 1 and 3.

Since \( S_{2n-2} \) is symplectic and \( T_0S_{2n-2} = T_0(\tilde{S} \cap \{ y_1 = 0 \}) \), we have \( (\iota^* \alpha)_0^{n-1} = (\tilde{\iota}^* \omega)_0^{n-1} \neq 0 \). Hence by Lemma 2.4, we obtain

\[
\delta^* \iota^*(d\alpha)^{n-1} = dx_2 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n,
\]

where \( \delta : (\tilde{S}, 0) \to (\tilde{S}, 0) \) is a diffeomorphism which preserves \( \tilde{S} \cap \{ y_1 = 0 \} \). Therefore

\[
\iota^* d(\Delta^* \alpha)^{n-1} = dx_2 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n,
\]

where \( \Delta \in G_S \) and

\[
\Delta(x, y) = (x_1, \delta(x_2, \ldots, x_n, y_1, \ldots, y_n)).
\]

If \( \Delta^* \alpha \) fails to satisfy condition 2, then we replace it by the 1-form \( \Delta^* \alpha + dy_1 \), which satisfies all the conditions.

From condition 2 it follows that a vector field \( \mathbf{X} \) which satisfies the conditions

\[
X \cap \alpha \wedge (d\alpha)^{n-1} = 0, \quad X(0) \neq 0,
\]

meets \( \tilde{S} \) transversally at 0. Hence \( \mathbf{X} \) also meets \( S_{2n-1} \) transversally at 0. Therefore by means of elements from \( G_S \), one can reduce \( \mathbf{X} \) to the form \( \pm \frac{\partial \psi}{\partial x_1} \). Thus \( \tilde{S} \) is locally a graph.
of a smooth function \( \theta : (S_{2n-1}, 0) \to (R, 0) \). Hence \((x_2, \ldots, x_n, \ldots, y_1, \ldots, y_n)\) define a coordinate system on \( \tilde{S} \). From 2 and 3 it follows that \( \iota^* \alpha \) satisfies the assumptions of Lemma 2.5. Therefore we have

\[
\phi^* \iota^* \alpha = dy_1 + dy_2 + \sum_{i=2}^{n} x_i dy_i,
\]

where \( \phi : (\tilde{S}, 0) \to (\tilde{S}, 0) \) is a diffeomorphism which preserves \( \tilde{S} \cap \{y_1 = 0\} \). Let \( \Phi \in G_{\tilde{S}} \) be such that

\[
\Phi(x, y) = (x_1, \phi(x_2, \ldots, x_n, y_1, \ldots, y_n)).
\]

Hence we obtain

\[
\iota^* \Phi^* \alpha = dy_1 + dy_2 + \sum_{i=2}^{n} x_i dy_i.
\]

It is easy to check that the vector field \( X \) satisfies the following conditions:

\[(8) \quad X |\alpha = 0 \quad \text{and} \quad X |d\alpha = \varphi \alpha,\]

where \( \varphi : R^{2n} \to R \) is a smooth function. Thus by Lemma 2.3, we obtain

\[
\alpha = h\left(dy_1 + dy_2 + \sum_{i=2}^{n} x_i dy_i\right),
\]

where \( h : R^n \to R \) is a smooth function such that \( h_{|\tilde{S}} = 1 \). We have

\[
(d\alpha)^n = n! h^{n-1} \frac{\partial h}{\partial x_1} \Omega.
\]

On the other hand, by Lemma 2.1, \( \omega^n = \pm (x_1 - g) \Omega \). Hence \( n! h^{n-1} \frac{\partial h}{\partial x_1} = \pm (x_1 - g) \), and

\[
\frac{\partial h^n}{\partial x_1} = \pm \frac{1}{(n-1)!} \frac{1}{(x_1 - g)^{2(n-1)}}
\]

with an extra condition \( h_{|\{x_1 = g\}} = 1 \). Solving this equation we get

\[
h = \sqrt[2n]{\pm \frac{1}{2(n-1)!} (x_1 - g)^2 + 1}.
\]

By the diffeomorphism \( A^{-1} \in G_\Sigma \), where

\[
A(x, y) = (x_1, h(x, y)x_2, \ldots, h(x, y)x_n, y_1, \ldots, y_n),
\]

we reduce \( \alpha \) to

\[
\alpha = h(dy_1 + dy_2 + \sum_{i=2}^{n} x_i dy_i),
\]

The diffeomorphism

\[
\Upsilon(x, y) = \left((x_1 - \zeta)\sqrt{\frac{n-1}{n-1} \sum_{i=0}^{n-1} \left( \frac{n}{i+1} \left(\frac{1}{2} (x_1 - \zeta)^2\right)^i - g, y_1, \ldots, x_n, y_n \right)}\right),
\]

where \( \zeta \) is a function which does not depend on \( x_1 \) and satisfies

\[
\sqrt[2n]{\pm \frac{1}{2(n-1)!} g^2 + 1} = \pm \frac{\zeta^2}{2} + 1,
\]
Since $\Sigma$ exists, we obtain which contradicts the fact that $\epsilon$ is a critical value of $\psi$. We can find a diffeomorphism $\Phi$ on closed 2-forms such that if $\beta$ is in $V$, then there is a neighbourhood of 0 which preserves $S$. Hence we obtain

$$\alpha = \left(1 \pm \frac{1}{2}(x_1 - \psi)^2\right)(dy_1 + dy_2) + \sum x_idy_i.$$ 

Therefore

$$\omega = d\alpha = \pm(x_1 - \psi)d(x_1 - \psi) \land dy_1 + d\left(x_2 \pm \frac{1}{2}(x_1 - \psi)^2\right) \land dy_2 + \sum dx_i \land dy_i.$$ 

Finally, by means of $\Xi \in G_\Sigma$, where

$$\Xi(x, y) = \left(x_1, x_2 \pm \frac{1}{2}(x_1 - \psi)^2, x_3, \ldots, x_n, \pm y_1, y_2, y_3, \ldots, y_n\right),$$

we reduce $\omega$ to the form 7. 

Now we pass to the investigation of stability properties of symplectic structures on $S$.

**Definition 2.2.** Let $\omega$ be a symplectic form on $S$. Then $\omega$ is stable at $p \in S_{2n-2}$ if for any neighbourhood $U$ of $p$ in $S_{2n-2}$ there is a neighbourhood $V$ of $\omega$ (in the $C^\infty$ topology on closed 2-forms) such that if $\beta$ is in $V$, then there is a point $q \in U$ and a germ of a diffeomorphism $\Phi: (R^{2n}, q) \to (R^{2n}, p)$ which preserves $S$ and $\Phi^*\beta = \omega$.

It is easy to see that the Darboux form on $S$ is stable.

**Proposition 2.3.** Let $\omega$ be a symplectic structure on $S$. Assume $f(0) = 0$ and $df_0 \neq 0$. Then $\omega$ is not stable at 0.

**Proof.** From Theorem 2.1 it follows that $\omega$ can be reduced to the form

$$(x_1 - \psi)d(x_1 - \psi) \land dy_1 + \sum dx_i \land dy_i.$$ 

Suppose the proposition is false. Let $U$ be a neighbourhood of 0 in $R^{2n}$, $\psi(0) = 0 \in R$ is a critical value of $\psi|_U$. From the Sard theorem we see that there is $\epsilon \in R$ which is not a critical value of $\psi|_U$, in any neighbourhood of 0 $\in R$. Let $\beta = \alpha + ed(x_1 - \psi) \land dy_1$. Then we can find a diffeomorphism $\Phi$ which preserves $S$ and $\Phi^*\beta = \omega$. Hence

$$\Phi^*\beta^n = \Phi^*((x_1 - \psi + \epsilon)\Omega) = \omega^n = (x_1 - \psi)\Omega.$$ 

Since $\Sigma_2(\omega)$ is tangent to $S_{2n-1}$ at 0, $\Sigma_2(\beta)$ is tangent to $S_{2n-1}$ at $q = \Phi(0) \in S_{2n-2}$. Therefore, we obtain

$$\psi(q) = \epsilon, \quad d\psi_q = 0,$$

which contradicts the fact that $\epsilon$ is not a critical value of $\psi|_U$. 


2.1. Remark. Let us consider the following semialgebraic set:

\[ S = S_{2n} \cup S_{2n-2} \subset R^{2n}; \]
\[ S_{2n} = \{ (x, y) \in R^{2n} : x_1^3 > y_1^2 \}, \]
\[ S_{2n-2} = \{ (x, y) \in R^{2n} : x_1 = 0, y_1 = 0 \}. \]

We notice the difference with the previous space: \( \partial S_{2n} \) is a singular set.

We endow \( S \) with a symplectic structure \( \omega \). As before \( G_S \) denotes the group of diffeomorphisms \( (R^{2n}, 0) \to (R^{2n}, 0) \) preserving \( S \). Let \( \omega_1, \omega_2 \) be two symplectic structures on \( S \). We say that \( \omega_1 \) and \( \omega_2 \) are \( S \)-equivalent if and only if \( \Phi^* \omega_1 = \omega_2 \) for some \( \Phi \in G_S \).

Now we can show the following

**Proposition 2.4.** Let \( \omega \) be a symplectic structure on \( S \). Assume \( f(0) = 0 \) and \( df_0 \neq 0 \).

Then \( \omega \) is a singular form of type \( \Sigma_{2,0} \) at zero.

**Proof.** By straightforward use of the proof of Proposition 2.1.

An analogous Darboux theorem for the space \( S \) is proved by Arnold ([2]): Let \( \omega \) be a symplectic structure on \( R^{2n} \). Then \( \omega \) is \( S \)-equivalent with respect to formal equivalence to the Darboux form:

\[ \omega \sim \sum_{i=1}^{n} dx_i \wedge dy_i. \]

**References**