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THE MILNOR NUMBER OF FUNCTIONS ON SINGULAR HYPERSURFACES

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Abstract. The behaviour of a holomorphic map germ at a critical point has always been an important part of the singularity theory. It is generally known (cf. [5]) that we can associate an integer invariant—called the *multiplicity*—to each isolated critical point of a holomorphic function of many variables. Several years later it was noticed that similar invariants exist for function germs defined on isolated hypersurface singularities (see [1]). The present paper aims to show a simple approach to critical points of maps defined on the A_k-type singular hypersurfaces. After some changes it can probably be adopted to other isolated hypersurface singularities.

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1. Let $\mathcal{O}_n$ denote the ring of germs of holomorphic functions $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$.

Definition. The *multiplicity* of the critical point of f at zero is $\mu(f) := \dim_{\mathbf{C}} \mathcal{O}_n / J_f$, where $J_f = \langle \partial f/\partial z_1, \ldots, \partial f/\partial z_n \rangle_{\mathcal{O}_n}$ is the *Jacobian ideal* of f.

The number $\mu(f)$ is also called the *Milnor number* of f, because it was first introduced by J. Milnor in 1968 [4].

Proposition. *Let $f \in \mathcal{O}_n$ have an isolated critical point at zero and let $f(0) = 0$. Then $\mu(f)$ is finite and the preimage of each sufficiently small non-zero complex number intersects a small open disk in a smooth manifold, which is homotopy equivalent to a bouquet of $\mu(f)$ $(n-1)$-dimensional spheres.*

$$(\exists \varepsilon, \delta \in \mathbf{R}_+)(\forall t \in \mathbf{C}) \quad 0 < |t| < \varepsilon \Rightarrow f^{-1}(t) \cap D_\delta \sim \bigvee_{\mu(f)} S^{n-1}.$$

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The paper is in final form and no version of it will be published elsewhere.

For a proof the reader is referred to Milnor's original article [4] as well as [3] and [5]—a survey article dealing with various definitions of multiplicity.

2. Let Ω^i $(1 \le i \le n)$ denote the $\mathcal{O}_n$-module of germs of holomorphic i-forms at 0. The explicit expressions for the most relevant cases: $i = n-1$ and $i = n$ are

$$\Omega^n = \{g \cdot dz_1 \wedge \ldots \wedge dz_n \mid g \in \mathcal{O}_n\}$$

and

$$\Omega^{n-1} = \Big\{ \sum_{k=1}^{n} g_k \cdot dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_n \,\Big|\, g_k \in \mathcal{O}_n \Big\},$$

where the hat over dz_k means "skip". Moreover, for all $f \in \mathcal{O}_n$ we have

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} \cdot dz_i.$$

Hence

$$df \wedge \Big(\sum_{k=1}^{n} g_k \cdot dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_n \Big) = \Big(\sum_{k=1}^{n} (-1)^{k-1} g_k \cdot \frac{\partial f}{\partial z_k} \Big) \cdot dz_1 \wedge \ldots \wedge dz_n$$

and denoting $df \wedge \Omega^{n-1} := \{df \wedge \omega \mid \omega \in \Omega^{n-1}\}$, we obtain immediately

$$\Omega^n/(df \wedge \Omega^{n-1}) \cong \mathcal{O}_n/J_f \quad \text{and} \quad \dim_{\mathbf{C}} \Omega^n/(df \wedge \Omega^{n-1}) = \mu(f).$$

Example 1. We shall find the Milnor number of the function $f : \mathbf{C}^2 \to \mathbf{C}$ given by

$$f(z) = z_1^{e_1} + z_2^{e_2},$$

where $e_1, e_2 \ge 2$. Computing the Jacobian ideal gives us at once

$$\mu(f) = e_1 e_2 - e_1 - e_2 + 1 = (e_1 - 1)(e_2 - 1),$$

but the result could also be obtained by looking at fibres. Our function is quasi-homogeneous, hence $f^{-1}(t_1)$ is homeomorphic to $f^{-1}(t_2)$, for all $t_1 \ne 0 \ne t_2$. We can therefore consider the fibre $F := f^{-1}(1)$. Let $\xi_1 = \exp(2\pi i/e_1)$, $\xi_2 = \exp(2\pi i/e_2)$ and put

$$P_r = (\xi_1^r, 0) \qquad \text{for } r = 0, \ldots, e_1 - 1,$$
$$Q_s = (0, \xi_2^s) \qquad \text{for } s = 0, \ldots, e_2 - 1.$$

Given r and s, there is an arc $\gamma_{r,s} \subset F$ between P_r and Q_s described by

$$\gamma_{r,s} : [0,1] \to F, \qquad t \mapsto ((1 - t^{e_2})^{1/e_1} \cdot \xi_1^r, t \cdot \xi_2^s).$$

It can be shown that the graph $G := \bigcup_{r,s} \gamma_{r,s}$ is a retract of F (see [5], p. 434). Moreover G is homotopy equivalent to a bouquet of $(e_1 - 1)(e_2 - 1)$ circles.

3. For a fixed integer $k > 1$ we shall consider the following singular hypersurface

$$X := \{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_1^k = z_2 z_3\}.$$

Denote also $X' := X \backslash \{(0,0,0)\}$.

Proposition. *X' is a two-dimensional complex manifold.*

Proof. Observe that $X' = U_1 \cup U_2$, where

$$U_1 = \{(z_1, z_2, z_3) \in X \mid z_2 \ne 0\}, \qquad U_2 = \{(z_1, z_2, z_3) \in X \mid z_3 \ne 0\}.$$

Both sets U_i are homeomorphic to $\mathbf{C} \times (\mathbf{C}\backslash\{0\})$; the homeomorphisms can be

$$h_1 : U_1 \to \mathbf{C} \times (\mathbf{C}\backslash\{0\}), \qquad (z_1, z_2, z_3) \mapsto (z_1, z_2),$$
$$h_2 : U_2 \to \mathbf{C} \times (\mathbf{C}\backslash\{0\}), \qquad (z_1, z_2, z_3) \mapsto (z_1, z_3).$$

Moreover, the transition function

$$h_2 \circ h_1^{-1} : (\mathbf{C}\backslash\{0\})^2 \to (\mathbf{C}\backslash\{0\})^2, \qquad (s,t) \mapsto (s, s^k/t),$$

is holomorphic.

PROPOSITION. *The mapping $m : \mathbf{C}^2 \to X$ defined by $m(s,t) := (st, s^k, t^k)$ induces a holomorphic covering $m' : \mathbf{C}^2\backslash\{(0,0)\} \to X'$. The preimage of any $x' \in X'$ consists of k points of the form $(\xi^i s, \xi^{-i} t)$, where $0 \le j \le k-1$ and $\xi = \exp(2\pi i/k)$.*

COROLLARY. *Every holomorphic function $f' : X' \to \mathbf{C}$ comes from a holomorphic function $\tilde{f} : \mathbf{C}^2\backslash\{(0,0)\} \to \mathbf{C}$ satisfying $\tilde{f}(\xi s, \xi^{-1} t) = \tilde{f}(s,t)$.*

The following diagram is then commutative

$$\begin{array}{ccc} \mathbf{C}^2\backslash\{(0,0)\} & & \\ {\scriptstyle m}\downarrow & \searrow {\scriptstyle \tilde{f}} & \\ X' & \xrightarrow{f'} & \mathbf{C} \end{array}$$

A well-known theorem of Hartogs implies that every holomorphic function defined on $\mathbf{C}^2\backslash\{(0,0)\}$ extends to $\mathbf{C}^2$. Hence the preceding considerations result in the following

PROPOSITION. *If $f : X \to \mathbf{C}$ is a continuous function such that $f|_{X'}$ is holomorphic then there exists a holomorphic function $\tilde{f} : \mathbf{C}^2 \to \mathbf{C}$ making the following diagram commutative*:

$$\begin{array}{ccc} \mathbf{C}^2 & & \\ {\scriptstyle m}\downarrow & \searrow {\scriptstyle \tilde{f}} & \\ X & \xrightarrow{f} & \mathbf{C} \end{array}$$

$\tilde{f}$ is invariant under the action of the covering group, i.e.

$$\tilde{f}(\xi s, \xi^{-1} t) = \tilde{f}(s,t). \tag{1}$$

On the other hand, if $\tilde{f} = \sum a_{pq} s^p t^q$ (1) is equivalent to

$$a_{pq} = 0 \quad \text{if } p \not\equiv q \pmod{k}.$$

Therefore if $\tilde{f} : \mathbf{C}^2 \to \mathbf{C}$ satisfies (1) then all non-zero components $a_{pq} s^p t^q$ can be written as $a \cdot (s^k)^{m_1} \cdot (st)^{m_2}$ or $a \cdot (t^k)^{m_1} \cdot (st)^{m_2}$. In this manner $\tilde{f}$ induces a function $f : X \to \mathbf{C}$. If $\tilde{f}$ is holomorphic then f is continuous on X and holomorphic on X'. In the sequel we shall sometimes identify corresponding functions $f : X \to \mathbf{C}$ and $\tilde{f} : \mathbf{C}^2 \to \mathbf{C}$.

4. Let $f : X \to \mathbf{C}$ and $\tilde{f} : \mathbf{C}^2 \to \mathbf{C}$ be corresponding functions in the sense of the above diagram. If $f((0,0,0)) = 0$ then for $t \neq 0$

$$m|_{\tilde{f}^{-1}(t)} : \tilde{f}^{-1}(t) \to f^{-1}(t)$$

is an unramified covering of degree k. If we define F_t as we have above for functions $\mathbf{C}^2 \to \mathbf{C}$ then we can easily see that

$$\chi(\tilde{F}_t) = k \cdot \chi(F_t), \tag{2}$$

where χ denotes the Euler characteristic.

THEOREM. *If* 0 *is an isolated critical point of* $\tilde{f}$ *then for all sufficiently small* $t \neq 0$

$$F_t \sim \bigvee_{\mu_X(f)} S^1,$$

where

$$\mu_X(f) = \frac{\mu(\tilde{f}) - 1}{k} + 1.$$

Proof. The fibre $\tilde{F}_t$ is a Riemann surface. Viewed as a subset of $\mathbf{CP}^2$ it becomes a compact Riemann surface (a sphere with handles) with a finite number of disks removed. Using properties of coverings we see that F_t is also a surface of this type, hence it is homotopy equivalent to a bouquet of a certain number of circles. We know that

$$\tilde{F}_t \sim \bigvee_{\mu(\tilde{f})} S^1 \quad \text{and} \quad \chi(\tilde{F}_t) = \mu(\tilde{f}) - 1.$$

Equation (2) gives now the required result.

EXAMPLE 2. Consider $f : X \to \mathbf{C}$ induced by

$$\tilde{f}(s,t) = s^{k\cdot e_1} + t^{k\cdot e_2},$$

where $e_1, e_2 \geq 2$. The above theorem yields together with Example 1

$$\mu_X(f) = \frac{(ke_1 - 1)(ke_2 - 1) - 1}{k} + 1 = k \cdot e_1 e_2 - e_1 - e_2 + 1.$$

On the other hand, the reader can check that the graph $m(\Gamma)$ is homotopy equivalent to a bouquet of

$$\mu = (e_1 - 1)(e_2 - 1) + (k - 1) \cdot e_1 e_2$$

circles. Obviously $\mu_X(f) = \mu$.

5. Although the Jacobian ideal of a function $f : X \to \mathbf{C}$ cannot be defined in the previous way, we can proceed using the language of differential forms. Let $\mathcal{F}, \mathcal{G}^1, \mathcal{G}^2$ be the sheaves of holomorphic functions, 1-forms and 2-forms on X', respectively. If $j : X' \hookrightarrow X$ is the natural inclusion, then the stalks

$$\mathcal{O}_X := (j_*\mathcal{F})_{(0,0,0)}, \quad \Omega^1_X := (j_*\mathcal{G}^1)_{(0,0,0)} \quad \text{and} \quad \Omega^2_X := (j_*\mathcal{G}^2)_{(0,0,0)}$$

are analogues of the ring of germs of holomorphic functions and the modules of germs of holomorphic 1-forms and 2-forms. This enables us to define the *algebraic multiplicity* in the current context by the formula

$$\mu_a(f) := \dim_{\mathbf{C}} \Omega^2_X / (df \wedge \Omega^1_X).$$

Also differential forms on X are induced by those forms on $\mathbf{C}^2$ which satisfy certain invariance conditions. In fact Ω^2_X is generated as an $\mathcal{O}_X$-module by the form induced by $ds \wedge dt$, while Ω^1_X needs four generators: $s^{k-1} \cdot ds$, $t \cdot ds$, $s \cdot dt$ and $t^{k-1} \cdot dt$. Therefore we have

$$\mu_a(f) = \dim_{\mathbf{C}} \mathcal{O}_X \Big/ \left\langle s^{k-1}\frac{\partial \tilde{f}}{\partial t}, t\frac{\partial \tilde{f}}{\partial t}, s\frac{\partial \tilde{f}}{\partial s}, t^{k-1}\frac{\partial \tilde{f}}{\partial s} \right\rangle_{\mathcal{O}_X}.$$

Example 3. Let us return to the function $f : X \to \mathbf{C}$ induced by

$$\tilde{f}(s,t) = s^{k\cdot e_1} + t^{k\cdot e_2},$$

where $e_1, e_2 \geq 2$. Now

$$\mu_a(f) = \dim_{\mathbf{C}} \mathcal{O}_X / \langle (st)^{k-1} \cdot (t^k)^{e_2-1}, t^{k\cdot e_2}, s^{k\cdot e_1}, (st)^{k-1} \cdot (s^k)^{e_1-1} \rangle_{\mathcal{O}_X}.$$

An explicit calculation shows that $\mu_a(f) = k \cdot e_1 e_2 - e_1 - e_2 + 1$.

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