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L_∞-ESTIMATE FOR SOLUTIONS OF NONLINEAR PARABOLIC SYSTEMS

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Abstract. We prove existence of weak solutions to nonlinear parabolic systems with p-Laplacians terms in the principal part. Next, in the case of diagonal systems an L_∞-estimate for weak solutions is shown under additional restrictive growth conditions. Finally, L_∞-estimates for weakly nondiagonal systems (where nondiagonal elements are absorbed by diagonal ones) are proved. The L_∞-estimates are obtained by the Di Benedetto methods.

1. Introduction. In this paper we consider the following initial boundary value problem:

$$
\begin{aligned}
(1.1)\qquad & u_{it} - \sum_{j=1}^m \nabla\cdot(a_{ij}(x,t,u,\nabla u)\cdot\nabla u_j) + R_i(x,t,u,\nabla u)u_i \\
&\qquad = f_i(x,t,u,\nabla u), \quad i=1,\dots,m, \quad \text{in } \Omega^T=\Omega\times(0,T),\\
& u_i|_{t=0} = u_{0i}, \quad i=1,\dots,m, \quad \text{in } \Omega,\\
& u_i = u_{bi}, \quad i=1,\dots,m, \quad \text{on } S^T = S\times(0,T),
\end{aligned}
$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain, $T\in(0,\infty)$, $S=\partial\Omega$, $u=(u_1,\dots,u_m)\in\mathbb{R}^m$, $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and dot denotes the scalar product in $\mathbb{R}^n$.

The aim of this paper is to prove the existence of weak solutions to (1.1) and next to show that the weak solutions are bounded under some restrictions.

To this end we assume the following structure conditions:

$$a_{ij}:\Omega^T\times\mathbb{R}^m\times\mathbb{R}^{mn}\to\mathbb{R}^{n^2}, \qquad i,j=1,\dots,m,$$

satisfy the Carathéodory condition and

$$(1.2)\qquad \alpha_1|\nabla u|^p \le \sum_{i,j=1}^m a_{ij}(x,t,u,\nabla u)\nabla u_j\cdot\nabla u_i \le \alpha_2|\nabla u|^p,$$

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where α_1, α_2 are positive constants and $p \ge 2$; sometimes we use also the inequality

$$\sum_{i,j=1}^{m} (a_{ij}(x,t,u,\nabla u_1)\cdot\nabla u_{1j} - a_{ij}(x,t,u,\nabla u_2)\cdot\nabla u_{2j})\cdot(\nabla u_{1i}-\nabla u_{2i}) \ge \overline{\alpha}|\nabla u_1 - \nabla u_2|^p, \tag{1.3}$$

where $\overline{\alpha}$ is a positive constant. Moreover,

$$R_i : \Omega^T \times \mathbb{R}^m \times \mathbb{R}^{mn} \to \mathbb{R}, \quad i = 1,\dots,m,$$

satisfy the Carathéodory condition and

$$R_i(x,t,u,\nabla u) = R_{1i}(x,t,u,\nabla u) + R_{2i}(x,t,u,\nabla u), \tag{1.4}$$

where

$$\begin{gathered} \beta_1|u|^{p_0-2} \le R_{1i}(x,t,u) \le \beta_2|u|^{p_0-2}, \\ \beta_0|u-v|^{p_0} \le \sum_{i=1}^{m}\big(R_{1i}(x,t,u)u_i - R_{1i}(x,t,v)v_i\big)(u_i-v_i), \end{gathered} \tag{1.5}$$

where β_0,β_1,β_2 are positive constants, $p_0 \ge 2$, and

$$\gamma_1|\nabla u|^{q_0} \le R_{2i}(x,t,u,\nabla u) \le \gamma_2|\nabla u|^{q_0}, \tag{1.6}$$

where γ_1,γ_2 are positive constants and $q_0 \ge 0$. Next,

$$f_i : \Omega^T \times \mathbb{R}^m \times \mathbb{R}^{mn} \to \mathbb{R}, \quad i = 1,\dots,m,$$

satisfy the Carathéodory condition and

$$|f_i(x,t,u,\nabla u)| \le \delta_1(|u|) + \delta_2(|u|)|\nabla u|^{\nu}, \quad i=1,\dots,m, \tag{1.7}$$

where δ_1,δ_2 are positive increasing functions and $\nu \ge 0$ will be chosen later.

Finally we assume the following restrictions:

$$\frac{q_0}{p} + \frac{2}{p_*} \le 1, \quad p_* = \max\{q,p_0\}, \quad q = p\frac{n+2}{n}, \tag{1.8}$$

and

$$\delta_1(|u|) \le c(|u|^{\mu_1}+1), \quad \delta_2(|u|) \le c|u|^{\mu_2}, \tag{1.9}$$

where

$$\mu_1 + 1 < p^*, \quad p^* = \max\{p,p_0\}, \tag{1.10}$$

and

$$\frac{\mu_2+1}{p^*} + \frac{\nu}{p} < 1. \tag{1.11}$$

DEFINITION 1.1. We denote by (P.1) the problem (1.1) with relations (1.2)–(1.11).

DEFINITION 1.2. By a *weak solution* of problem (P.1) we mean a solution $u_i \in L_\infty(0,T;L_2(\Omega)) \cap L_p(0,T;W_p^1(\Omega)) \cap L_{p_0}(\Omega^T)$, $i=1,\dots,m$, of the following integral identity:

$$\begin{aligned} -\sum_{i=1}^{m}\int_{\Omega^T} u_i\varphi_{it}\,dx\,dt + \sum_{i,j=1}^{m}\int_{\Omega^T} a_{ij}\cdot\nabla u_j\cdot\nabla\varphi_i\,dx\,dt \\ + \sum_{i=1}^{m}\int_{\Omega^T} R_iu_i\varphi_i\,dx\,dt = \sum_{i=1}^{m}\int_{\Omega^T} f_i\varphi_i\,dx\,dt - \sum_{i=1}^{m}\int_{\Omega} u_{0i}\varphi_i(x,0)\,dx, \end{aligned} \tag{1.12}$$

which holds for any φ_i such that $\varphi_i|_S = 0$, $\varphi_i|_{t=T} = 0$, $\varphi_{it} \in L_2(\Omega^T)$, $\varphi_i \in L_\infty(0,T; L_2(\Omega)) \cap L_p(0,T; W_p^1(\Omega)) \cap L_{p_0}(\Omega^T)$, $i = 1,\dots,m$, and

$$\int_{\Omega^T} \partial_t u\zeta\,dx\,dt = -\int_{\Omega^T} (u-u_0)\partial_t\zeta\,dx\,dt \tag{1.13}$$

valid for any $\zeta \in L_p(\Omega^T)$, $\partial_t\zeta \in L_{p'}(\Omega^T)$, $1/p + 1/p' = 1$, such that $\zeta(T) = 0$. To show boundedness of weak solutions to problem (P.1) we have to obtain first an estimate in $L_\infty(0,T; L_2(\Omega)) \cap L_p(0,T; W_p^1(\Omega)) \cap L_{p_0}(\Omega^T)$ and next applying the technique of truncations we are able to get a sup-estimate. This procedure follows from the growth condition (1.7) with $\mu_1 \ge 0$, $\mu_2 \ge 0$ and $\nu \ge 0$, because we need such an estimate for weak solutions to obtain the well known recursive inequalities (see (3.16)) which imply the sup-estimate.

Generally to prove existence of weak solutions and to obtain necessary estimates we need the following identity with Steklov averages (see the end of this section):

$$\begin{aligned}\sum_{i=1}^m \int_{\Omega\times(h,T)} \Big[\partial_t u_{hi}\varphi_i + \sum_{j=1}^m (a_{ij}\cdot\nabla u_j)_h\cdot\nabla\varphi_i + (R_iu_i)_h\varphi_i\Big]\,dx\,dt \\ = \sum_{i=1}^m \int_{\Omega\times(h,T)} f_{hi}\varphi_i\,dx\,dt.\end{aligned} \tag{1.14}$$

Assuming now the growth conditions on the r.h.s. of (1.1) in the form

$$|f_i(x,t,u,\nabla u)| \le d_1|\nabla u_i|^{\frac{p}{2}} + d_2|u_i|^\sigma + d_3, \qquad i = 1.\dots,m, \tag{1.15}$$

where d_1, d_2, d_3 are constants, $\sigma \le \frac{p_*}{2}$, $p_* = \max\{q, p_0\}$, $q = p\frac{n+2}{n}$, we can generalize the growth conditions (1.2), (1.5), (1.6) in the following way:

(1.16) $\alpha_2, \beta_2, \gamma_2$ from (1.2), (1.5) and (1.6), respectively, are increasing functions of $|u|$.

Now we can introduce

Definition 1.3. By (P.2) we denote the problem (1.1) with the growth conditions (1.2)–(1.6), (1.15), (1.16).

Then to prove existence of solutions to problem (P.2) we have to consider instead of (1.1) the following truncated problem:

$$\begin{aligned}&u_{it} - \sum_{j=1}^m \nabla\cdot(a_{ij}(x,t,u^{(l_1,l_2)},\nabla u)\cdot\nabla u_j) + R_i(x,t,u^{(l_1,l_2)},\nabla u)u_i \\ &\qquad\qquad = f_i(x,t,u,\nabla u), \quad i = 1,\dots,m, \quad \text{in } \Omega^T, \\ &u_i|_{t=0} = u_{0i}, \quad i = 1,\dots,m, \quad \text{in } \Omega, \\ &u_i = u_{bi}, \quad i = 1,\dots,m, \quad \text{on } S^T,\end{aligned} \tag{1.17}$$

where

$$v^{(l_1,l_2)} = \begin{cases} l_1 & \text{for } v > l_1, \\ v & \text{for } l_2 \le v \le l_1, \\ l_2 & \text{for } v < l_2, \end{cases} \tag{1.18}$$

where $l_2 \le l_1$ are constants, $v \in \mathbb{R}^1$.

By $u^{(l_1,l_2)}$, where $u=(u_1,\dots,u_m)$, we indicate that each of the coordinates is of the form (1.18). The truncated solutions were considered in [8].

DEFINITION 1.4. By (P.3) we denote the problem (1.17) with (1.2)–(1.6) and (1.15), (1.16).

Remark 1.5. Generally any solution of problem (P.3) depends on (l_1,l_2), so we should write $u=u_{(l_1,l_2)}$, but to simplify notation we omit the index (l_1,l_2).

DEFINITION 1.6. By a weak solution of the problem (P.3) we mean functions $u_i\in L_\infty(0,T;L_2(\Omega))\cap L_p(0,T;W_p^1(\Omega))\cap L_{p_0}(\Omega^T)$, $i=1,\dots,m$, which satisfy the following integral identity:

$$
(1.19)\qquad -\sum_{i=1}^m\int_{\Omega^T}u_i\varphi_{it}\,dx\,dt+\sum_{i,j=1}^m\int_{\Omega^T}a_{ij}^{(l_1,l_2)}\cdot\nabla u_j\cdot\nabla\varphi_i\,dx\,dt+\sum_{i=1}^m\int_{\Omega^T}R_i^{(l_1,l_2)}u_i\varphi_i\,dx\,dt
=\sum_{i=1}^m\int_{\Omega^T}f_i\varphi_i\,dx\,dt-\sum_{i=1}^m\int_{\Omega}u_{0i}\varphi_i(x,0)\,dx,
$$

which holds for any φ_i such that $\varphi_i|_S=0$, $\varphi_i|_{t=T}=0$, $\varphi_{it}\in L_2(\Omega^T)$, $\nabla\varphi_i\in L_p(\Omega^T)$, $\varphi_i\in L_{p_0}(\Omega^T)$, $i=1,\dots,m$. Moreover,

$$
a_{ij}^{(l_1,l_2)}=a_{ij}(x,t,u^{(l_1,l_2)},\nabla u),\qquad R_i^{(l_1,l_2)}=R_i(x,t,u^{(l_1,l_2)},\nabla u).
$$

Now we introduce some notation. Let $k>0$. Then $(u-k)_+=\max\{u-k,0\}$, $(u-k)_-=\max\{-(u-k),0\}$, $A_{k,i}^+(t)=\{x\in\Omega:u_i(x,t)>k\}$, $A_{k,i}^-(t)=\{x\in\Omega:u_i(x,t)<k\}$. We introduce the Steklov averages

$$
v_h(x,t)=\begin{cases}\frac{1}{h}\int_{t-h}^t v(x,\tau)\,d\tau, & t\in(h,T],\\ 0, & t<h.\end{cases}
$$

By $|\Omega|$ we denote the measure of Ω. The dot $\cdot$ denotes the scalar product in $\mathbb{R}^n$ and $\overset{\circ}{W}{}_p^1(\Omega)=\{u\in W_p^1(\Omega):u=0\text{ on }\partial\Omega\}$.

Now we formulate the well known result used in this paper. The following interpolation inequality is satisfied (see [3], Ch.1):

$$
(1.20)\qquad \int_{\Omega^t}|v|^q dxdt\le c_*\Big(\operatorname*{ess\,sup}_t\int_\Omega|v|^2\,dx\Big)^{\frac{p}{n}}\int_{\Omega^t}|\nabla v|^p\,dx\,dt,
$$

which holds for any $v\in V_0^{2,p}(\Omega^t)$ and $q=p\frac{n+2}{n}$, where $V_0^{2,p}(\Omega^T)$ is a Banach space with the norm

$$
\|v\|_{V^{2,p}(\Omega^T)}=\operatorname*{ess\,sup}_{t\le T}\|v(t)\|_{L_2(\Omega)}+\|\nabla v\|_{L_2(\Omega^T)},
$$

and $v|_s=0$.

Now we present some information about the results of this paper. In Sections 2 and 3 the existence of bounded solutions to diagonal problem (P.1) is proved. In Sections 4 and 5 we show existence of bounded solutions to the diagonal problem (P.2) in which the r.h.s. has very strong growth restrictions with respect to u. Finally in Section 6 we prove existence of bounded solutions to nondiagonal problem (P.1).

Finally we add some remarks concerning the results of this paper. We proved sup-estimates for solutions of problem (1.1) under very strong growth restrictions (see (3.24), (5.8) and (6.11)). These restrictions follow from the used cut-off functions $(u_i-k)_+$, $i=1,\dots,m$. Much less restrictions can be expected in the case of cut-off functions $(|u|-k)_+$ which are used in [3], Ch. 8, Sect. 2. However in [3] there are considered only systems with the same matrices in the main terms, $a_i=a$, $i=1,\dots,m$ (see (3.1)).

Moreover, we can expect much less restrictions on the growth of the r.h.s. in the case when Stampaccia's idea of getting sup-estimates is used (see [3], Ch. 5, Sect. 17). However in the last case the coefficients a_i, R_i and f_i, $i=1,\dots,m$, must be either continuous or Hölder continuous with respect to x and t or must satisfy some additional structure conditions.

We think that the method presented in this paper (the proof of existence of weak solutions and then showing L_∞-estimates) is appropriate for systems with measurable coefficients with respect to x and t.

2. Existence of weak solutions to problem (P.1). First we obtain an estimate.

LEMMA 2.1. *Let* (1.2)–(1.11) *hold. Let S be Lipschitz continuous. Let* $p^*=\max\{p,p_0\}$.

Let $u_{bt}\in L_2(\Omega^t)$, $u_b\in L_p(0,t;W_p^1(\Omega))\cap L_{p_0}(\Omega^t)\cap L_{\frac{2p}{p-q_0}}(\Omega^t)\cap L_2(\Omega^t)$, $u_b|_{t=0}\in L_2(\Omega)$, $u_0\in L_2(\Omega)$, $t\le T$. *Then for solutions of problem* (P.1) *the following estimate holds*:

$$
\tag{2.1}
\begin{aligned}
&\int\limits_\Omega |u|^2\,dx+\int\limits_{\Omega^t}|\nabla u|^p\,dx\,dt+\int\limits_{\Omega^t}|u|^{p_0}\,dx\,dt+\int\limits_{\Omega^t}|\nabla u|^{q_0}|u|^2\,dx\,dt\\
&\quad\le c_1\Big(1+\int\limits_{\Omega^t}(|u_{bt}|^2+|\nabla u_b|^p+|u_b|^p+|u_b|^{p_0}+|u_b|^{\frac{2p}{p-q_0}}+|u_b|^2)\,dx\,dt\\
&\qquad+\int\limits_\Omega u_b^2(0)\,dx+\int\limits_\Omega u_0^2\,dx\Big)\le c_0.
\end{aligned}
$$

Proof. Putting $\varphi_i=u_i-u_{bi}$, $i=1,\dots,m$, into (1.14), performing integration with respect to time, passing with h to zero and using the growth conditions (1.2), (1.4)–(1.7) we obtain

$$
\tag{2.2}
\begin{aligned}
&\frac12\int\limits_\Omega (u-u_b)^2\,dx+\alpha_1\int\limits_{\Omega^t}|\nabla u|^p\,dx\,dt+\beta_1\int\limits_{\Omega^t}|u|^{p_0}\,dx\,dt+\gamma_1\int\limits_{\Omega^t}|\nabla u|^{q_0}|u|^2\,dx\,dt\\
&\le\int\limits_{\Omega^t}|u_{bt}|\,|u-u_b|\,dx\,dt+\alpha_2\int\limits_{\Omega^t}|\nabla u|^{p-1}|\nabla u_b|\,dx\,dt+\beta_2\int\limits_{\Omega^t}|u|^{p_0-2}|u||u_b|\,dx\,dt\\
&\quad+\gamma_2\int\limits_{\Omega^t}|\nabla u|^{q_0}|u||u_b|\,dx\,dt+\frac12\int\limits_\Omega (u_0-u_b(0))^2\,dx\\
&\quad+\int\limits_{\Omega^t}(\delta_1(|u|)+\delta_2(|u|)|\nabla u|^\nu)|u-u_b|\,dx\,dt.
\end{aligned}
$$

In view of the Hölder and Young inequalities the r.h.s. of (2.2) is estimated by

$$\begin{aligned}(2.3)\qquad &\frac{1}{2}\int\limits_{\Omega}|u-u_b|^2\,dx+\frac{1}{2}\int\limits_{\Omega^t}|u_{bt}|^2\,dx\,dt+\varepsilon_1\int\limits_{\Omega^t}|\nabla u|^p\,dx\,dt+c(\varepsilon_1)\int\limits_{\Omega^t}|\nabla u_b|^p\,dx\,dt\\&+\varepsilon_2\int\limits_{\Omega^t}|u|^{p_0}\,dx\,dt+c(\varepsilon_2)\int\limits_{\Omega^t}|u_b|^{p_0}\,dx\,dt+\varepsilon_3\int\limits_{\Omega^t}|\nabla u|^{q_0}|u|^2\,dx\,dt\\&+c(\varepsilon_3)\int\limits_{\Omega^t}|\nabla u|^{q_0}|u_b|^2\,dx\,dt+\frac{1}{2}\int\limits_{\Omega}(u_0-u_b(0))^2\,dx\\&+\int\limits_{\Omega^t}(\delta_1(|u|)+\delta_2(|u|)|\nabla u|^{\nu})|u-u_b|\,dx\,dt,\end{aligned}$$

where $\varepsilon_i\in(0,1)$, $i=1,2,3$.

Since $q_0<p$ the third from the end term in (2.3) is bounded by

$$\varepsilon_4\int\limits_{\Omega^t}|\nabla u|^p\,dx\,dt+c(\varepsilon_4)\int\limits_{\Omega^t}|u_b|^{\frac{2p}{p-q_0}}\,dx\,dt,\qquad \varepsilon_4\in(0,1).$$

In view of (1.7) and (1.9) the last term in (2.3) is bounded by

$$c\int\limits_{\Omega^t}(|u-u_b|+|u|^{\mu_1}|u-u_b|+|u|^{\mu_2}|\nabla u|^{\nu}|u-u_b|)\,dx\,dt\equiv I_0+I_1+I_2,$$

where

$$I_1\le\int\limits_{\Omega^t}(|u-u_b|^{\mu_1+1}+|u_b|^{\mu_1}|u-u_b|)\,dx\,dt\equiv I_{11}.$$

Let $p^*=p_0$ and $\mu_1+1<p_0$. Then

$$\begin{aligned}I_{11}&\le\varepsilon_5\int\limits_{\Omega^t}|u-u_b|^{p_0}\,dx\,dt+c(\varepsilon_5)\Big(1+\int\limits_{\Omega^t}|u_b|^{\frac{\mu_1p_0}{p_0-1}}\,dx\,dt\Big)\\&\le\varepsilon_5\int\limits_{\Omega^t}|u|^{p_0}\,dx\,dt+c(\varepsilon_5)\Big(1+\int\limits_{\Omega^t}(|u_b|^{p_0}+|u_b|^{\frac{\mu_1p_0}{p_0-1}})\,dx\,dt\Big),\qquad\varepsilon_5\in(0,1).\end{aligned}$$

Let $p^*=p$ and $\mu_1+1<p$. Then

$$\begin{aligned}I_{11}&\le\varepsilon_6\int\limits_{\Omega^t}|u-u_b|^{p}\,dx\,dt+c(\varepsilon_6)\Big(1+\int\limits_{\Omega^t}|u_b|^{\frac{\mu_1p}{p-1}}\,dx\,dt\Big)\\&\le\varepsilon_6c\int\limits_{\Omega^t}|\nabla u|^{p}\,dx\,dt+c(\varepsilon_6)\Big(1+\int\limits_{\Omega^t}(|u_b|^{\frac{\mu_1p}{p-1}}+|\nabla u_b|^p)\,dx\,dt\Big),\qquad\varepsilon_6\in(0,1).\end{aligned}$$

Moreover, $\mu_1p^*/(p^*-1)<p^*$.

Now we estimate I_2. Hence we have

$$I_2\le\int\limits_{\Omega^t}(|u-u_b|^{\mu_2+1}|\nabla u|^{\nu}+|u_b|^{\mu_2}|u-u_b||\nabla u|^{\nu})\,dx\,dt\equiv I_3+I_4.$$

First we examine I_3. Let $p^*=p_0$, $\mu_2+1<p_0$, $\frac{\mu_2+1}{p_0}+\frac{\nu}{p}<1$. Then

$$I_3\le\varepsilon_7\int\limits_{\Omega^t}(|u-u_b|^{p_0}+|\nabla u|^p)\,dx\,dt+c(\varepsilon_7)$$

$$\le \varepsilon_7 \int\limits_{\Omega^t} (|u|^{p_0} + |\nabla u|^p)\,dx\,dt + c(\varepsilon_7)\Big(1 + \int\limits_{\Omega^t} |u_b|^{p_0}\,dx\,dt\Big), \qquad \varepsilon_7 \in (0,1).$$

Let $p^* = p$, $\nu + \mu_2 + 1 < p$. Then

$$\begin{aligned} I_3 &\le \varepsilon_8 \int\limits_{\Omega^t} |u-u_b|^p\,dx\,dt + c(\varepsilon_8)\int\limits_{\Omega^t} |\nabla u|^{\frac{\nu p}{p-(\mu_2+1)}}\,dx\,dt \\ &\le \varepsilon_8 c \int\limits_{\Omega^t} |\nabla(u-u_b)|^p\,dx\,dt + \varepsilon_8 \int\limits_{\Omega^t} |\nabla u|^p\,dx\,dt + c(\varepsilon_8) \\ &\le \varepsilon_9 \int\limits_{\Omega^t} |\nabla u|^p\,dx\,dt + c(\varepsilon_9)\Big(\int\limits_{\Omega^t} |\nabla u_b|^p\,dx\,dt + 1\Big), \qquad \varepsilon_9 \in (0,1). \end{aligned}$$

Finally we estimate I_4. Let $p^* = p_0$, $\frac{\nu}{p} + \frac{1}{p_0} < 1$. Then

$$\begin{aligned} I_4 &\le \varepsilon_{10} \int\limits_{\Omega^t} |u-u_b|^{p_0}\,dx\,dt \\ &\quad + c(\varepsilon_{10})\Big(\int\limits_{\Omega^t} |\nabla u|^p\,dx\,dt\Big)^{\frac{\nu}{p}\frac{p_0}{p_0-1}}\Big(\int\limits_{\Omega^t} |u_b|^{\frac{\mu_2}{1-1/p_0-\frac{\nu}{p}}}\,dx\,dt\Big)^{\frac{p_0}{p_0-1}(1-\frac{1}{p_0}-\frac{\nu}{p})} \\ &\le \varepsilon_{10} \int\limits_{\Omega^t} (|u|^{p_0} + |\nabla u|^p)\,dx\,dt + c(\varepsilon_{10})\int\limits_{\Omega^t} |u_b|^{p_0}\,dx\,dt, \qquad \varepsilon_{10} \in (0,1). \end{aligned}$$

Let $p^* = p$ and $p > \nu + 1$. Then

$$\begin{aligned} I_4 &\le \varepsilon_{11} \int\limits_{\Omega^t} |u-u_b|^p\,dx\,dt + c(\varepsilon_{11})\Big(\int\limits_{\Omega^t} |\nabla u|^p\,dx\,dt\Big)^{\frac{\nu}{p-1}}\Big(\int\limits_{\Omega^t} |u_b|^{\frac{\mu_2}{1-\frac{1+\nu}{p}}}\,dx\,dt\Big)^{(1-\frac{\nu+1}{p})\frac{p}{p-1}} \\ &\le \varepsilon_{12} \int\limits_{\Omega^t} |\nabla u|^p + c(\varepsilon_{12})\int\limits_{\Omega^t} \big(|\nabla u_b|^p + |u_b|^{\frac{\mu_2}{1-\frac{1+\nu}{p}}}\big)\,dx\,dt, \qquad \varepsilon_{12} \in (0,1). \end{aligned}$$

In view of (1.11) we have

$$\frac{\mu_2}{1-\frac{1}{p_0}-\frac{\nu}{p}} < p_0, \qquad \frac{\mu_2}{1-\frac{1+\nu}{p}} < p.$$

Applying the Gronwall lemma and using the above considerations in (2.2) we obtain (2.1) for sufficiently small ε_i, $i = 1, \ldots, 12$. This concludes the proof.

Now applying the ideas from [1, 4, 9] we prove existence of weak solutions to problem (P.1). Hence we have

THEOREM 2.2. *Let the assumptions of Lemma* 2.1 *be satisfied. Let either*

(a) $p_0 \le q$ *and* $p > q_0 + \frac{n}{n+2}$, *or*
(b) $p_0 > q$ *and* $\frac{n}{n+2} + q_0 < p(1 - \frac{1}{p_0})$.

Let either

(c) $p_0 \le q$ *and* $p > \max\{(1+\mu_1)\frac{n}{n+2}, (1+\mu_2)\frac{n}{n+2} + \nu\}$, *or*
(d) $p_0 > q$ *and* $p > \max\{\frac{n}{n+2}/(1-\frac{\mu_1}{p_0}), (\frac{n}{n+2}+\nu)/(1-\frac{\mu_2}{p_0})\}$.

Then there exists a solution of problem (P.1) *such that* $u \in L_\infty(0,T;L_2(\Omega)) \cap L_p(0,T; W_p^1(\Omega)) \cap L_{p_0}(\Omega^T)$ *and the estimate* (2.1) *holds.*

Proof. To prove existence of solutions to problem (P.1) we assume that coefficients in $(1.1)_1$ do not depend on t. The case with time dependent coefficients can be treated in the same way as in Remark 3.32 of [5].

Then we replace $\partial_t u$ by the backward difference quotient

$$\partial_t^{-h} u = \frac{1}{h}[u(t) - u(t-h)].$$

Thus instead of the parabolic problem (1.1) we obtain an elliptic problem which we solve by applying the Galerkin method. To do this we choose linearly independent functions $e_i \in \overset{\circ}{W}{}^1_p(\Omega)$ such that their linear combinations are dense in $\overset{\circ}{W}{}^1_p(\Omega)$. Similarly to [1, 4, 9] we are looking for an approximate solution of (1.14) in the form

$$u_{h\lambda} = u_{bh} + \sum_{i=1}^{\lambda} \alpha_{h\lambda i}(t) e_i(x), \tag{2.4}$$

with step functions $\alpha_{h\lambda i} \in L_\infty(0,T)$, where u_{bh} is time independent in each interval $((k-1)h, kh)$, $k = 0, 1, \ldots$,

$$u_{bh}(x,t) := \frac{1}{h} \int_{(k-1)h}^{kh} u_b(x,s)\, ds \quad \text{for } (k-1)h \le t \le kh, \tag{2.5}$$

where for simplicity it is assumed that $\frac{T}{h}$ is an integer, and $u_{h\lambda}$ satisfies the equality

$$\begin{aligned} S_{h\lambda}(u_{h\lambda}, \varphi) := \sum_{i=1}^{m} \int_\Omega \partial_t^{-h} u_{h\lambda i} \varphi_i \, dx + \sum_{i,j=1}^{m} \int_\Omega a_{ij} \nabla u_{h\lambda j} \cdot \nabla \varphi_i \, dx \\ + \sum_{i=1}^{m} \int_\Omega R_i u_{h\lambda i} \varphi_i \, dx - \sum_{i=1}^{m} \int_\Omega f_i \varphi_i \, dx = 0, \end{aligned} \tag{2.6}$$

which holds for all test functions $\varphi \in V_\lambda := \operatorname{span}\{e_1, \ldots, e_\lambda\}$. We take initial data

$$u_{h\lambda}(t) := u_{0h}(t) \quad \text{for } -h < t < 0, \tag{2.7}$$

where u_{0h} is bounded,

$$u_{0h} := \min\left(1, \frac{1}{h|u_0|}\right) u_0.$$

Hence the choice of u_{0h} and u_{bh} imply that we can determine $u_{h\lambda}(t)$ inductively for $t \in ((k-1)h, kh)$ as a solution of an elliptic problem. In fact, if $u_{h\lambda}(t-h)$ is known the l.h.s. of (2.6) defines a continuous mapping $\Phi_{h\lambda} : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$, where the λ parameters are the unknown coefficients of $u_{h\lambda}(t)$.

To prove the existence of $u_{h\lambda}(t)$ for $t \in (0, kh)$ we assume that $u_{h\lambda}(t)$ is already known in $\big(0, (k-1)h\big)$. Hence we have to determine $\alpha = \{\alpha_i\}_{i=1,\ldots,\lambda} \equiv \{\alpha_{h\lambda i}\}_{i=1,\ldots,\lambda}$ for $t \in (0, kh)$. Denote $\phi = \sum_{i=1}^{m} \alpha_i e_i$ and consider a continuous mapping $\Phi_{h\lambda} : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ such that $\Phi_{h\lambda i}(\alpha) = S_{h\lambda}(\phi + u_{bh}, e_i)$, $i = 1, \ldots, \lambda$. Using (2.6) we obtain

$$\Phi_{h\lambda}(\alpha) \cdot \alpha = \sum_{i=1}^{\lambda} \Phi_{h\lambda i}(\alpha) \alpha_i = \sum_{i=1}^{\lambda} S_{h\lambda}(\phi + u_{bh}, e_i) \alpha_i = S_{h\lambda}(\phi + u_{bh}, \phi) \tag{2.8}$$

$$= \sum_{i=1}^{m} \int_{\Omega} \Big[\frac{1}{h}(u_{h\lambda i}(t) - u_{h\lambda i}(t-h))(u_{h\lambda i}(t) - u_{bhi}) + \sum_{j=1}^{m} a_{ij} \cdot \nabla u_{h\lambda j}(t) \cdot \nabla(u_{h\lambda i}(t) - u_{bhi}) + R_i u_{h\lambda i}(t)(u_{h\lambda i}(t) - u_{bhi}) - f_i(u_{h\lambda i}(t) - u_{bhi}) \Big]\, dx.$$

Using the structure conditions (1.2)–(1.11) we obtain

$$\begin{aligned} \Phi_{h\lambda}(\alpha)\cdot\alpha \ge{}& \frac{1}{2h}\int_\Omega u_{h\lambda}^2(t)\,dx + \alpha_1 \int_\Omega |\nabla u_{h\lambda}|^p\,dx + \beta_1 \int_\Omega |u_{h\lambda}|^{p_0}\,dx \\ &+ \gamma_1 \int_\Omega |\nabla u_{h\lambda}|^{q_0}|u_{h\lambda}|^2\,dx - \alpha_2 \int_\Omega |\nabla u_{h\lambda}|^{p-1}|u_{bh}|\,dx - \beta_2 \int_\Omega |u_{h\lambda}|^{p_0-1}|u_{bh}|\,dx \\ &- \gamma_2 \int_\Omega |\nabla u_{h\lambda}|^{q_0}|u_{h\lambda}|\quad |u_{bh}|\,dx - c\int_\Omega \big(u_{bh}^2 + u_{h\lambda}^2(t-h)\big)\,dx \\ &- c\int_\Omega |u_{h\lambda} - u_{bh}|\,dx - c\int_\Omega (|u_{h\lambda}(t)|^{\mu_1} + |u_{h\lambda}(t)|^{\mu_2}|\nabla u_{h\lambda}(t)|^{\nu})|u_{h\lambda} - u_{bh}|\,dx. \end{aligned} \tag{2.9}$$

In view of the Hölder and Young inequalities and proceeding exactly as in Lemma 2.1 we get

$$\begin{aligned} \Phi_{h\lambda}(\alpha)\cdot\alpha \ge{}& \frac{1}{2h}\int_\Omega u_{h\lambda}^2\,dx + \frac{\alpha_1}{2}\int_\Omega |\nabla u_{h\lambda}|^p\,dx + \frac{\beta_1}{2}\int_\Omega |u_{h\lambda}|^{p_0}\,dx \\ &+ \frac{\gamma_1}{2}\int_\Omega |\nabla u_{h\lambda}|^{q_0}|u_{h\lambda}|^2\,dx \\ &- c\int_\Omega (1 + |u_{bh}|^p + |u_{bh}|^{p_0} + |u_{bh}|^2 + |u_{bh}|^{\frac{2p}{p-q_0}} + |\nabla u_{bh}|^p)\,dx \\ &- \frac{c}{h}\int_\Omega (|u_{bh}|^2 + |u_{h\lambda}(t-h)|^2)\,dx > 0, \end{aligned} \tag{2.10}$$

where for sufficiently large $|\alpha|$ the second inequality in (2.10) holds. Therefore there exists $\alpha_0 \in \mathbb{R}^\lambda$ such that $\Phi_{h\lambda}(\alpha_0) = 0$. Thus we have proved the existence of solutions to (2.6).

Now we obtain an estimate for solutions of (2.6). We put $\varphi = u_{h\lambda} - u_{bh}$ into (2.6) and integrate the result over t from 0 to t. We have

$$\frac{1}{h}\int_\Omega (u_{h\lambda}(t) - u_{h\lambda}(t-h))u_{h\lambda}(t)\,dx \ge \frac{1}{2h}\int_\Omega (u_{h\lambda}^2(t) - u_{h\lambda}^2(t-h))\,dx$$

and

$$\begin{aligned} \frac{1}{2h}\int_0^t\int_\Omega (u_{h\lambda}^2(t) - u_{h\lambda}^2(t-h))\,dx\,dt &= \frac{1}{2h}\int_{t-h}^t\int_\Omega u_{h\lambda}^2(t)\,dx\,dt - \frac{1}{2h}\int_{-h}^0\int_\Omega u_{h\lambda}^2(t)\,dx\,dt \\ &= \frac{1}{2}\int_\Omega u_{h\lambda}^2(t)\,dx - \frac{1}{2}\int_\Omega u_{0h}^2(t)\,dx, \end{aligned}$$

where we used the fact that $u_{h\lambda}(t)$ are independent of t in any interval $(ih,(i+1)h)$, $i=0,\dots,\frac{T}{h}-1$, where $\frac{T}{h}$ is an integer, and $u_{h\lambda}(t)=u_{0h}(t)$ for $t\in(-h,0)$. Using the above considerations and the proof of Lemma 2.1 we obtain

$$\int_\Omega u_{h\lambda}^2(t)\,dx+\int_{\Omega^t}(|\nabla u_{h\lambda}|^p+|u_{h\lambda}|^{p_0}+|\nabla u_{h\lambda}|^{q_0}|u_{h\lambda}|^2\,dx\,dt\le c, \tag{2.11}$$

where c depends on the norms of data functions. From (2.11) we can choose a subsequence of $\{u_{h\lambda}\}$ still denoted by $\{u_{h\lambda}\}$ such that

$$u_{h\lambda}\to u\quad\text{weakly in } L_p(0,T;W_p^1(\Omega))\cap L_{p_0}(\Omega^T)$$

and

$$u_{h\lambda}\to u\quad\text{weak star in } L_\infty(0,T;L_2(\Omega))$$

as $(h,\lambda)\to(0,\infty)$.

Now we show almost everywhere convergence of $u_{h\lambda}\to u$ in Ω^T. Changing variables in (2.6) from t to $t+h$ and integrating the result over t from 0 to $T-h$ we obtain

$$\begin{aligned}\sum_{j=1}^m\frac{1}{h}\int_0^{T-h}\int_\Omega(u_{h\lambda j}(t+h)-u_{h\lambda j}(t))\varphi_j\,dx\,dt\\+\sum_{j=1}^m\int_0^{T-h}\int_\Omega\Big(\sum_{k=1}^m a_{kj}\nabla u_{h\lambda j}(t+h)\cdot\nabla\varphi_k+R_ju_{h\lambda j}(t+h)\varphi_j-f_j\varphi_j\Big)\,dx\,dt=0,\end{aligned} \tag{2.12}$$

where the coefficients a_{jk}, R_j and f_j, $j,k=1,\dots,m$, depend on $u_{h\lambda}(t+h)$.

Since $\varphi|_s=0$ we put $\varphi=\frac{1}{h}(u_{h\lambda}(t+h)-u_{h\lambda}(t))-\frac{1}{h}(u_{bh}(t+h)-u_{bh}(t))$ into (2.12). Hence in view of (2.11) we obtain

$$\int_0^{T-h}\int_\Omega(u_{h\lambda}(t+h)-u_{h\lambda}(t))^2\,dx\,dt\le ch \tag{2.13}$$

hence

$$u_{h\lambda}\to u\quad\text{in } L_1(\Omega^T) \tag{2.14}$$

so

$$u_{h\lambda}\to u\quad\text{almost everywhere in } \Omega^T. \tag{2.15}$$

Next from Lemma 6.3 of [6, Ch. 5, Sect. 6] we get

$$u_{h\lambda}\to u\quad\text{strongly in } L_r(\Omega^T) \tag{2.16}$$

where $r<q=p\frac{n+2}{n}$.

Finally we prove strong convergence of $\nabla u_{h\lambda}$ to ∇u. To show this we put $\varphi=u_{h\lambda}-u_{bh}-v_{h\lambda}\equiv\omega$ into (2.6), where $v_{h\lambda}\in L_p(0,T;V_\lambda)\cap L_{p_0}(\Omega^T)$ are approximations of $u-u_b$ in $L_p(0,T;W_p^1(\Omega))\cap L_{p_0}(\Omega^T)$ which are time independent in each interval $((k-1)h,kh)$. Therefore

$$v_{h\lambda} \to u - u_b \quad \text{strongly in } L_p(0,T;W_p^1(\Omega)) \cap L_{p_0}(\Omega^T). \tag{2.17}$$

Now from (2.6) we obtain

$$\begin{aligned} \sum_{i=1}^m \int_0^t \int_\Omega \partial_t^{-h} u_{h\lambda i}\omega_i\,dx\,dt + \sum_{i,j=1}^m \int_0^t \int_\Omega a_{ij}(u_{h\lambda}, \nabla u_{h\lambda}) \cdot \nabla u_{h\lambda j} \cdot \nabla \omega_i\,dx\,dt \\ + \sum_{i=1}^m \int_0^t \int_\Omega R_i(u_{h\lambda}, \nabla u_{h\lambda}) u_{h\lambda i}\omega_i\,dx\,dt \\ = \sum_{i=1}^m \int_0^t \int_\Omega f_i(u_{h\lambda}, \nabla u_{h\lambda})\omega_i\,dx\,dt. \end{aligned} \tag{2.18}$$

Repeating the considerations from [1] in the case $\Phi = \frac{1}{2}(u_1^2+\ldots+u_m^2)$, $b = (u_1,\ldots,u_m) = \nabla\Phi$, $B(u) = \sum_{i=1}^m \int_0^{u_i}(u_i - s_i)ds_i = \frac{1}{2}(u_1^2+\ldots+u_m^2)$, we obtain

$$\begin{aligned} \sum_{i=1}^m \int_0^t \int_\Omega \partial_t^{-h} u_{h\lambda i}\omega_i\,dx\,dt \\ \ge \frac{1}{h}\int_{t-h}^t \int_\Omega B(u_{h\lambda}(t))\,dx\,dt - \int_\Omega B(u(t))\,dx + 0(h,\lambda), \end{aligned} \tag{2.19}$$

where $0(h,\lambda)$ converges to zero as $(h,\lambda) \to (0,\infty)$. The second term in (2.18) we write in the form

$$\begin{aligned} \sum_{i,j=1}^m \Big[\int_0^t \int_\Omega [a_{ij}(u_{h\lambda}, \nabla u_{h\lambda}) \cdot \nabla u_{h\lambda j} \\ - a_{ij}(u_{h\lambda}, \nabla(u_{bh} + v_{h\lambda})) \cdot \nabla(u_{bhj} + v_{h\lambda j})] \cdot \nabla\omega_i\,dx\,dt \\ + \int_0^t \int_\Omega [a_{ij}(u_{h\lambda}, \nabla(u_{bh} + v_{h\lambda})) \cdot \nabla(u_{bhj} + v_{h\lambda j}) - a_{ij}(u_{h\lambda}, \nabla u) \cdot \nabla u_j] \cdot \nabla\omega_i\,dx\,dt \\ + \int_0^t \int_\Omega [a_{ij}(u_{h\lambda}, \nabla u) - a_{ij}(u, \nabla u)] \cdot \nabla u_j \cdot \nabla\omega_i\,dx\,dt + \int_0^t \int_\Omega a_{ij}(u, \nabla u) \cdot \nabla u_j \cdot \nabla\omega_i\,dx\,dt\Big] \\ \equiv I_1 + I_2 + I_3. \end{aligned} \tag{2.20}$$

Using the ellipticity condition (1.3) we have $I_1 \ge \overline{\alpha}|\nabla\omega|^p$. In view of the Hölder and Young inequalities we obtain

$$\begin{aligned} I_2 \le \varepsilon \int_0^t \int_\Omega |\nabla\omega|^p\,dx\,dt \\ + c(\varepsilon) \int_0^t \int_\Omega |a_{ij}(u_{h\lambda}, \nabla(u_{bh} + v_{h\lambda})) \cdot \nabla(u_{bhj} + v_{h\lambda j}) - a_{ij}(u_{h\lambda}, \nabla u) \cdot \nabla u_j|^{\frac{p}{p-1}}\,dx\,dt, \end{aligned}$$

where $\varepsilon \in (0,1)$ and the second integral converges to zero as $(h,\lambda) \to (0,\infty)$ because of the strong convergence of $u_{bh} + v_{h\lambda}$ to u in $L_p(0,T;W_p^1(\Omega))$ and of the fact that $a_{ij}(u_{h\lambda}, \nabla(u_{b\lambda} + v_{h\lambda})) \cdot \nabla(u_{bhj} + v_{h\lambda j}) \in L_{\frac{p}{p-1}}(\Omega^T)$ (see [2], Th. 2, Ch. 1, Sect. 4).

Similarly we have

$$I_3 \le \varepsilon \int_0^t \int_\Omega |\nabla\omega|^p \, dx\, dt + c(\varepsilon) \int_0^t \int_\Omega \big|[a_{ij}(u_{h\lambda}, \nabla u) - a_{ij}(u, \nabla u)]|\nabla u_j\big|^{\frac{p}{p-1}} \, dx\, dt$$
$$+ \Big| \int_0^t \int_\Omega a_{ij}(u, \nabla u) \cdot \nabla u_j \cdot \nabla\omega_i \, dx\, dt \Big|,$$

where $\varepsilon \in (0,1)$ and the second term converges to zero because of the strong convergence of $u_{h\lambda} \to u$ in $L_r(\Omega^T)$, $r < q$.

Next we consider the third term on the l.h.s. of (2.18). First we examine

$$\sum_{i=1}^m \int_0^t \int_\Omega R_{1i}(u_{h\lambda}) u_{h\lambda i} \omega_i \, dx\, dt$$
$$= \sum_{i=1}^m \int_0^t \int_\Omega [R_{1i}(u_{h\lambda}) u_{h\lambda i} - R_{1i}(u_{bh} + v_{h\lambda})(u_{bhi} + v_{h\lambda i})]\omega_i \, dx\, dt$$
$$+ \sum_{i=1}^m \int_0^t \int_\Omega [R_{1i}(u_{bh} + v_{h\lambda})(u_{bhi} + v_{h\lambda i}) - R_{1i}(u) u_i]\omega_i \, dx\, dt$$
$$+ \sum_{i=1}^m \int_0^t \int_\Omega R_{1i}(u) u_i \omega_i \, dx\, dt \equiv I_4 + I_5 + I_6.$$

In view of $(1.5)_2$ it follows that

$$I_4 \ge \beta_0 \int_0^t \int_\Omega |\omega|^{p_0} \, dx\, dt.$$

In virtue of the Hölder and Young inequalities one gives

$$I_5 \le \varepsilon \int_0^t \int_\Omega |\omega|^{p_0} \, dx\, dt$$
$$+ c(\varepsilon) \int_0^t \int_\Omega |R_{1i}(u_{bh} + v_{h\lambda})(u_{bhi} + v_{h\lambda i}) - R_{1i}(u) u_i|^{\frac{p_0}{p_0-1}} \, dx\, dt,$$

where $\varepsilon \in (0,1)$ and the second term converges to zero because $u_{bh} + v_{h\lambda}$ converges strongly to u in $L_p(0,T; W_p^1(\Omega)) \cap L_{p_0}(\Omega^T)$ (see also [2], Th. 2, Ch. 1, Sect. 4).

Finally I_6 converges to zero because ω converges to zero weakly in $L_p(0,T; W_p^1(\Omega)) \cap L_{p_0}(\Omega^T)$.

Consider the second part of the third term on the l.h.s. of (2.18). In view of (1.6) and the Hölder inequality we obtain

$$\Big|\sum_{i=1}^m \int_0^t \int_\Omega R_{2i}(u_h, \nabla u_h) u_{hi} \omega_i \, dx\, dt\Big| \le c \int_0^t \int_\Omega |\nabla u_h|^{q_0} |u_h|\, |\omega| \, dx\, dt$$
$$\le c\Big(\int_{\Omega^t} |\nabla u_h|^p \, dx\, dt\Big)^{q_0/p} \Big(\int_{\Omega^t} |u_h|^{p_*} \, dx\, dt\Big)^{1/p_*} \Big(\int_{\Omega^t} |\omega|^\sigma \, dx\, dt\Big)^{1/\sigma} \equiv I_7,$$

where $p_* = \max\{p_0, q\}$, $q = p\frac{n+2}{n}$ and $\sigma = \frac{1}{1 - \frac{q_0}{p} - \frac{1}{p_*}}$.

Let the assumption (a) of the theorem hold. Then $p_* = q$, $\sigma < q$ and ω converges to 0 strongly in $L_\sigma(\Omega^T)$, so I_7 converges also to zero. Let the assumption (b) hold. Then $p_* = p_0$, $\sigma < q$ and I_7 converges also to zero.

Finally we pass to the limit on the r.h.s. of (2.18). In view of (1.7) and the Hölder inequality we get

$$\begin{aligned}
&\Big|\sum_{i=1}^m \int_{\Omega^t} f_i(u_{h\lambda}, \nabla u_{h\lambda})\omega_i \,dx\,dt\Big| \\
&\qquad \le c\int_{\Omega^t} (|u_{h\lambda}|^{\mu_1} + |\nabla u_{h\lambda}|^\nu |u_{h\lambda}|^{\mu_2})|\omega|\,dx\,dt + c\int_{\Omega^t}|\omega|\,dx\,dt \\
&\qquad \le c\Big(\int_{\Omega^t} |u_{h\lambda}|^{p_*}\,dx\,dt\Big)^{\mu_1/p_*}\Big(\int_{\Omega^t}|\omega|^{\sigma_1}\,dx\,dt\Big)^{1/\sigma_1} \\
&\qquad\quad + c\Big(\int_{\Omega^t} |\nabla u_{h\lambda}|^p\,dx\,dt\Big)^{\nu/p}\Big(\int_{\Omega^t} |u_{h\lambda}|^{p_*}\,dx\,dt\Big)^{\mu_2/p_*}\Big(\int_{\Omega^t}|\omega|^{\sigma_2}\,dx\,dt\Big)^{1/\sigma_2} \\
&\qquad\quad + c\int_{\Omega^t}|\omega|\,dx\,dt,
\end{aligned}$$

where $\sigma_1 = \frac{1}{1-\frac{\mu_1}{p_*}}$, $\sigma_2 = \frac{1}{1-\frac{\nu}{p}-\frac{\mu_2}{p_*}}$.

Let the assumption (c) of the theorem hold. Then

$$\sigma_1 = \frac{1}{1-\frac{\mu_1}{q}}, \quad \sigma_2 = \frac{1}{1-\frac{\nu}{p}-\frac{\mu_2}{q}} \quad \text{and} \quad \sigma_i < q, \quad i = 1,2,$$

so $\|\omega\|_{L_{\sigma_i}(\Omega^T)} \to 0, i = 1,2,$ as $(h,\lambda) \to (0,\infty)$. If the assumption (d) is valid then $\sigma_1 = \frac{1}{1-\frac{\mu_1}{p_0}}$, $\sigma_2 = \frac{1}{1-\frac{\nu}{p}-\frac{\mu_2}{p_0}}$, $\sigma_i < q$ and also $\|\omega\|_{L_{\sigma_i}(\Omega^T)} \to 0$, $i = 1,2$, as $(h,\lambda) \to (0,\infty)$. Summarizing the above considerations instead of (2.18) we obtain

$$\frac{1}{h}\int_{t-h}^{t}\int_\Omega B(u_{h\lambda}(t))\,dx\,dt - \int_\Omega B(t)\,dx + c\int_{\Omega^t}|\nabla\omega|^p\,dx\,dt \le 0(h,\lambda) \tag{2.21}$$

if ε is sufficiently small.

In view of (2.15) and the Fatou lemma

$$\liminf_{\substack{h\to 0\\ \lambda\to\infty}} \int_\Omega (B(u_{h\lambda}(t)) - B(u(t)))\,dx \ge 0$$

so (2.21) implies

$$\nabla u_{h\lambda} \to \nabla u \quad \text{strongly in } L_p(\Omega^t),\ t \le T. \tag{2.22}$$

Hence (2.15) and (2.22) yield

$$\begin{aligned}
&a_{ij}(u_{h\lambda}, \nabla u_{h\lambda}) \to a_{ij}(u,\nabla u), && i,j = 1,\dots,m,\\
&R_i(u_{h\lambda}, \nabla u_{h\lambda})u_{h\lambda i} \to R_i(u,\nabla u)u_i, && i = 1,\dots,m,\\
&f_i(u_{h\lambda}, \nabla u_{h\lambda}) \to f_i(u,\nabla u), && i = 1,\dots,m,
\end{aligned} \tag{2.23}$$

almost everywhere convergence in Ω^T and also weak convergence in $L_{\frac{p}{p-1}}(\Omega^t)$, $t \le T$.

Hence the above considerations imply that u satisfies the identity (1.12).

Finally the approximate solution satisfies

$$\int_{\Omega^T} \partial_t^{-h} u_{h\lambda} \zeta \, dx \, dt = - \int_{\Omega^{T-h}} (u_{h\lambda} - u_{0h}) \partial_t^h \zeta \, dx \, dt, \tag{2.24}$$

which holds for any ζ such that $\zeta(t) = 0$ for $t > T - h$ and $\zeta \in L_p(\Omega^T)$, $\zeta_t \in L_{p'}(\Omega^T)$, $1/p + 1/p' = 1$. Since $u \in L_p(0, T; W_p^1(\Omega))$ we have a weak convergence of $\partial_t^{-h} u_{h\lambda} \to \partial_t u$ in $L_{p'}(0, T; W_{p'}^{-1}(\Omega))$. Hence the limit function u satisfies (1.13), so u is a solution of problem (P.1) defined by Definition 1.2. This concludes the proof.

In the case of vanishing boundary conditions we obtain

LEMMA 2.3. *Let $u_b = 0$. Let (1.7)–(1.11) hold. Let $p^* = \max\{p, p_0\}$ and $\mu_i + 1 < p^*$, $i = 1, 2$, $\frac{\nu}{p} + \frac{\mu_2+1}{p^*} < 1$. Moreover, let $u_0 \in L_2(\Omega)$ and $u_0|_S = 0$. Then*

$$\int_{\Omega} u^2 \, dx + \int_{\Omega^t} (\alpha_1 |\nabla u|^p + \beta_1 |u|^{p_0} + \gamma_1 |\nabla u|^{q_0} |u|^2) \, dx \, dt \le \int_{\Omega} u_0^2 \, dx + c \le c_0. \tag{2.25}$$

Proof. Putting $\varphi_i = u_i$, $i = 1, \ldots, m$, into (1.12) and using the growth conditions (1.2)–(1.7) we obtain

$$\begin{aligned} \frac{1}{2} \int_{\Omega} u^2 \, dx + \int_{\Omega^t} (\alpha_1 |\nabla u|^p + \beta_1 |u|^{p_0} + \gamma_1 |\nabla u|^{q_0} |u|^2) \, dx \, dt \\ \le \frac{1}{2} \int_{\Omega} u_0^2 \, dx + c \int_{\Omega} (|u|^{\mu_1} + |u|^{\mu_2} |\nabla u|^{\nu}) |u| \, dx \, dt. \end{aligned} \tag{2.26}$$

Let $\mu_1 + 1 < p^*$. Then $\int_{\Omega^t} |u|^{\mu_1+1} \, dx \, dt \le \varepsilon_1 \int_{\Omega^t} (|u|^{p_0} + |\nabla u|^p) \, dx \, dt + c(\varepsilon_1)$, $\varepsilon_1 \in (0, 1)$. Assuming $\mu_2 + 1 < p^*$, $\frac{\nu}{p} + \frac{\mu_2+1}{p^*} < 1$ yields

$$\int_{\Omega^t} |u|^{\mu_2+1} |\nabla u|^{\nu} \, dx \, dt \le \varepsilon_2 \int_{\Omega^t} (|u|^{p_0} + |\nabla u|^p) \, dx \, dt + c(\varepsilon_2).$$

Using the above inequalities in (2.26) and assuming $\varepsilon_1, \varepsilon_2$ sufficiently small we obtain (2.25). This concludes the proof.

THEOREM 2.4. *Let the assumptions of Lemma* 2.3 *and the assumptions* (a)–(d) *of Theorem* 2.2 *hold. Then there exists a solution of problem* (P.1) *such that $u \in L_\infty(0, T; L_2(\Omega)) \cap L_p(0, T; \mathring{W}_p^1(\Omega)) \cap L_{p_0}(\Omega^T)$ and estimate* (2.25) *is valid.*

3. L_∞-estimate for solutions of diagonal problem (P.1). In this section we consider the following diagonal system:

$$\begin{aligned} & u_{it} - \nabla \cdot (a_i(x, t, u, \nabla u) \nabla u_i) + R_i(x, t, u, \nabla u) u_i = f_i(x, t, u, \nabla u) \quad \text{in } \Omega^T, \\ & u_i|_{t=0} = u_{0i} \quad \text{in } \Omega, \\ & u_i = u_{bi} \quad \text{on } S^T, \end{aligned} \tag{3.1}$$

where $i = 1, \ldots, m$ and instead of (1.2), (1.3) we assume that

$$a_i : \Omega^T \times \mathbb{R}^m \times \mathbb{R}^{mn} \to \mathbb{R}^{n^2}, \quad i = 1, \ldots, m,$$

satisfy the Carathéodory condition and

$$\alpha_1|\nabla u|^{p-2}|\nabla u_i|^2 \le a_i(x,t,u,\nabla u)\cdot\nabla u_i\cdot\nabla u_i \le \alpha_2|\nabla u|^{p-2}|\nabla u_i|^2,\ p\ge 2,\ i=1,\dots,m, \tag{3.2}$$

where α_1,α_2 are the same as in (1.2), and (1.3) is replaced by

$$\sum_{i=1}^m (a_i(x,t,u,\nabla u_1)\cdot\nabla u_{1i} - a_i(x,t,u,\nabla u_2)\cdot\nabla u_{2i})\cdot(\nabla u_{1i}-\nabla u_{2i}) \ge \overline{\alpha}|\nabla u_1-\nabla u_2|^p, \tag{3.3}$$

where $\overline{\alpha}$ is the same as before.

To show an L_∞-estimate for solutions of problem (3.1) we use the following weak formulation with Steklov averages:

$$\sum_{i=1}^m \int_h^T\int_\Omega [\partial_t u_{hi}\varphi_i + (a_i(x,t,u,\nabla u)\cdot\nabla u_i)_h\cdot\nabla\varphi_i + (R_i(x,t,u,\nabla u)u_i)_h\varphi_i - (f_i(x,t,u,\nabla u))_h\varphi_i]\,dx\,dt = 0, \tag{3.4}$$

which holds for all $\varphi\in L_2(0,T;\overset{\circ}{W}{}^1_p(\Omega))$. First we prove

Lemma 3.1. *Let $\overline{k}>0$ and let*

$$|u_b|_{L_\infty(\Omega^T)} < \overline{k}, \qquad |u_0|_{L_\infty(\Omega)} < \overline{k}. \tag{3.5}$$

Let $q = p\frac{n+2}{n}$, $p_* = \max\{p_0,q\}$. Let

$$1-\frac{\mu_1 d}{p_*(d-1)} > 0, \qquad d<q, \tag{3.6}$$

and

$$1-\left(\frac{\mu_2}{p_*}+\frac{\nu}{p}\right)\frac{d}{d-1} > 0, \qquad d<q. \tag{3.7}$$

Let $p^* = \max\{p,p_0\}$,

$$\mu_i + 1 < p^*, \qquad i=1,2, \tag{3.8}$$

and

$$\frac{\nu}{p}+\frac{\mu_2+1}{p^*} < 1. \tag{3.9}$$

Moreover, let the other assumptions of Lemma 2.1 and Theorem 2.2 hold. Then

$$\begin{aligned}\sum_{i=1}^m\Big[\int_\Omega (u_i-\overline{k})_+^2\,dx + \alpha_1\int_{\Omega^t}|\nabla(u_i-\overline{k})_+|^p\,dx\,dt \\ + \beta_1\int_{\Omega^t}(u_i-\overline{k})_+^{p_0}\,dx\,dt + \gamma_1\int_{\Omega^t}|\nabla u_i|^{q_0}(u_i-\overline{k})_+^2\,dx\,dt\Big] \\ \le c_2\sum_{i=1}^m\Big[\int_{\Omega^t}(u_i-\overline{k})_+\,dx\,dt + \int_{\Omega^t}(u_i-\overline{k})_+^d\,dx\,dt + \Big(\int_0^t |A^\pm_{\overline{k},i}(t)|\,dt\Big)^{1-\frac{\mu_1 d}{p_*(d-1)}} \\ + \Big(\int_0^t |A^\pm_{\overline{k},i}(t)|\,dt\Big)^{1-\left(\frac{\mu_2}{p_*}+\frac{\nu}{p}\right)\frac{d}{d-1}}\Big],\end{aligned} \tag{3.10}$$

where $d<q$, c_2 depends on the r.h.s. of (2.1) *and $A^\pm_{\overline{k},i}(t) = \operatorname{meas}\{x\in\Omega : u_i(x,t) > \overline{k}\}$.*

Proof. Putting $\varphi_i = (u_{hi} - \overline{k})_+$ into (3.4), using (3.2), (1.7) and (1.9) and letting $h \to 0$ we obtain

$$\sum_{i=1}^m \int_\Omega (u_i - \overline{k})_+^2\, dx + \alpha_1 \sum_{i=1}^m \int_{\Omega^t} |\nabla (u_i - \overline{k})_+|^p\, dx\, dt + \sum_{i=1}^m \int_{\Omega^t} R_i(u, \nabla u) u_i (u_i - \overline{k})_+\, dx\, dt \le c \sum_{i=1}^m \int_{\Omega^t} \big(|u|^{\mu_1} + |u|^{\mu_2} |\nabla u|^\nu\big)(u_i - \overline{k})_+\, dx\, dt + c \sum_{i=1}^m \int_{\Omega^t} (u_i - \overline{k})_+\, dx\, dt. \tag{3.11}$$

Using $(1.5)_1$ and the fact that $u_i(x,t) > \overline{k} > 0$ for $x \in A^{\pm}_{\overline{k},i}(t)$ we have

$$\begin{aligned}\int_0^t dt \int_\Omega R_{1i}(u) u_i (u_i - \overline{k})_+\, dx &= \int_0^t dt \int_{A^{\pm}_{\overline{k},i}(t)} R_{1i}(u) u_i (u_i - \overline{k})_+\, dx \\ &\ge \beta_1 \int_0^t dt \int_{A^{\pm}_{\overline{k},i}(t)} |u|^{p_0 - 2} u_i (u_i - \overline{k})\, dx \ge \beta_1 \int_0^t dt \int_{A^{\pm}_{\overline{k},i}(t)} |u|^{p_0-2} (u_i - \overline{k})^2\, dx \\ &\ge \beta_1 \int_0^t dt \int_{A^{\pm}_{\overline{k},i}(t)} |u_i|^{p_0-2} (u_i - \overline{k})^2\, dx \ge \beta_1 \int_0^t dt \int_{A^{\pm}_{\overline{k},i}(t)} (u_i - \overline{k})^{p_0}\, dx \\ &= \beta_1 \int_0^t dt \int_{\Omega^t} (u_i - \overline{k})_+^{p_0}\, dx\, dt, \quad i = 1, \ldots, m,\end{aligned} \tag{3.12}$$

and in view of (1.6) we get

$$\begin{aligned}\int_{\Omega^t} R_{2i}(u, \nabla u) u_i (u_i - \overline{k})_+\, dx\, dt &= \int_0^t dt \int_{A^{\pm}_{\overline{k},i}(t)} R_{2i}(u, \nabla u) u_i (u_i - \overline{k})\, dx \\ &\ge \gamma_1 \int_0^t dt \int_{A^{\pm}_{\overline{k},i}(t)} |\nabla u_i|^{q_0} u_i (u_i - \overline{k})\, dx \ge \gamma_1 \int_0^t dt \int_{A^{\pm}_{\overline{k},i}(t)} |\nabla u_i|^{q_0} (u_i - \overline{k})^2\, dx \\ &= \gamma_1 \int_{\Omega^t} |\nabla u_i|^{q_0} (u_i - \overline{k})_+^2\, dx\, dt, \quad i = 1, \ldots, m.\end{aligned} \tag{3.13}$$

Now we examine the r.h.s. of (3.11). Using the Hölder and Young inequalities we have

$$\begin{aligned}\int_{\Omega^t} |u|^{\mu_1} (u_i - \overline{k})_+\, dx\, dt &= \int_0^t dt \int_{A^{\pm}_{\overline{k},i}(t)} |u|^{\mu_1} (u_i - \overline{k})\, dx \\ &\le \frac{d-1}{d} \int_0^t dt \int_{A^{\pm}_{\overline{k},i}(t)} |u|^{\frac{\mu_1 d}{d-1}}\, dx + \frac{1}{d} \int_{\Omega^t} (u_i - \overline{k})_+^d\, dx\, dt\end{aligned} \tag{3.14}$$

$$\leq \frac{d-1}{d}\Big(\int\limits_{\Omega^t}|u|^{p_*}\,dx\,dt\Big)^{\frac{\mu_1 d}{p_*(d-1)}}\Big(\int\limits_0^t|A^{\pm}_{\overline{k},i}(t)|\,dt\Big)^{1-\frac{\mu_1 d}{p_*(d-1)}}+\frac{1}{d}\int\limits_{\Omega^t}(u_i-\overline{k})^d_+\,dx\,dt,$$

where $p_* = \max\{q,p_0\}$, $q = p\frac{n+2}{n}$, $1 < d < q$.

Similarly we have

$$\int\limits_{\Omega^t}|u|^{\mu_2}|\nabla u|^{\nu}(u_i-\overline{k})_+\,dx\,dt \tag{3.15}$$

$$\leq \frac{d-1}{d}\int\limits_0^t dt\int\limits_{A^{\pm}_{\overline{k},i}(t)}|u|^{\frac{\mu_2 d}{d-1}}|\nabla u|^{\frac{\nu d}{d-1}}\,dx+\frac{1}{d}\int\limits_{\Omega^t}(u_i-\overline{k})^d_+\,dx\,dt$$

$$\leq \frac{d-1}{d}\Big(\int\limits_{\Omega^t}|u|^{p_*}\,dx\,dt\Big)^{\frac{\mu_2}{p_*}\frac{d}{d-1}}\Big(\int\limits_{\Omega^t}|\nabla u|^{p}\,dx\,dt\Big)^{\frac{\nu}{p}\frac{d}{d-1}}\Big(\int\limits_0^t|A^{\pm}_{\overline{k},i}(t)|\,dt\Big)^{1-\left(\frac{\mu_2}{p_*}+\frac{\nu}{p}\right)\frac{d}{d-1}}$$

$$+\frac{1}{d}\int\limits_{\Omega^t}(u_i-\overline{k})^d_+\,dx\,dt.$$

Using (3.12)–(3.15) in (3.11) and the estimate (2.1) for the weak solution we obtain (3.10). This concludes the proof.

Now we obtain the well known iterative inequality of the type

$$Y_{s+1}\leq cb^sY_s^{1+\alpha}, \tag{3.16}$$

where $s=0,1,\ldots$, $\alpha>0$ (see [3], Ch. 1, Lemma 4.1; [6], Ch. 2, Lemma 5.7; [7], Ch. 2, Lemma 4.7) which implies an L_∞-estimate.

LEMMA 3.2. *Let the assumptions of either Lemma* 2.1 *or Lemma* 2.3 *hold. Let*

$$Y_s=\sum_{i=1}^m\int\limits_{\Omega^t}(u_i-k_s)^{\delta}_+\,dx\,dt, \tag{3.17}$$

where $d<\delta<q=p\frac{n+2}{n}$, $k_s=k_*+k-\frac{k}{2^s}$, $k_*=\max\{\|u_0\|_{L_\infty(\Omega)},\|u_b\|_{L_\infty(S^t)}\}$, $t\leq T$, $k\in\mathbb{R}^+$, $s=0,1,\ldots,$ *Then there exist positive constants* c_3, a_*, a^*, σ *such that*

$$Y_{s+1}\leq c_3\frac{2^{a^*s}}{k^{a_*}}Y_s^{1+\sigma\frac{\delta}{q}}, \tag{3.18}$$

where $c_3=c_3(c_0)$, $a^*=\max\{a_1,a_2,\delta\alpha_1,\delta\alpha_2\}$, $a_*=\min\{a_1,a_2,\delta\alpha_1,\delta\alpha_2\}$, $a_1=(\delta-1)\frac{p}{n}\frac{\delta}{q}+\delta(1-\frac{1}{q})$, $a_2=(\delta-d)\frac{p}{n}\frac{\delta}{q}+(q-d)\frac{\delta}{q}$, $\alpha_i=1+[\frac{p}{n}(1-\gamma_i)-\gamma_i]\frac{\delta}{d}\equiv 1+\sigma_i\frac{\delta}{q}$, $i=1,2$, $\gamma_1=\frac{\mu_1 d}{p^*(d-1)}$, $\gamma_2=(\frac{\mu_2}{p^*}+\frac{\nu}{p})\frac{d}{d-1}$, $\sigma=\min\{\sigma_1,\sigma_2\}$, $k>1$. *Moreover, we assume that* $\gamma_1<1$, $\gamma_2<1$.

P r o o f. Putting $\overline{k}=k_{s+1}$ into (3.10) and using the estimates (see [3], Ch. 5, Sect. 7),

$$\int\limits_0^t|A^+_{k_{s+1},i}(t)|\,dt\leq\frac{2^{\sigma(s+1)}}{k^{\sigma}}\int\limits_{\Omega^t}(u_i-k_s)^{\sigma}_+\,dx\,dt, \tag{3.19}$$

$$\int\limits_{\Omega^t}(u_i-k_{s+1})^{\rho}_+\,dx\,dt\leq c\frac{2^{(\delta-\rho)s}}{k^{\delta-\rho}}\int\limits_{\Omega^t}(u_i-k_s)^{\delta}_+\,dx\,dt,\qquad \rho<\delta, \tag{3.20}$$

we obtain

$$
\begin{aligned}
(3.21)\qquad &\sum_{i=1}^m\Big[\int_\Omega (u_i-k_{s+1})_+^2\,dx+\alpha_1\int_{\Omega^t}|\nabla(u_i-k_{s+1})_+|^p\,dx\,dt\\
&\quad+\beta_1\int_{\Omega^t}(u_i-k_{s+1})_+^{p_0}\,dx\,dt\Big]\\
&\le c\sum_{i=1}^m\bigg[\frac{2^{(\delta-1)s}}{k^{\delta-1}}\int_{\Omega^t}(u_i-k_s)_+^\delta\,dx\,dt+\frac{2^{(\delta-d)s}}{k^{\delta-d}}\int_{\Omega^t}(u_i-k_s)_+^\delta\,dx\,dt\\
&\quad+\bigg(\frac{2^{\delta s}}{k^\delta}\int_{\Omega^t}(u_i-k_s)_+^\delta\,dx\,dt\bigg)^{\beta_1}+\bigg(\frac{2^{\delta s}}{k^\delta}\int_{\Omega^t}(u_i-k_s)_+^\delta\,dx\,dt\bigg)^{\beta_2}\bigg],
\end{aligned}
$$

where $\beta_1=1-\frac{\mu_1 d}{p^*(d-1)}=1-\gamma_1$, $\beta_2=1-(\frac{\mu_2}{p^*}+\frac{\nu}{p})\frac{d}{d-1}=1-\gamma_2$.

In view of (3.17) and the Hölder inequality we have

$$
(3.22)\qquad Y_{s+1}\le\sum_{i=1}^m\Big(\int_{\Omega^t}(u_i-k_{s+1})_+^q\,dx\,dt\Big)^{\delta/q}\Big(\int_0^t|A^+_{k_{s+1},i}(t)|\,dt\Big)^{1-\delta/q}.
$$

Using (3.19) with $\sigma=\delta$ and (3.21) in (3.22) yields

$$
(3.23)\qquad Y_{s+1}\le c\bigg[\bigg(\frac{2^{a_1 s}}{k^{a_1}}+\frac{2^{a_2 s}}{k^{a_2}}\bigg)Y_s^{1+\frac{p}{n}\frac{\delta}{q}}+\frac{2^{\delta s\alpha_1}}{k^{\delta\alpha_1}}Y_s^{1+\sigma_1\frac{\delta}{q}}+\frac{2^{\delta s\alpha_2}}{k^{\delta\alpha_2}}Y_s^{1+\sigma_2\frac{\delta}{q}}\bigg].
$$

In view of either (2.1) or (2.25) we have

$$
Y_s\le\sum_{i=1}^m\int_{\Omega^t}|u_i|^q\,dx\,dt\le c_0,
$$

where c_0 depends on the norms of the data functions (u_0 and u_b) (see either (2.1) or (2.25)).

Then instead of (3.23) we obtain (3.18). This concludes the proof.

Finally we show the boundedness of weak solutions.

LEMMA 3.3. *Let the assumptions of either Lemma* 2.1 *or Lemma* 2.3 *be satisfied. Let* σ_i, $i=1,2$, *be positive*, *so*

$$
(3.24)\qquad \frac{p}{n}>\frac{\gamma_i}{1-\gamma_i},\qquad i=1,2,
$$

where $\gamma_1=\frac{\mu_1 d}{p^*(d-1)}<1$, $\gamma_2=(\frac{\mu_2}{p^*}+\frac{\nu}{p})\frac{d}{d-1}<1$, $d<q$. *Then*

$$
(3.25)\qquad \sup_i|u_i|_{L_\infty(\Omega^T)}\le k_*+k_0,
$$

where

$$
(3.26)\qquad k_0=[c_0c_3^{\frac{q}{\sigma\delta}}2^{a^*(\frac{q}{\sigma\delta})^2}]^{\frac{\sigma\delta}{a_*q}}.
$$

P r o o f. In view of either Lemma 4.1 of [3], Ch. 1, or Lemma 5.6 of [6], Ch. 2, or Lemma 4.7 of [7], Ch. 2, we find that Y_s converges to zero as $s\to\infty$ if

$$
Y_0\le c_3^{-\frac{q}{\sigma\delta}}k^{a_*q/(\sigma\delta)}2^{-a^*q^2/(\sigma^2\delta^2)}.
$$

We have

$$Y_0=\sum_{i=1}^m\int\limits_{\Omega^t}(u_i-k_*)_+^\delta\,dx\,dt\le\sum_{i=1}^m\int\limits_{\Omega^t}|u_i|^\delta\,dx\,dt,\qquad t\le T,$$

and the r.h.s. of the above inequality is bounded by c_0 in view of either (2.1) or (2.25). Therefore $k=k_0$, where k_0 is determined by (3.26). Similar considerations can be applied to the function $(u_i-k)_-$ also, $i=1,\dots,m$. In this way the lemma has been proved.

Remark 3.4. We find restrictions on μ_1, μ_2 and ν which satisfy relations (1.10) and (1.11):

$$\mu_1<p^*-1,\qquad\frac{\mu_2+1}{p^*}+\frac{\nu}{p}<1,\tag{3.27}$$

where $p^*=\max\{p,p_0\}$, and (3.24) gives

$$\mu_1<\frac{d-1}{d}\frac{p_*p}{n+p},\qquad\frac{\mu_2}{p_*}+\frac{\nu}{p}<\frac{d-1}{d}\frac{p}{n+p},\tag{3.28}$$

where $p_*=\max\{q,p_0\}$, $d<q=p\frac{n+2}{n}$.

Let $p_0>q>p$. Then

$$\mu_1<\frac{d-1}{d}\frac{pp_0}{n+p},\qquad\frac{\mu_2}{p_0}+\frac{\nu}{p}<\frac{d-1}{d}\frac{p}{n+p}.\tag{3.29}$$

Let $q\ge p_0>p$. Then

$$\mu_1<\min\left\{p_0-1,\frac{d-1}{d}\frac{pq}{n+p}\right\},\qquad\frac{\mu_2}{p_0}+\frac{\nu}{p}<1-\frac{1}{p_0},\qquad\frac{\mu_2}{q}+\frac{\nu}{p}<\frac{d-1}{d}\frac{p}{n+p}.\tag{3.30}$$

Finally for $p_0\le p$ we have

$$\mu_1<\min\left\{p-1,\frac{d-1}{d}\frac{pq}{n+p}\right\},\qquad\frac{\mu_2}{p}+\frac{\nu}{p}<1-\frac{1}{p},\qquad\frac{\mu_2}{q}+\frac{\nu}{p}<\frac{d-1}{d}\frac{p}{n+p}.\tag{3.31}$$

4. Existence of weak solutions to problem (P.3)**.** First we obtain an estimate for solutions of problem (P.3).

LEMMA 4.1. *Assume the growth conditions* (1.2)–(1.6), (1.15), (1.16). *Assume that* $u_{bt}\in L_2(\Omega^T)$, $u_b\in L_p(0,T;W_p^1(\Omega))\cap L_{p_0}(\Omega^T)\cap L_{\frac{2p}{p-q_0}}(\Omega^T)\cap L_2(\Omega^T)$, $u_0\in L_2(\Omega)$, $u_b(0)\in L_2(\Omega)$. *Then for solutions of problem* (P.3) *the following estimate holds*

$$\begin{aligned}\int\limits_\Omega|u|^2\,dx+\int\limits_{\Omega^t}(|\nabla u|^p+|u|^{p_0}+|\nabla u|^{q_0}|u|^2)\,dx\,dt\\ \le c_1\Big[1+\int\limits_{\Omega^t}(|u_{bt}|^2+|\nabla u_b|^p+|u_b|^{p_0}+|u_b|^{2p/(p-q_0)}+|u_b|^2)\,dx\,dt\\ +\int\limits_\Omega(|u_b(0)|^2+|u_0|^2)\,dx\Big],\qquad t\le T,\end{aligned}\tag{4.1}$$

where $c_1=c_1(l_1,l_2,d_1,d_2,d_3,t)$ *is an increasing function of its arguments.*

P r o o f. To obtain the estimate, the Steklov averages should be used so instead of (1.19) we examine the following integral identity:

$$\sum_{i=1}^{m}\int_{h}^{t}\int_{\Omega}\Big[u_{iht}\varphi_i+\sum_{j=1}^{m}(a_{ij}^{(l_1,l_2)}\cdot\nabla u_j)_h\cdot\nabla\varphi_i+(R_i^{(l_1,l_2)}u_i)_h\varphi_i-f_{ih}\varphi_i\Big]\,dx\,dt=0. \tag{4.2}$$

Putting $\varphi_i=u_{hi}-u_{bi}$ in (4.2), integrating with respect to time in the first term, letting $h\to 0$ and using the conditions (1.2)–(1.6), (1.15), (1.16) yields

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{m}\int_{\Omega}(u_i-u_{bi})^2\,dx+\sum_{i=1}^{m}\int_{\Omega^t}[\alpha_1|\nabla u_i|^p+\beta_1|u_i|^{p_0}+\gamma_1|\nabla u_i|^{q_0}|u_i|^2]\,dx\,dt\\
&\le\frac{1}{2}\sum_{i=1}^{m}\int_{\Omega}(u_{0i}-u_{bi})^2\,dx\\
&\quad+\sum_{i=1}^{m}\int_{\Omega^t}[\alpha_2|\nabla u|^{p-2}|\nabla u_i|\,|\nabla u_{bi}|+\beta_2|u|^{p_0-2}|u_i|\,|u_{bi}|+\gamma_2|\nabla u|^{q_0}|u_i|\,|u_{bi}|]\,dx\,dt\\
&\quad+\sum_{i=1}^{m}\int_{\Omega^t}\Big[\frac{1}{2}|u_{bit}|^2+\frac{1}{2}(u_i-u_{bi})^2+(d_1|\nabla u_i|^{\frac{p}{2}}+d_2|u_i|^{\sigma}+d_3)|u_i-u_{bi}|\Big]\,dx\,dt.
\end{aligned} \tag{4.3}$$

Using the Hölder and Young inequalities in (4.3) implies (4.1). This concludes the proof.

Now we formulate the result on existence.

THEOREM 4.2. *Let the assumptions of Lemma* 4.1 *be satisfied. Let*

$$p>\frac{n}{n+2}+q_0. \tag{4.4}$$

Then there exists a solution of problem (P.3) *such that* $u\in L_\infty(0,T;L_2(\Omega))\cap L_p(0,T;W_p^1(\Omega))\cap L_{p_0}(\Omega^T)$ *and the estimate* (4.1) *holds.*

P r o o f. The proof is similar to the proof of Theorem 2.2. The difference is only in passing to the limit in the third term on the l.h.s. of (2.18). We first consider the expression

$$\begin{aligned}
&\Big|\sum_{i=1}^{m}\int_{\Omega^t}R_{1i}^{(l_1,l_2)}(u_{h\lambda})u_{h\lambda_i}\omega_i\,dx\,dt\Big|\\
&\qquad\le c(l_1,l_2)\Big(\int_{\Omega^t}|u_{h\lambda}|^2\,dx\,dt\Big)^{1/2}\Big(\int_{\Omega^t}|\omega|^2\,dx\,dt\Big)^{1/2}\equiv J_1,
\end{aligned}$$

where J_1 converges to zero since ω converges strongly to zero in $L_{q'}(\Omega^T)$, $q'<q=p\frac{n+2}{n}$, $p\ge 2$.

Next we examine

$$\Big|\sum_{i=1}^{m}\int_{\Omega^t}R_{2i}^{(l_1,l_2)}(u_{h\lambda},\nabla u_{h\lambda})u_{h\lambda_i}\omega_i\,dx\,dt\Big|$$

$$
\begin{aligned}
&\le c(l_1,l_2)\int\limits_{\Omega^t}|\nabla u_{h\lambda}|^{q_0}|\omega|\,dx\,dt\\
&\le c(l_1,l_2)\Big(\int\limits_{\Omega^t}|\nabla u_{h\lambda}|^p\,dx\,dt\Big)^{q_0/p}\Big(\int\limits_{\Omega^t}|\omega|^\sigma\,dx\,dt\Big)^{1/\sigma}\equiv J_2,
\end{aligned}
$$

where $\sigma=1/(1-q_0/p)$. The assumption (4.4) implies that $\sigma<q$ so J_2 converges to zero.

In view of the growth condition (1.15) we can easily pass to the limit on the r.h.s. of (2.18). This concludes the proof.

5. Existence of solutions to diagonal problem (P.2)**.** First we consider the following diagonal and truncated system:

$$
\begin{aligned}
&u_{it}-\nabla\cdot(a_i(x,t,u^{(l_1,l_2)},\nabla u)\cdot\nabla u_i)\\
&\qquad+R_i(x,t,u^{(l_1,l_2)},\nabla u)\nabla u_i=f_i(x,t,u,\nabla u) && \text{in } \Omega^T,\\
&u_i|_{t=0}=u_{0i} && \text{in } \Omega,\\
&u_i=u_{bi} && \text{on } S^T,
\end{aligned}
\tag{5.1}
$$

where $i=1,\dots,m$, which is the truncated version of problem (3.2) and where the growth condition (1.16) holds.

To show an L_∞-estimate for solutions to problem (5.1) we use the following weak formulation of (5.1) with the Steklov averages

$$
\begin{aligned}
\sum_{i=1}^m\int\limits_h^t\int\limits_\Omega&[\partial_t u_{hi}\varphi_i+(a_i(x,t,u^{(l_1,l_2)},\nabla u)\cdot\nabla u_i)_h\cdot\nabla\varphi_i\\
&+((R_i(x,t,u^{(l_1,l_2)},\nabla u)u_i)_h\varphi_i-(f_i(x,t,u,\nabla u))_h\varphi_i]\,dx\,dt=0,
\end{aligned}
\tag{5.2}
$$

which holds for all $\varphi\in L_2(0,T;\overset{\circ}{W}{}^1_2(\Omega))$.

First we show

Lemma 5.1. *Let* $k_*=\max\{\|u_0\|_{L_\infty(\Omega)},\|u_b\|_{L_\infty(\Omega^T)}\}$, *let* $\overline{k}>0$ *be such that*

$$
\|u_b\|_{L_\infty(\Omega^T)}<\overline{k},\qquad \|u_0\|_{L_\infty(\Omega)}<\overline{k}.
\tag{5.3}
$$

Let assumptions (1.2)–(1.6), (1.15), (1.16) *hold. Then for weak solutions of problem* (5.1) *the following inequality holds*:

$$
\begin{aligned}
\sum_{i=1}^m\int\limits_\Omega(u_i-\overline{k})_+^2\,dx&+\sum_{i=1}^m\int\limits_{\Omega^t}\Big[\frac{\alpha_1}{2}|\nabla(u_i-\overline{k})_+|^p+\beta_1(u_i-\overline{k})_+^{p_0}\\
&+\gamma_1|\nabla u_i|^{q_0}(u_i-\overline{k})_+^2\Big]\,dx\,dt\\
&\le\sum_{i=1}^m\int\limits_{\Omega^t}\Big[\frac{d_1^2}{2\alpha_1}(u_i-\overline{k})_+^2+(d_2|u_i-k_*|^\sigma+d_2k_*^\sigma+d_{3i})(u_i-\overline{k})_+\Big]\,dx\,dt.
\end{aligned}
\tag{5.4}
$$

Proof. Putting $\varphi_i = (u_{ih} - \overline{k})_+$ into (5.2), integrating with respect to time in the first term, letting $h \to 0$ and using conditions (1.2)–(1.6), (1.15) yields

$$(5.5)\quad \sum_{i=1}^m \int_\Omega (u_i - \overline{k})_+^2\,dx + \sum_{i=1}^m \int_{\Omega^t} [\alpha_1|\nabla(u_i - \overline{k})_+|^p + \beta_1(u_i - \overline{k})_+^{p_0} + \gamma_1|\nabla u_i|^{q_0}(u_i - \overline{k})_+^2]\,dx\,dt \le \sum_{i=1}^m \int_{\Omega^t} (d_1|\nabla u_i|^{\frac{p}{2}} + d_2|u_i|^\sigma + d_{3i})(u_i - \overline{k})_+\,dx\,dt.$$

In view of the Hölder and Young inequalities in (5.5) we obtain (5.4). This concludes the proof.

We need a bound for weak solutions of problem (5.1) which does not depend on l_1 and l_2. Hence we have

Lemma 5.2. *Let k_* be defined in Lemma 5.1. Let assumptions (1.2)–(1.5), (1.15), (1.16) hold. Then for weak solutions of problem (5.1) the following estimate is valid*:

$$(5.6)\quad \sum_{i=1}^m \int_\Omega (u_i - k_*)_+^2\,dx + \sum_{i=1}^m \int_{\Omega^t} (\alpha_1|\nabla(u_i - k_*)_+|^p + \beta_1(u_i - k_*)_+^{p_0} + \gamma_1|\nabla u_i|^{q_0}(u_i - k_*)_+^2)\,dx\,dt \le tc_2(e^{c_3 t} + 1) \equiv c_4,$$

where $c_2 = |\Omega|(d_2 k_*^\sigma + d_3)^2$, $c_3 = c_1(d_1^2 + d_2^2 + 1)$.

Proof. Putting $\overline{k} = k_*$ into (5.4) and using the Hölder and Young inequalities yields

$$(5.7)\quad \sum_{i=1}^m \int_\Omega (u_i - k_*)_+^2\,dx + \sum_{i=1}^m \int_{\Omega^t} [\alpha_1|\nabla(u_i - k_*)_+|^p + \beta_1(u_i - k_*)_+^{p_0} + \gamma_1|\nabla u_i|^{q_0}(u_i - k_*)_+^2]\,dx\,dt \le c_1(\alpha_1, \beta_1, c_*)(d_1^2 + d_2^2 + 1) \sum_{i=1}^m \int_{\Omega^t} (u_i - k_*)_+^2\,dx\,dt + t|\Omega|(d_2 k_*^\sigma + d_3)^2,$$

where c_* is the constant from imbedding (1.20) and $|\Omega|$ denotes the volume of Ω.

In view of the Gronwall lemma we get

$$\sum_{i=1}^m \int_\Omega (u_i - k_*)_+^2\,dx \le e^{c_1(d_1^2 + d_2^2 + 1)t} t|\Omega|(d_2 k_*^\sigma + d_3)^2.$$

Using this inequality in (5.7) implies (5.6). This concludes the proof.

Next we prove a result analogous to Lemma 3.2.

Lemma 5.3. *Let the assumptions of Lemma 5.2 hold. Let*

$$Y_s = \sum_{i=1}^m \int_{\Omega^t} (u_i - k_s)_+^\delta\,dx\,dt,$$

where $k_s = k_* + k - \frac{k}{2^s}$, $s = 0, 1, \ldots,$ $\delta < q = p\frac{n+2}{n}$, k_* *is defined in Lemma 5.1. Let*

$$(5.8)\quad \frac{\sigma}{p_*} < 1 - \frac{1}{\delta}, \quad \frac{\sigma}{p_*} < \frac{p}{n+p}, \quad p_* = \max\{p_0, q\}.$$

Then we have the recursive inequalities

$$Y_{s+1} \le c_6 \frac{2^{a^* s}}{k^{a_*}} Y_s^{1+\sigma_0 \frac{\delta}{q}}, \tag{5.9}$$

where $a_* = \min\{a_1, a_2, a_3\}$, $a^* = \max\{a_1, a_2, a_3\}$, *and* a_i, $i = 1, 2, 3$, σ_0, *are defined by* (5.14), $k > 1$ *and* c_6 *depends on* c_5, c_4.

Proof. Putting $\overline{k} = k_{s+1}$ into (5.4) and using the Hölder inequality yields

$$\begin{aligned} \sum_{i=1}^m \int_\Omega (u_i - k_{s+1})_+^2 \, dx + \sum_{i=1}^m \int_{\Omega^t} \Big[\frac{\alpha_1}{2} |\nabla (u_i - k_{s+1})_+|^p + \beta_1 (u_i - k_{s+1})_+^{p_0}\Big] \, dx \, dt \\ \le \sum_{i=1}^m \Big[\frac{d_1^2}{2\alpha_1} \int_{\Omega^t} (u_i - k_{s+1})_+^2 \, dx \, dt \\ + d_2 \Big(\int_{\Omega^t} (u_i - k_*)^{p_*} \, dx \, dt \Big)^{\sigma/p_*} \Big(\int_{\Omega^t} (u_i - k_{s+1})_+^{\frac{p_*}{p_* - \sigma}} \, dx \, dt \Big)^{\frac{p_* - \sigma}{p_*}} \\ + (k_*^\sigma + d_3) \int_{\Omega^t} (u_i - k_{s+1})_+ \, dx \, dt \Big]. \end{aligned} \tag{5.10}$$

Using (5.6) and (3.20) in (5.10) implies

$$\begin{aligned} \sum_{i=1}^m \Big[\int_\Omega (u_i - k_{s+1})_+^2 \, dx + \alpha_1 \int_{\Omega^t} |\nabla (u_i - k_{s+1})_+|^p \, dx \, dt \\ + \beta_1 \int_{\Omega^t} (u_i - k_{s+1})_+^{p_0} \, dx \, dt \Big] \\ \le c \sum_{i=1}^m \Big[\frac{d_1^2}{2\alpha_1} \frac{2^{(\delta - 2)s}}{k^{\delta - 2}} \int_{\Omega^t} (u_i - k_s)_+^\delta \, dx \, dt \\ + d_2 c_4^{\frac{\sigma}{p_*}} \Big(\frac{2^{(\delta - \frac{p_*}{p_* - \sigma})s}}{k^{\delta - \frac{p_*}{p_* - \sigma}}} \int_{\Omega^t} (u_i - k_s)_+^\delta \, dx \, dt \Big)^{\frac{p_* - \sigma}{p_*}} \\ + (k_*^\sigma + d_3) \frac{2^{(\delta - 1)s}}{k^{\delta - 1}} \int_{\Omega^t} (u_i - k_s)_+^\delta \, dx \, dt \Big], \end{aligned} \tag{5.11}$$

where in view of $(5.8)_1$, $\frac{p_*}{p_* - \sigma} < \delta < q$ so for $p_* = q$ we get the relation $p > (\sigma + 1)\frac{n}{n+2}$ and for $p_* = p_0$ the relation $\sigma < \big(p - \frac{n}{n+2}\big)\frac{p_0}{p}$.

From (3.22), (3.19), (1.20) and (5.11) we obtain

$$Y_{s+1} \le c_5 \Big[\frac{2^{(\delta - 2)s}}{k^{\delta - 2}} Y_s + \Big(\frac{2^{(\delta - \frac{p_*}{p_* - \sigma})}}{k^{\delta - \frac{p_*}{p_* - \sigma}}} Y_s \Big)^{\frac{p_* - \sigma}{p_*}} + \frac{2^{(\delta - 1)s}}{k^{\delta - 1}} Y_s \Big]^{(1 + \frac{p}{n})\frac{\delta}{q}} \cdot \Big(\frac{2^{\delta s}}{k^\delta} Y_s \Big)^{1 - \frac{\delta}{q}}. \tag{5.12}$$

Continuing calculations (5.12) implies

$$Y_{s+1} \le c_5 \Big[\Big(\frac{2^{a_1 s}}{k^{a_1}} + \frac{2^{a_2 s}}{k^{a_2}} \Big) Y_s^{1 + \frac{p}{n}\frac{\delta}{q}} + \frac{2^{a_3 s}}{k^{a_3}} Y_s^{1 + \sigma_0 \frac{\delta}{q}} \Big], \tag{5.13}$$

where

$$
\begin{aligned}
&a_1 = (\delta-1)\frac{p}{n}\frac{\delta}{q} + \delta\left(1-\frac{1}{q}\right), \qquad a_2 = (\delta-2)\frac{p}{n}\frac{\delta}{q} + \delta\left(1-\frac{2}{q}\right),\\
&a_3 = a_1 - \frac{\delta\sigma}{p_*}\left(1+\frac{p}{n}\right)\frac{\delta}{q} = \left(\delta-1-\frac{\delta\sigma}{p_*}\right)\frac{p}{n}\frac{\delta}{q} + \left(q-1-\frac{\delta\sigma}{p_*}\right)\frac{\delta}{q},\\
&\sigma_0 = \frac{p}{n}\left(1-\frac{\sigma}{p_*}\right) - \frac{\sigma}{p_*}
\end{aligned}
\tag{5.14}
$$

To obtain the iterative of inequalities of type (3.16) we have to assume that a_i, $i=1,2,3$, σ_0 are positive, which follows from the assumption (5.8). Using $Y_s \le Y_0 \le c_4$, where the last inequality follows from Lemma 5.2 and in view of the definitions of a^*, a_* and the assumption that $k>1$, instead of (5.13) we obtain (5.9). This concludes the proof.

Finally we show boundedness of weak solutions.

LEMMA 5.4. *Let the assumptions of Lemmas* 5.2 *and* 5.3 *be satisfied. Then*

$$
\sup_i |u_i|_{L_\infty(\Omega^T)} \le k_* + k_0, \tag{5.15}
$$

where

$$
k_0 = \left[c_4 c_6^{\frac{q}{\sigma_0\delta}} 2^{a^*(\frac{q}{\sigma_0\delta})^2}\right]^{\frac{\sigma_0\delta}{a_* q}}. \tag{5.16}
$$

Proof. The proof is the same as the proof of Lemma 3.3.

Summarizing the above considerations we obtain the main result of this section.

THEOREM 5.5. *Let the assumptions of Theorem* 4.2, *Lemmas* 5.2 *and* 5.3 *be satisfied. Put in place of* c_4 *in* (5.16) *a constant* $c_7 \ge c_4$ *such that* $|l_1|$, $|l_2|$ *are less than* $k_* + k_0$. *Then there exists a bounded solution of problem* (P.2) *such that* $u \in L_\infty(0,T;L_2(\Omega)) \cap L_{p_0}(\Omega^T) \cap L_p(0,T;W_p^1(\Omega))$.

6. L_∞-estimate for weakly nondiagonal problem (P.1)**.** In this section we prove an L_∞-estimate for weak solutions to problem (1.1) in the case when

$$
a_{ij}(x,t,u,\nabla u) = a_i(x,t,u,\nabla u)\delta_{ij} + A_{ij}(x,t,u,\nabla u), \qquad i,j=1,\dots,m, \tag{6.1}
$$

where A_{ij} is a matrix with vanishing diagonal elements.

To obtain the sup-estimate we have to repeat the proof of Lemma 3.1, i.e., to prove inequality (3.10).

LEMMA 6.1. *Assume* (1.2)–(1.11). *Assume that*

$$
|A_{ij}| \le c_1(|u|^{d_1}|\nabla u|^b + |u|^{d_2}), \qquad i,j=1,\dots,m, \tag{6.2}
$$

where c_1, b, d_1, d_2 *are nonnegative constants. Assume that* $\overline{k}>0$ *satisfies*

$$
\|u_0\|_{L_\infty(\Omega)} < \overline{k}, \qquad \|u_b\|_{L_\infty(\Omega^T)} < \overline{k}. \tag{6.3}
$$

Assume also that

$$
p > b+2, \qquad \frac{d_1 p}{p_*(p-1)} + \frac{b+1}{p-1} < 1, \qquad \frac{d_2 p}{p_*(p-1)} + \frac{1}{p-1} < 1. \tag{6.4}
$$

Then

$$\begin{aligned}(6.5)\quad & \sum_{i=1}^m \Big[\int\limits_\Omega (u_i-\overline{k})_+^2\,dx + \alpha_1 \int\limits_{\Omega^t} |\nabla(u_i-\overline{k})_+|^p\,dx\,dt + \beta_1 \int\limits_{\Omega^t} (u_i-\overline{k})_+^{p_0}\,dx\,dt \\ &\quad + \gamma_1 \int\limits_{\Omega^t} |\nabla u_i|^{q_0}(u_i-\overline{k})_+^2\,dx\,dt\Big] \\ &\le c_2 \sum_{i=1}^m \Big[\int\limits_{\Omega^t} (u_i-\overline{k})_+\,dx\,dt + \int\limits_{\Omega^t} (u_i-\overline{k})_+^d\,dx\,dt + \sum_{l=1}^4 \Big(\int\limits_0^t |A^\pm_{\overline{k},i}(t)|\,dt\Big)^{1-\gamma_l}\Big],\end{aligned}$$

where $d<q$, $q=p\frac{n+2}{n}$, $\gamma_1=\frac{\mu_1 d}{p_*(d-1)}$, $\gamma_2=\big(\frac{\mu_2}{p_*}+\frac{\nu}{p}\big)\frac{d}{d-1}$, $\gamma_3=\frac{d_1 p}{p_*(p-1)}+\frac{b+1}{p-1}$, $\gamma_4=\frac{d_2 p}{p_*(p-1)}+\frac{1}{p-1}$.

Proof. The proof is very close to the proof of Lemma 3.1, where in the integral identity (3.4) the diagonal matrix $a_i\delta_{ij}$ is replaced by the matrix defined by (6.1). Another difference is that we have to add on the r.h.s. of (3.10) the term

$$(6.6)\qquad \Big|\sum_{i,j=1}^m \int\limits_{\Omega^t} A_{ij}\nabla u_i \nabla(u_j-\overline{k})_+\,dx\,dt\Big|.$$

We shall treat the term in the similar way to the expression on the r.h.s. of (3.11).

In view of (6.2) to estimate (6.6) we have to examine the integrals

$$(6.7)\quad \sum_{i=1}^m \Big(\int\limits_0^t dt \int\limits_{A^\pm_{\overline{k},i}(t)} |u|^{d_1p'}|\nabla u|^{(b+1)p'}\,dx + \int\limits_0^t dt \int\limits_{A^\pm_{\overline{k},i}(t)} |u|^{d_2p'}|\nabla u|^{p'}\,dx\Big) \equiv K_1+K_2,$$

where $1/p+1/p'=1$.

We shall restrict our considerations to K_1. By the Hölder inequality we have

$$\begin{aligned}(6.8)\qquad K_1 \le & \sum_{i=1}^m \Big(\int\limits_0^t dt \int\limits_{A^\pm_{\overline{k},i}(t)} |u|^{p_*}\,dx\Big)^{\frac{d_1p'}{p_*}} \Big(\int\limits_0^t dt \int\limits_{A^\pm_{\overline{k},i}(t)} |\nabla u|^p\,dx\Big)^{\frac{(b+1)p'}{p}} \\ & \times \Big(\int\limits_0^t |A^\pm_{\overline{k},i}(t)|\,dt\Big)^{1-\gamma_3},\end{aligned}$$

where $(6.4)_1$ has to be used.

Similarly, we have

$$(6.9)\qquad K_2 \le c\sum_{i=1}^m \Big(\int\limits_0^t |A^\pm_{\overline{k},i}(t)|\,dt\Big)^{1-\gamma_4},$$

where $(6.4)_2$ was used.

Therefore (6.5) has been proved. This concludes the proof.

Repeating the proof of Lemma 3.2 yields

LEMMA 6.2. *Let the assumptions of Lemma* 3.2 *and Lemma* 6.1 *be satisfied. Then there exist positive constants* $\overline{c}$, a_0, a^0, $\overline{\sigma}$ *such that*

$$(6.10)\qquad Y_{s+1} \le \overline{c}\frac{2^{a^0}}{k^{a_0}} Y_s^{1+\overline{\sigma}\frac{\delta}{q}},$$

where $\overline{c} = \overline{c}(c_0)$, c_0 *is defined either in* (2.1) *or in* (2.25), $d < \delta < q$, $a^0 = \max\{a_1, a_2, \delta\alpha_1, \delta\alpha_2, \delta\alpha_3, \delta\alpha_4\}$, $a_0 = \min\{a_1, a_2, \delta\alpha_1, \delta\alpha_2, \delta\alpha_3, \delta\alpha_4\}$, $\alpha_i = 1 + \sigma_i \frac{\delta}{q}$, $\sigma_i = \frac{p}{n}(1-\gamma_i) - \gamma_i$, $i = 1, \dots, 4$, $\overline{\sigma} = \min\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, $k > 1$ *and* γ_3, γ_4 *are defined in* (6.5).

Similarly to the case of Lemma 3.3 we have

LEMMA 6.3. *Let the assumptions of either Lemma* 2.1 *or Lemma* 2.3 *be satisfied. Let the assumptions of Lemma* 6.1 *hold. Let*

$$\frac{p}{n} > \frac{\gamma_i}{1-\gamma_i}, \quad \gamma_i < 1, \quad i = 1, \dots, 4. \tag{6.11}$$

Then

$$\sup_i |u_i|_{L_\infty(\Omega^T)} \le k_* + \overline{k}_0, \tag{6.12}$$

where

$$\overline{k}_0 = [c_0 \overline{c}^{\frac{q}{\overline{\sigma}\delta}} 2^{a^0 (\frac{q}{\overline{\sigma}\delta})^2}]^{\frac{\overline{\sigma}\delta}{a_0 q}}. \tag{6.13}$$

Remark 6.4. To prove Lemma 6.3 the following restrictions must be imposed:

$$\mu_1 + 1 < p^*, \quad \frac{\mu_2 + 1}{p^*} + \frac{\nu}{p} < 1, \quad \gamma_i < \frac{p}{n+p}, \quad i = 1, \dots, 4,$$

where $\gamma_1 = \frac{\mu_1 d}{p_*(d-1)}$, $\gamma_2 = \left(\frac{\mu_2}{p_*} + \frac{\nu}{p}\right)\frac{d}{d-1}$, $\gamma_3 = \frac{d_1 p}{p_*(p-1)} + \frac{b+1}{p-1}$, $\gamma_4 = \frac{d_2 p}{p_*(p-1)} + \frac{1}{p-1}$, and $p^* = \max\{p_0, p\}$, $p_* = \max\{p_0, q\}$, $d < q$.

Remark. The method of getting an L_∞-estimate presented in this paper is much more restrictive that the one given in [3], Ch. 8, Sect. 2. However, it seems that our method can be applied more successfully to some anisotropic cases and for systems with different matrices a_i, $i = 1, \dots, m$.

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