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# CLASSIFICATION OF (1,1) TENSOR FIELDS AND BIHAMILTONIAN STRUCTURES

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**Abstract.** Consider a $(1,1)$ tensor field $J$, defined on a real or complex $m$-dimensional manifold $M$, whose Nijenhuis torsion vanishes. Suppose that for each point $p \in M$ there exist functions $f_1, \ldots, f_m$, defined around $p$, such that $(df_1 \wedge \ldots \wedge df_m)(p) \neq 0$ and $d(df_j(J(\ )))(p) = 0$, $j = 1, \ldots, m$. Then there exists a dense open set such that we can find coordinates, around each of its points, on which $J$ is written with affine coefficients. This result is obtained by associating to $J$ a bihamiltonian structure on $T^*M$.

**Introduction.** Consider a $(1,1)$ tensor field $J$, defined on a real or complex $m$-dimensional manifold $M$, whose Nijenhuis torsion vanishes. Suppose that for each point $p \in M$ there exist functions $f_1, \ldots, f_m$, defined around $p$, such that $(df_1 \wedge \ldots \wedge df_m)(p) \neq 0$ and $d(df_j \circ J)(p) = 0$ , $j = 1, \ldots, m$ [here $df \circ J$ means $df(J(\ ))$]. In this paper we give a complete local classification of $J$ on a dense open set that we call the regular open set. Moreover, near each regular point, i.e. each element of the regular open set, $J$ is written with affine coefficients on a suitable coordinate system.

To express the condition about functions $f_1, \ldots, f_m$, stated above, in a simple computational way we introduce the invariant $P_J$ (see section 1). This invariant only depends on the 1-jet of $J$ at each point, and $P_J(p) = 0$ iff functions $f_1, \ldots, f_m$ as before exist. When $J$ defines a G-structure, the first-order structure function being zero implies $P_J = 0$ and $N_J = 0$ (this last property is well known). Besides all points of $M$ are regular; therefore this work generalizes the main result of [5]. On the other hand $N_J$ and $P_J$ both together can be considered as a generalization of the first-order structure function.

This kind of tensor fields appear in a natural way in Differential Geometry. For example, on the base space of a bilagrangian fibration (see [1]) there exists a tensor field $J$,

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with $N_J = 0$, such that if $(x_1, \ldots, x_m)$ are action coordinates then each $dx_j \circ J$ is closed; so $P_J = 0$. From a wider viewpoint, when $N_J = 0$, we can study the equation:

$$d(df \circ J) = 0; \tag{1}$$

i.e. the existence of conservation laws for $J$. Our classification shows that the existence, close to $p$, of $m$ functionally independent solutions to equation (1) is equivalent to $P_J = 0$ near $p$.

Partial answers to the foregoing question may be found in [2], [6] and [7]. In [4], by using eigenvalues and eigenspaces, J. Grifone and M. Mehdi give an elegant necessary and sufficient condition for the existence of enough local solutions to equations (1) when $J$ is real analytic. With the Grifone-Mehdi condition all points are regular and a calculation shows that it implies $P_J = 0$. Therefore the Grifone-Mehdi result follows from ours.

Finally, let us sketch the way for classifying $J$. As $N_J = 0$ we can construct a bi-hamiltonian structure on $T^*M$ and from it a $(1,1)$ tensor field $J^*$, prolongation of $J$ to $T^*M$ (see [8]). The main result of [9] gives us the local model of $J^*$ on a dense open set and now a $J^*$-invariant cross section of $T^*M$ allows us to find a model of $J$. This cross section exists because $P_J = 0$ implies that the behaviour of $J^*$ does not change along each fiber of $T^*M$.

In a forthcoming paper we will study some cases where $P_J \neq 0$.

**1. The first step.** Consider a $(1,1)$ tensor field $J$ on a real or complex manifold $M$ of dimension $m$. We recall that the Nijenhuis torsion of $J$ is the $(1,2)$ tensor field given by the formula

$$N_J(X,Y) = [JX, JY] + J^2[X,Y] - J[X, JY] - J[JX, Y].$$

If $\tau$ is a 1-form $\tau \circ J$ will mean the 1-form defined by $(\tau \circ J)(X) = \tau(JX)$.

For each $p \in M$ let $F(2,J)(p)$ be the vector subspace of all the 2-forms $\beta_\sigma$ defined by $\beta_\sigma(v,w) = \sigma(Jv,w) - \sigma(v,Jw)$ where $v, w \in T_pM$ and $\sigma$ is a symmetric bilinear form on $T_pM$. Observe that $F(2,J^k)(p) \subset F(2,J)(p)$ for each $k \in \mathbb{N}$. Set

$$F_J(p) = \frac{\Lambda^2 T_p^* M}{F(2,J)(p)}.$$

Given $\alpha \in T_p^*M$ and a function $f$ defined around $p$ such that $df(p) = \alpha$, the class of $d(df \circ J)(p)$ on $F_J(p)$ only depends on $\alpha$. That defines a linear map $P_J(p) : T_p^*M \to F_J(p)$ or, from a global viewpoint, $P_J : T^*M \to F_J$ where $F_J$ is the disjoint union of all $F_J(p)$.

Note that $P_J(p) = 0$ if and only if there exist functions $f_1, \ldots, f_m$, defined around $p$, such that $(df_1 \wedge \ldots \wedge df_m)(p) \neq 0$ and $d(df_j \circ J)(p) = 0$, $j = 1, \ldots m$. When the characteristic polynomial of $J(p)$ equals its minimal polynomial, i.e. when $T_pM$ is cyclic, then $F(2,J)(p) = \Lambda^2 T_p^*M$ and automatically $P_J(p) = 0$. If $J^2 = -\mathrm{Id}$ a straightforward calculation shows that $N_J = 0$ implies $P_J = 0$. However $J$ can be semisimple, $N_J = 0$ and $P_J \neq 0$; e.g. on $\mathbb{R}^m$, $m \geq 2$, $J = e^{x_1}\,\mathrm{Id}$.

Let $\mathbb{K}_N[t]$ be the polynomial algebra in one variable over the ring of differentiable functions on a manifold $N$. Here differentiable means $C^\infty$ if $N$ is a real manifold ($\mathbb{K} = \mathbb{R}$) and holomorphic in the complex case ($\mathbb{K} = \mathbb{C}$). A polynomial $\varphi \in \mathbb{K}_N[t]$ is called *irreducible* if it is irreducible at each point of $N$. We shall say that $\varphi, \rho \in \mathbb{K}_N[t]$ are *relatively prime* if they are at each point. Consider an endomorphism field $H$ of a vector bundle $\pi : V \to N$, i.e. a cross section of $V \otimes V^*$. We will say that $H$ has *constant algebraic type* if there exist relatively prime irreducible polynomials $\varphi_1, \ldots, \varphi_\ell \in \mathbb{K}_N[t]$ and natural numbers $a_{ij}$, $i = 1, \ldots, r_j$, $j = 1, \ldots, \ell$, such that for each $p \in N$ the family $\{\varphi_j^{a_{ij}}(p)\}$, $i = 1, \ldots, r_j$, $j = 1, \ldots, \ell$, is the family of elementary divisors of $H(p)$.

Suppose that $J$ defines a G-structure, i.e. $J$ has constant algebraic type on $M$ and $\varphi_1, \ldots, \varphi_\ell \in \mathbb{K}[t]$. If its first-order structure function vanishes then $P_J = 0$. Indeed, around each point $p \in M$ there exists a linear connection $\nabla$, whose torsion at $p$ vanishes, such that $\nabla J = 0$. Let $f_1, \ldots, f_m$ be normal coordinates with origin $p$; then $d(df_j \circ J)(p) = 0$ and $P_J(p) = 0$. Conversely $N_J = 0$ and $P_J = 0$ imply that the first-order structure function equals zero. In a word, *the invariants $N_J$ and $P_J$ can be seen as a generalization of the first-order structure function to the case where $J$ does not define a G-structure.*

*Henceforth we shall suppose $N_J = 0$.* Set $g_k = \mathrm{trace}(J^k)$ and $E = \bigcap_{j=1}^m \operatorname{Ker} dg_j$. It is well known that $(k+1)dg_k \circ J = k dg_{k+1}$ and $JE \subset E$ (see [9]).

We say that a point $p \in M$ is *regular* if there exists an open neighbourhood $A$ of $p$ such that:

(1) $J$ has constant algebraic type on $A$,

(2) $E$, restricted to $A$, is a vector subbundle of $TA$.

(3) The restriction of $J$ to $E$ has constant algebraic type on $A$.

The set of all regular points is a dense open set of $M$ which we shall call the *regular open set*. Our local classification of $J$ only refers to the regular open set.

Now suppose that on an open neighbourhood of a regular point $p$ the characteristic polynomial $\varphi$ of $J$ is the product $\varphi_1 \cdot \varphi_2$ of two monic relatively prime polynomials $\varphi_1$ and $\varphi_2$. Then around $p$ the structure $(M, J)$ decomposes into a product of two similar structures $(M_1, J_1) \times (M_2, J_2)$, where $\varphi_1$ is the characteristic polynomial of $J_1$ (more exactly $\varphi_1$ is the pull-back of the characteristic polynomial of $J_1$) and $\varphi_2$ that of $J_2$ (see [3] and [9]). Moreover $N_{J_1} = 0$, $N_{J_2} = 0$, and $p_1$ and $p_2$ are regular points where $p = (p_1, p_2)$. On the other hand $P_{J_1} = 0$ and $P_{J_2} = 0$ if $P_J = 0$.

This splitting property reduces the classification to the case where the characteristic polynomial $\varphi$ of $J$ is a power of an irreducible one. Therefore we have only two possibilities: $\varphi = (t + f)^m$, or $\varphi = (t^2 + ft + g)^n$ where $m = 2n$ and $M$ is a real manifold.

## 2. The case $\varphi = (t + f)^m$.

In this section, by associating to $J$ a bihamiltonian structure on $T^*M$, we prove the following result:

THEOREM 1. *Consider a $(1,1)$ tensor field $J$ such that $N_J = 0$ and $P_J = 0$. Suppose that its characteristic polynomial is $(t+f)^m$. Then around each regular point $p$ there exist coordinates $((x_i^j), y)$ with origin $p$, i.e. $p \equiv 0$, such that:*

(a) *$i = 1, \ldots, r_j$ and $r_1 \geq r_2 \geq \ldots \geq r_\ell$. Moreover we also consider the case with no coordinates $(x_i^j)$, i.e. $\ell = 0$, and the case with coordinates $(x_i^j)$ only.*

(b) $J = (y+a)\,\mathrm{Id} + H + Y \otimes dy$ *where*

$$H = \sum_{j=1}^{\ell}\left(\sum_{i=1}^{r_j-1} \frac{\partial}{\partial x_{i+1}^j} \otimes dx_i^j\right) \quad \textit{and} \quad Y = \frac{\partial}{\partial x_1^1} + \sum_{j=1}^{\ell}\left(\sum_{i=2}^{r_j}(1-i)x_i^j \frac{\partial}{\partial x_i^j}\right).$$

Remark. In the first special case $m = 1$ and $J = (y+a)\,\mathrm{Id}$; in the second one $m = r_1 + \ldots + r_\ell$ and $J = a\,\mathrm{Id} + \sum_{j=1}^{\ell}(\sum_{i=1}^{r_j-1} \partial/\partial x_{i+1}^j \otimes dx_i^j)$. The elementary divisors of $J$ determine its model completely. If there is no coordinate $y$, i.e. if $J$ defines a G-structure, they are: $\{(t-a)^{r_j}\}$, $j=1,\ldots,\ell$. Otherwise they are: $(t-(y+a))^{r_1+1}$; $\{(t-(y+a))^{r_j}\}$, $j = 2,\ldots,\ell$.

Let $c_J : T^*M \to T^*M$ be the morphism of $T^*M$ defined by $c_J(\tau) = \tau \circ J$ and let $\omega$ be the Liouville symplectic form of $T^*M$. Set $\omega_1 = (c_J)^*\omega$ where $c_J$ is regarded as a differentiable map. Consider the $(1,1)$ tensor field $J^*$, on $T^*M$, defined by $\omega_1(X,Y) = \omega(J^*X,Y)$. Then $N_{J^*} = 0$, because $N_J = 0$, and $\{\omega,\omega_1\}$ is a bihamiltonian structure (see [8]). If $(x_1,\ldots,x_m)$ are coordinates on $M$, $(x_1,\ldots,x_m,z_1,\ldots,z_m)$ the associated coordinates on $T^*M$, and $J = \sum_{i,j=1}^m f_{ij}\partial/\partial x_i \otimes dx_j$ then

$$J^* = \sum_{i,j=1}^m f_{ij}\left(\frac{\partial}{\partial x_i} \otimes dx_j + \frac{\partial}{\partial z_j} \otimes dz_i\right) + \sum_{i,j,k=1}^m z_i\left(\frac{\partial f_{ij}}{\partial x_k} - \frac{\partial f_{ik}}{\partial x_j}\right)\frac{\partial}{\partial z_j} \otimes dx_k.$$

Hence $\pi_* \circ J^* = J \circ \pi_*$.

Throughout the rest of this section $J$ is as in theorem 1. By the local expression of $J^*$ given above, its characteristic polynomial is $\varphi^* = (t + f\circ\pi)^{2m}$. Since $P_J = 0$, around each regular point $p \in M$ there exist coordinates $(x_1,\ldots,x_m)$ such that $d(dx_i \circ J)(p) = 0$, $i = 1,\ldots,m$. Even more if $df(p) \neq 0$ [regularity implies $df(p) = 0$ iff $f$ is constant near $p$] we can suppose $f = x_1$ because $g_1 = -mf$ and $dg_1 \circ J = \frac{dg_2}{2}$. But $dx_i \circ J = \sum_{j=1}^m f_{ij}dx_j$, then $\frac{\partial f_{ij}}{\partial x_k}(p) = \frac{\partial f_{ik}}{\partial x_j}(p)$ and

$$J^*(p,z) = \sum_{i,j=1}^m f_{ij}(p)\left(\frac{\partial}{\partial x_i} \otimes dx_j + \frac{\partial}{\partial z_j} \otimes dz_i\right)(p,z).$$

Therefore the elementary divisors of $J(p)$ and $(J_{|E})(p)$ determine those of $J^*(p,z)$ and $(J^*{}_{|E^*})(p,z)$ completely, and the pull-back of the regular open set of $J$ is included in the regular open set of $J^*$. This is the role of the assumption $P_J = 0$ while $N_J = 0$ assures us that $\{\omega,\omega_1\}$ is bihamiltonian.

The zero cross section allows us to consider $M$ as a submanifold of $T^*M$. Take a regular point $p \in M$ such that $df(p) = 0$, i.e. $f$ constant near $p$. By theorem 3 of [9] there exist coordinates $(y_1,\ldots,y_{2m})$ on an open neighbourhood $A$ of $p$, with origin this point, on which $\omega$ and $\omega_1$ are written with constant coefficients and $J^*$ as well. By rearranging coordinates $(y_1,\ldots,y_{2m})$ if necessary, we can suppose that $\{\frac{\partial}{\partial y_1}(p),\ldots,\frac{\partial}{\partial y_m}(p)\}$ spans $T_pM$ and $\{\frac{\partial}{\partial y_{m+1}}(p),\ldots,\frac{\partial}{\partial y_{2m}}(p)\}$ spans the vertical subspace $\mathrm{Ker}\,\pi_*(p)$ at $p$. Both subspaces are $J^*$-invariant as the local expression of $J^*$ shows. Set $A_0 = \{y \in A : y_{m+1} = \ldots = y_{2m} = 0\}$. As $\mathrm{rank}((\pi_{|A_0})(p)) = m$ we can choose an open neighbourhood $B$ of $p$ on $A_0$ such that $\pi(B)$ is open and $\pi : B \to \pi(B)$ a diffeomorphism.

By construction $J^*(TA_0) \subset TA_0$. Let $J'$ be the restriction of $J^*$ to $A_0$. The tensor field $J'$ is written with constant coefficients on $A_0$. Moreover $(\pi_{|A_0})_* \circ J' = J \circ (\pi_{|A_0})_*$ since $\pi_* \circ J^* = J \circ \pi_*$. Then $J$ is written with constant coefficients on $\pi(B)$, *which proves theorem 1 when* $df(p) = 0$.

The proof of the other case is basically the same but we have to rearrange coordinates in a more sophisticated way. Let $V$ be a real or complex vector space of dimension $2n$. Consider $\alpha, \alpha_1 \in \Lambda^2 V^*$ such that $\alpha^n \neq 0$. Let $\tilde{J}$ be the endomorphism of $V$ given by $\alpha_1(v,w) = \alpha(\tilde{J}v, w)$. *Suppose* $\tilde{J}$ *nilpotent* (see proposition 1 of [9] for the model of $\{\alpha, \alpha_1\}$). An $n$-dimensional vector subspace $W$ of $V$ is called *bilagrangian* if $\alpha(v,w) = \alpha_1(v,w) = 0$ for all $v, w \in W$; in other words $W$ is lagrangian for $\alpha$ and $JW \subset W$. When $W$ is bilagrangian and there exists another bilagrangian subspace $W'$ such that $V = W \oplus W'$ we shall say that $W$ is *superlagrangian.* A bilagrangian subspace $W$ is superlagrangian if and only if the elementary divisors of $J_{|W}$ are half those of $J$; i.e. if $\{t^{r_j}\}$, $j = 1, \dots, \ell$, are the elementary divisors of $J_{|W}$ then $\{t^{r_j}, t^{r_j}\}$, $j = 1, \dots, \ell$, are those of $J$.

Lemma 1. *Consider a basis* $\{e_i^j\}$, $i = 1, \dots, 2r_j$, $j = 1, \dots, \ell$, *of* $V$ *such that*

$$\alpha = \sum_{j=1}^{\ell} \Big( \sum_{k=1}^{r_j} e_{2k-1}^{*j} \wedge e_{2k}^{*j} \Big) \quad \textit{and} \quad \alpha_1 = \sum_{j=1}^{\ell} \Big( \sum_{k=1}^{r_j - 1} e_{2k-1}^{*j} \wedge e_{2k+2}^{*j} \Big).$$

*Let* $W$ *be the vector subspace spanned by* $\{e_{2k-1}^j\}$, $k = 1, \dots, r_j$, $j = 1, \dots, \ell$. *Then for each superlagrangian subspace* $W'$ *of* $V$ *there exists* $T \in GL(V)$ *such that* $T^*\alpha = \alpha$, $T^*\alpha_1 = \alpha_1$ *and* $W \cap TW' = \{0\}$. *Moreover if* $e_{2r_1 - 1}^1 \notin W'$ *we can choose* $T$ *in such a way that* $Te_1^1 = e_1^1$.

Now take a regular point $p \in M$. *Suppose* $df(p) \neq 0$. By theorem 3 of [9] there exist coordinates $(x, y) = ((x_i^j), y_1, y_2)$, $i = 1, \dots, 2r_j$ and $r_1 \geq r_2 \geq \dots \geq r_\ell$, with origin $p$, such that

$$\omega = \sum_{j=1}^{\ell} \Big( \sum_{k=1}^{r_j} dx_{2k-1}^j \wedge dx_{2k}^j \Big) + dy_1 \wedge dy_2$$

and $\omega_1 = (y_2 + a)\omega + \tau + \alpha \wedge dy_2$ where

$$\tau = \sum_{j=1}^{\ell} \Big( \sum_{k=1}^{r_j - 1} dx_{2k-1}^j \wedge dx_{2k+2}^j \Big)$$

and

$$\alpha = dx_2^1 + \sum_{j=1}^{\ell} \Big( \sum_{k=1}^{r_j} [(k + 1/2) x_{2k}^j dx_{2k-1}^j + (k - 1/2) x_{2k-1}^j dx_{2k}^j] \Big).$$

Hence $J^* = (y_2 + a) \operatorname{Id} + H^* + \frac{\partial}{\partial y_1} \otimes \alpha - Z \otimes dy_2$ where

$$H^* = \sum_{j=1}^{\ell} \Bigg( \sum_{k=1}^{r_j - 1} \frac{\partial}{\partial x_{2k+1}^j} \otimes dx_{2k-1}^j + \sum_{k=2}^{r_j} \frac{\partial}{\partial x_{2k-2}^j} \otimes dx_{2k}^j \Bigg)$$

and

$$Z=\frac{\partial}{\partial x_1^1}+\sum_{j=1}^{\ell}\left(\sum_{k=1}^{r_j}\left[(k-1/2)x_{2k-1}^j\frac{\partial}{\partial x_{2k-1}^j}-(k+1/2)x_{2k}^j\frac{\partial}{\partial x_{2k}^j}\right]\right).$$

LEMMA 2. *The vector* $\frac{\partial}{\partial x^1_{2r_1-1}}(p)$ *does not belong to the vertical subspace* $\operatorname{Ker}\pi_*(p)$.

Proof. By the local expression of $J^*$ in the coordinates $(x_1,\dots,x_m,z_1,\dots,z_m)$ given at the beginning of this section, $\operatorname{Ker}\pi_*(p)$ and $T_pM$ are $J^*(p)$-invariant, and $J_{|\operatorname{Ker}\pi_*(p)}$ and $J_{|T_pM}$ have the same elementary divisors. As $p\equiv 0$ in coordinates $(x,y)$, the elementary divisors of $J^*(p)$ are $(t-a)^{r_1+1}$; $(t-a)^{r_1+1}$; $\{(t-a)^{r_j},(t-a)^{r_j}\}$, $j=2,\dots,\ell$. Therefore there exists $v\in T_pM$ spanning a cyclic subspace $U$ of dimension $r_1+1$ such that $U\cap\operatorname{Ker}\pi_*(p)=\{0\}$.

Moreover $v=a\frac{\partial}{\partial y_2}(p)+b\frac{\partial}{\partial x^1_{2r_1}}(p)+v_1$ where $(J^*(p)-a\operatorname{Id})^{r_1}v_1=0$.

By construction

$$(J^*(p)-a\operatorname{Id})^{r_1}v=a\frac{\partial}{\partial x^1_{2r_1-1}}(p)+b\frac{\partial}{\partial y_1}(p)$$

does not belong to $\operatorname{Ker}\pi_*(p)$. As $\omega(\partial/\partial y_1,\ )=dy_2=-d(f\circ\pi)$ and $\omega=\sum_{j=1}^m dz_j\wedge dx_j$ in coordinates $(x_1,\dots,x_m,z_1,\dots,z_m)$ of $T^*M$, the vector $\frac{\partial}{\partial y_1}(p)$ belongs to $\operatorname{Ker}\pi_*(p)$. So $\frac{\partial}{\partial x^1_{2r_1-1}}(p)\notin\operatorname{Ker}\pi_*(p)$. ■

Set $\omega'=\sum_{j=1}^{\ell}(\sum_{k=1}^{r_j}dx^j_{2k-1}\wedge dx^j_{2k})$.

LEMMA 3. *The vector subspace* $(\operatorname{Ker}\pi_*\cap\operatorname{Ker}dy_1\cap\operatorname{Ker}dy_2)(p)$, *regarded as a subspace of* $T_0\mathbb{K}^{2m-2}$, *is superlagrangian with respect to* $\{\omega'(0),\tau(0)\}$.

Proof. As $f\circ\pi=-(y_2+a)$, $\operatorname{Ker}\pi_*(p)\subset\operatorname{Ker}dy_2(p)=\operatorname{Ker}d(f\circ\pi)(p)$. Now note that $((J^*-a\operatorname{Id})^{r_1}\operatorname{Ker}\pi_*)(p)$ is a 1-dimensional subspace of $\operatorname{Ker}\pi_*(p)\cap\mathbb{K}\{\frac{\partial}{\partial x^1_{2r_1-1}}(p),\frac{\partial}{\partial y_1}(p)\}$ (here $\mathbb{K}\{v_1,\dots,v_s\}$ is the space spanned by $\{v_1,\dots,v_s\}$). So $((J^*-a\operatorname{Id})^{r_1}\operatorname{Ker}\pi_*)(p)=\mathbb{K}\{\frac{\partial}{\partial y_1}(p)\}$ since $\frac{\partial}{\partial x^1_{2r_1-1}}(p)\notin\operatorname{Ker}\pi_*(p)$.

On the other hand $T_0\mathbb{K}^{2m-2}$ can be seen as the quotient space $\operatorname{Ker}dy_2(p)/\mathbb{K}\{\frac{\partial}{\partial y_1}(p)\}$, which identifies $(\operatorname{Ker}\pi_*\cap\operatorname{Ker}dy_1\cap\operatorname{Ker}dy_2)(p)$ with $\operatorname{Ker}\pi_*(p)/\mathbb{K}\{\frac{\partial}{\partial y_1}(p)\}$, and $(H^*+a\operatorname{Id})(0)$ as the endomorphism induced by $J^*{}_{|\operatorname{Ker}dy_2(p)}$. Therefore the elementary divisors of $H^*{}_{|(\operatorname{Ker}\pi_*\cap\operatorname{Ker}dy_1\cap\operatorname{Ker}dy_2)(p)}$ are $\{t^{r_j}\}$, $j=1,\dots,\ell$. ■

LEMMA 4. *Let* $\{e^j_i\}$, $i=1,\dots,2r_j$, $j=1,\dots,\ell$, *be the canonical basis of* $\mathbb{K}^{2m-2}=\mathbb{K}^{2r_1}\times\ldots\times\mathbb{K}^{2r_\ell}$. *Set* $\alpha=\sum_{j=1}^{\ell}(\sum_{k=1}^{r_j}e^{*j}_{2k-1}\wedge e^{*j}_{2k})$ *and* $\alpha_1=\sum_{j=1}^{\ell}(\sum_{k=1}^{r_j-1}e^{*j}_{2k-1}\wedge e^{*j}_{2k+2})$. *Given* $T\in GL(\mathbb{K}^{2m-2})$ *if* $Te^1_1=e^1_1$; $T^*\alpha=\alpha$ *and* $T^*\alpha_1=\alpha_1$, *there exists a germ of diffeomorphism* $\tilde G:(\mathbf{K}^{2m},0)\to(\mathbf{K}^{2m},0)$ *such that* $\tilde G(x,y)=(G(x),y)$; $\tilde G^*\omega=\omega$; $\tilde G^*\omega_1=\omega_1$ *and* $G_*(0)=T$.

Proof. We will adapt to our case the proof of proposition 3 of [9]. Consider the map $G_T:\mathbb{K}^{2m}\to\mathbb{K}^{2m}$ given by $G_T(x,y)=(Tx,y)$. Then $G_T^*\omega=\omega$ and $G_T^*\omega_1=\omega_1+dg\wedge dy_2$ where $g$ is a quadratic function such that $d(dg\circ H^*)=0$. Indeed $G_T$ preserves $dx^1_2(0)=\omega(\frac{\partial}{\partial x^1_1},\ )(0)$ and $H^*$, and $d(\alpha\circ H^*)=-2\tau$.

Let $D$ and $\mathbb{L}$ be the exterior derivative and the Lie derivative *with respect to the variables $x$ only.* We begin searching for a vector field $X_t = \sum_{j=1}^{\ell}(\sum_{i=1}^{2r_j} \varphi_i^j(x,t)\frac{\partial}{\partial x_i^j})$, defined on an open neighbourhood of the compact $\{0\} \times [0,1] \subset \mathbb{K}^{2m-2} \times \mathbb{K}$, such that:

(1) $\mathbb{L}_{X_t}\omega' = \mathbb{L}_{X_t}\tau = 0$.

(2) $\mathbb{L}_{X_t}(\alpha + tDg) = Dg$ (remark that $dg = Dg$).

(3) For each $i = 1, \ldots, 2r_j$ and $j = 1, \ldots, \ell$, $\varphi_i^j$ and $D\varphi_i^j$ vanish on $\{0\} \times [0,1]$.

Consider the vector field $Z_t$ given by $\omega'(Z_t, \ ) = \alpha + tDg$. Take a function $f(x,t)$, defined around $\{0\} \times [0,1]$, such that:

(I) $Z_t f = -f - g$.

(II) $D(Df \circ H^*) = 0$.

(III) For all $i = 1, \ldots, 2r_j$, $j = 1, \ldots, \ell$, $k = 1, \ldots, 2r_s$ and $s = 1, \ldots, \ell$, the partial derivatives $\partial f/\partial x_i^j$ and $\partial^2 f/\partial x_k^s \partial x_i^j$ vanish on $\{0\} \times [0,1]$.

Let $X_t$ the vector field defined by $\omega'(X_t, \ ) = Df$. Then $X_t$ satisfies conditions (1), (2) and (3). By proposition 1.A (see the appendix) this kind of functions exists because $g$ is quadratic, $D(Dg \circ H^*) = 0$, $Z_t(0) = \partial/\partial x_1^1$, and $\mathbb{L}_{Z_t}H^* = -H^*$ since $\mathbb{L}_{Z_t}\omega' = D(\alpha + tDg) = -\omega'$ and $\mathbb{L}_{Z_t}\tau = D(\alpha \circ H^* + tDg \circ H^*) = -2\tau$.

By integrating the vector field $-X_t$ we obtain a germ of diffeomorphism $F : (\mathbb{K}^{2m-2}, 0) \to (\mathbb{K}^{2m-2}, 0)$ such that $F^*\omega' = \omega'$; $F^*\tau = \tau$; $F^*(\alpha + Dg) = \alpha$ and $F_*(0) = \mathrm{Id}$. Now set $\tilde{G} = \tilde{F} \circ G_T$ where $\tilde{F}(x,y) = (F(x), y)$. ■

Let $W$ be the subspace of $T_pT^*M$ spanned by $\{\frac{\partial}{\partial x_{2k-1}^j}(p)\}$, $k = 1, \ldots, r_j$, $j = 1, \ldots, \ell$. By lemmas 1, 2, 3 and 4 we can suppose, without loss of generality, $W \cap (\operatorname{Ker}\pi_* \cap dy_1 \cap dy_2)(p) = \{0\}$, which implies $(W \oplus \mathbb{K}\{\frac{\partial}{\partial y_2}(p)\}) \cap Ker\pi_*(p) = \{0\}$. Indeed $dim(\operatorname{Ker}\pi_* \cap dy_1 \cap dy_2)(p) = m-1$ (lemma 3) and $\frac{\partial}{\partial y_1}(p) \in \operatorname{Ker}\pi_*(p)$ (lemma 2, proof); then $\operatorname{Ker}\pi_*(p) = \mathbb{K}\{\frac{\partial}{\partial y_1}(p)\} \oplus (\operatorname{Ker}\pi_* \cap dy_1 \cap dy_2)(p)$.

Set $A_0 = \{(x,y) \in A : x_{2k}^j = y_1 = 0, k = 1, \ldots, r_j, j = 1, \ldots, \ell\}$ where $A$ is the domain of coordinates $(x,y)$. Then $J^*(TA_0) \subset TA_0$ and $T_pA_0 \oplus \operatorname{Ker}\pi_*(p) = T_pT^*M$. Finally, by reasoning as in the case $df(p) = 0$ we can state:

Proposition 1. *Under the assumptions of theorem 1, if $df(p) \neq 0$ then there exist coordinates $((x_i^j), y)$ as in this theorem such that $J = (y+a)\,\mathrm{Id} + H + Y \otimes dy$ where*

$$H = \sum_{j=1}^{\ell}\left(\sum_{i=1}^{r_j-1} \frac{\partial}{\partial x_{i+1}^j} \otimes dx_i^j\right) \quad \textit{and} \quad Y = \frac{\partial}{\partial x_1^1} + \sum_{j=1}^{\ell}\left(\sum_{i=1}^{r_j}(1/2 - i)x_i^j \frac{\partial}{\partial x_i^j}\right).$$

When $df(p) \neq 0$, proposition 1 shows that the local model of $J$ only depends on its elementary divisors.

Lemma 5. *Consider on $\mathbb{K}^m = \mathbb{K}^{r_1} \times \ldots \times \mathbb{K}^{r_\ell} \times \mathbb{K}$, with $r_1 \geq \ldots \geq r_\ell$ if $\ell > 0$, coordinates $((x_i^j), y)$. Let $\mathbb{L}$ be the Lie derivative with respect to variables $(x_i^j)$ only. Set $J = (y+a)\,\mathrm{Id} + H + Y \otimes dy$ where $Y$ is a vector field defined around the origin such that $dy(Y) = 0$ and $H = \sum_{j=1}^{\ell}\left(\sum_{i=1}^{r_j-1} \frac{\partial}{\partial x_{i+1}^j} \otimes dx_i^j\right)$. If $\mathbb{L}_Y H = H$ and $H^{r_1-1}Y(0) \neq 0$, then $N_J = 0$ and close to the origin $P_J = 0$ and $J$ has constant algebraic type.*

The elementary divisors of $J$, near the origin, are the same both for proposition 1 and lemma 5: $(t-(y+a))^{r_1+1}$; $\{(t-(y+a))^{r_j}\}$, $j=2,\ldots,\ell$. *So their models are equivalent.* We finish the proof of theorem 1 by taking

$$Y=\frac{\partial}{\partial x_1^1}+\sum_{j=1}^{\ell}\bigg(\sum_{i=2}^{r_j}(1-i)x_i^j\frac{\partial}{\partial x_i^j}\bigg).$$

The model announced by the author in a lecture at the Banach Center is obtained by setting

$$Y=\frac{\partial}{\partial x_1^1}-\sum_{j=1}^{\ell}\bigg(\sum_{i=1}^{r_j}ix_i^j\frac{\partial}{\partial x_i^j}\bigg).$$

Another interesting model is given by taking

$$Y=\frac{\partial}{\partial x_1^1}+\sum_{j=1}^{\ell}\bigg(\sum_{i=1}^{r_j}(r_j+1-i)x_i^j\frac{\partial}{\partial x_i^j}\bigg).$$

For this model the forms $dy\circ J=(y+a)dy$ and $dx_{r_j}^j\circ J=(y+a)dx_{r_j}^j+x_{r_j}^j dy+dx_{r_j-1}^j$ are closed. As $N_J=0$ all the forms $dx_{r_j}^j\circ J^k$ are closed too. Therefore if the characteristic polynomial of $J$ is $(t+f)^m$, for each regular point $p$ and for all $\lambda_0\in T_p^*M$ there exists a closed 1-form $\lambda$, defined near $p$, such that $\lambda(p)=\lambda_0$ and $d(\lambda\circ J)=0$; usually $\lambda$ is called a *conservation law*. In other words, the equation $d(df\circ J)=0$ has enough local solutions on the regular open set.

**3. The case $\varphi=(t^2+ft+g)^n$.** Since our problem is local we can suppose $M$ connected and all of its points regular. Set $J_0=2(4g-f^2)^{-\frac12}J+f(4g-f^2)^{-\frac12}\,\mathrm{Id}$ which makes sense because $f^2-4g<0$. By construction $J_0$ defines a G-structure and $(J_0^2+\mathrm{Id})^n=0$. Let $H$ be the semisimple part of $J_0$. Then $H$ is a complex structure, $J$ a holomorphic tensor field and $(t+h)^n$ its complex characteristic polynomial, where $h=\frac12(f-i(4g-f^2)^{\frac12})$ is holomorphic.

Indeed, consider $\{\omega,\omega_1\}$ and $J^*$ on $T^*M$ as in section 2. Now the characteristic polynomial of $J^*$ is $\varphi^*=(t^2+(f\circ\pi)t+(g\circ\pi))^{2n}$. Let $A$ be the regular open set of $J^*$. Set $J_0^*=2((4g-f^2)^{-\frac12}\circ\pi)J^*+((f(4g-f^2)^{-\frac12})\circ\pi)\,\mathrm{Id}$. On each connected component of $A$ the tensor field $J_0^*$ defines a G-structure; moreover $((J_0^*)^2+\mathrm{Id})^{2n}=0$. Let $H^*$ be the semisimple part of $J$. In section 6 of [9] we showed that $H^*$ is a complex structure, $J^*$ holomorphic and $(t+h^*)^{2n}$ its complex characteristic polynomial, where $h^*=\frac12(f\circ\pi-i(4g-f^2)^{\frac12}\circ\pi)$ is a holomorphic function. On the other hand $\pi_*\circ J_0^*=J_0\circ\pi_*$ and $\pi_*\circ H^*=H\circ\pi_*$ because $\pi_*\circ J^*=J\circ\pi_*$. So holomorphy holds on $\pi(A)$, and on $M$ as well since $A$ is dense on $T^*M$ and $\pi(A)$ on $M$.

The complex regular set of $J$ is $M$ (see section 6 of [9] again).

Suppose $P_J=0$. Let $f=f_1+if_2$ a holomorphic function. Then $d(df\circ J)=d(df_1\circ J)+i(d(df_2\circ J))$ is a holomorphic 2-form, so $d(df_1\circ J)(HX,Y)=d(df_1\circ J)(X,HY)$ and $d(df_2\circ J)(X,Y)=-d(df_1\circ J)(HX,Y)$. As $P_J(p)=0$ from the real viewpoint, there exists a real symmetric bilinear form $\sigma$ on $T_pM$ such that $d(df_1\circ J)(p)(v,w)=\sigma(J(p)v,w)-\sigma(v,J(p)w)$. Set $\sigma_1(v,w)=\frac12(\sigma(v,w)-\sigma(H(p)v,H(p)w))$ and $\tilde\sigma(v,w)=\sigma_1(v,w)-i\sigma_1(H(p)v,w)$. As $J$ and $H$ commute $\tilde\sigma$ is a complex symmetric bilinear form and $d(df \circ J)(p)(v,w) = \tilde{\sigma}(J(p)v,w) - \tilde{\sigma}(v,J(p)w)$. In other words $P_J = 0$ from the complex viewpoint. So to find a model of $J$, regard $M$ as a complex manifold of dimension $n$ and apply theorem 1. Then forget the complex structure and regard $J$ as a real tensor field.

THEOREM 2. *Suppose $N_J = 0$ and $P_J = 0$. Then the local model of $J$ around each regular point is a finite product of models chosen among*:

(a) *For a complex manifold*, *those of theorem* 1.

(b) *For a real manifold*, *those of theorem* 1 *and those obtained considering the complex models of that theorem from the real viewpoint.*

*The local model of $J$ is completely determined by its elementary divisors.*

Remark. Suppose $N_J = 0$. Let $p$ be a regular point. By theorem 2 there exist enough solutions to the equation $d(df \circ J) = 0$, i.e. conservation laws, near $p$ iff $P_J$ vanishes around this point. Nevertheless the existence of this kind of functions does not imply $N_J = 0$; e.g. on $\mathbb{K}^2$ consider $J = e^{x_2}\,\mathrm{Id} + \partial/\partial x_2 \otimes dx_1$; $f_1 = x_1 - e^{x_2}$ and $f_2 = x_2$.

**Appendix.** Consider an open set $A$ of $\mathbb{K}^n$ endowed with a nilpotent constant coefficient $(1,1)$ tensor field $H$. Let $B$ be a differentiable manifold (the parameter space). Elements of $A \times B$ will be denoted by $(x,y)$ while by $D$, $D^{(2)}$ and $\mathbb{L}$ we mean the exterior derivative, the second-order differential and the Lie derivative, all of them with respect to the variables $(x_1,\dots,x_n)$ only. Let $Z$ be a vector field on $A$ depending on the parameter $y \in B$. We say that $Z$ *is generic at a point* $(x,y)$ if the dimension of the cyclic subspace spanned by $Z(x,y)$ equals the degree of the minimal polynomial of $H$.

PROPOSITION 1.A. *Suppose given $p \in A$, a compact set $K \subset B$, a scalar $a \in \mathbb{K}$ and a function $g : A \times B \to \mathbb{K}$, such that*: (1) $\mathbb{L}_Z H = cH$ *where* $c \in \mathbb{K}$; (2) $Z$ *is generic on* $\{p\} \times K$; (3) $D(Dg \circ H) = 0$, $g(\{p\} \times B) = 0$ *and* $Dg(\{p\} \times B) = 0$.

*Then there exist an open neighbourhood $U$ of $p$, an open set $V \supset K$ and a function $f : U \times V \to \mathbb{K}$ such that*: (I) $Zf = af + g$; (II) $D(Df \circ H) = 0$; (III) $Df(\{p\} \times V) = 0$ *and* $D^{(2)}f(\{p\} \times V) = 0$. *Moreover if $Dg(\operatorname{Ker} H^r) = 0$ we can choose $f$ in such a way that $Df(\operatorname{Ker} H^r) = 0$.*

The proof of this result is essentially that of proposition 1.A of [9]. Before lemma 2.A no change is needed at all. This last result should be replaced with:

LEMMA 2′.A. *Consider a function $h_1 : A \times B \to \mathbb{K}$. Suppose $Dh_1(KerH) = 0$ and $D(Dh_1 \circ H) = 0$. Then there exist an open neighbourhood $U$ of $p$ and a function $h : U \times B \to \mathbb{K}$ such that*: (1) $Dh \circ H = Dh_1$; (2) $h(\{p\} \times B) = 0$; (3) $Dh(p,y) = 0$ *for all $y \in B$ such that $Dh_1(p,y) = 0$*; $D^{(2)}h(p,y) = 0$ *for each $y \in B$ such that $Dh_1(p,y) = 0$ and $D^{(2)}h_1(p,y) = 0$.*

Proof. There exist a vector subbundle $E$ of $TA$ and a morphism $\rho : TA \to TA$ such that $TA = E \oplus \operatorname{Ker} H$ and $(\rho \circ H)_{|E} = \mathrm{Id}$. Set $\alpha = Dh_1 \circ \rho$. Obviously $\alpha \circ H = Dh_1$. Let $C$ be the set of all $y \in B$ such that $Dh_1(p,y) = 0$ and $D^{(2)}h_1(p,y) = 0$. Suppose $\alpha = \sum_{j=1}^{n} g_j dx_j$. Then $g_j(\{p\} \times C) = 0$ and $Dg_j(\{p\} \times C) = 0$, $j = 1,\dots,n$.

By rearranging coordinates $(x_1, \ldots, x_n)$ we can suppose the foliation $\operatorname{Ker} H$ given by $dx_1 = \ldots = dx_k = 0$. From lemma 1.A, $D\alpha(\operatorname{Im} H, \operatorname{Im} H) = 0$ so $D\alpha = \sum_{j=1}^{k}(\sum_{i=1}^{n} f_{ij} dx_i) \wedge dx_j$ where each $f_{ij}$ equals zero on $\{p\} \times C$.

Let $U = \prod_{i=1}^{n} U_i$ be an open neighbourhood of $p$, product of intervals ($\mathbb{K} = \mathbb{R}$) or disks ($\mathbb{K} = \mathbb{C}$). As $D\alpha$ is closed, there exist functions $\tilde{f}_j : U \times B \to \mathbb{K}$ such that $\partial \tilde{f}_j / \partial x_i = f_{ij}$ and $\tilde{f}_j(U_1 \times \ldots \times U_k \times \{(p_{k+1}, \ldots, p_n)\} \times B) = 0$, $i = k+1, \ldots, n$, $j = 1, \ldots, k$, where $p = (p_1, \ldots, p_n)$. Therefore $\tilde{f}_j(\{p\} \times B) = 0$ and $D\tilde{f}_j(\{p\} \times C) = 0$.

Set $\beta = D\alpha - D(\sum_{j=1}^{k} \tilde{f}_j dx_j) = \sum_{i,\ell=1}^{k} e_{i\ell} dx_i \wedge dx_\ell$. As $D\beta = 0$, the functions $e_{i\ell}$ do not depend on $(x_{k+1}, \ldots, x_n)$. By construction $e_{i\ell}(\{p\} \times C) = 0$.

Now we can find functions $e_2, \ldots, e_k : U \times B \to \mathbb{K}$, which do not depend on $(x_{k+1}, \ldots \ldots, x_n)$, such that $\partial e_j / \partial x_1 = e_{1j}$ and $e_j(\{p_1\} \times U_2 \times \ldots \times U_n \times B) = 0$, $j = 2, \ldots, k$. So $e_j(\{p\} \times B) = 0$ and $De_j(\{p\} \times C) = 0$. Set $\beta' = \sum_{j=2}^{k} e_j dx_j$. Then $\beta_1 = \beta - D\beta'$ is closed and $\beta_1(\{p\} \times C) = 0$. Moreover $\beta_1$ only involves the variables $(x_2, \ldots, x_k)$ and differentials $dx_2, \ldots, dx_k$. By induction we construct $\tilde{\beta} = \sum_{j=1}^{k} \tilde{e}_j dx_j$ such that $D\tilde{\beta} = \beta$, $\tilde{e}_j(\{p\} \times B) = 0$ and $D\tilde{e}_j(\{p\} \times C) = 0$, $j = 1, \ldots, k$.

Set $\alpha_1 = \sum_{j=1}^{k} f_j dx_j$ where $f_j = \tilde{f}_j + \tilde{e}_j$. Again $f_j(\{p\} \times B) = 0$ and $Df_j(\{p\} \times C) = 0$, $j = 1, \ldots, k$. By construction $\alpha_1 \circ H = 0$ and $D(\alpha - \alpha_1) = 0$. Therefore there exists a function $h : U \times B \to \mathbb{K}$ such that $h(\{p\} \times B) = 0$ and $Dh = \alpha - \alpha_1$. Now $Dh \circ H = \alpha \circ H = Dh_1$ and $Dh(p, y) = \alpha(p, y) = (Dh_1 \circ \rho)(p, y)$, which proves (1), (2) and (3). Finally, note that $Dh = \sum_{j=1}^{k}(g_j - f_j) dx_j + \sum_{j=k+1}^{n} g_j dx_j$ so $D^{(2)} h(\{p\} \times C) = 0$. ∎

Beyond this point both propositions have the same proof (lemma $2'$.A assures us that $Dg_0(\{p\} \times B) = 0$).

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