SINGULARITIES AND DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 33
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1996

A SUPPLEMENT TO THE IOMDIN-LÊ THEOREM FOR SINGULARITIES WITH ONE-DIMENSIONAL SINGULAR LOCUS

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Abstract. To a germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ with one-dimensional singular locus one associates series of isolated singularities $f_N := f + l^N$, where l is a general linear function and $N \in \mathbf{N}$. We prove an attaching result of Iomdin-Lê type which compares the homotopy types of the Milnor fibres of f_N and f. This is a refinement of the Iomdin-Lê theorem in the general setting of a singular underlying space.

1. Introduction and main results. Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a function germ with 1-dimensional singular locus $\Sigma_f = \bigcup_{i\in I} \Sigma_i$, where Σ_i are the irreducible components. Let $l : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a general linear function. We denote by F the Milnor fibre of f and by F_N the Milnor fibre of the germ $f_N := f + l^N$, for $N \in \mathbf{N}$.

I. N. Iomdin proved the following:

Teorem [Io]. *If $N \gg 0$, then f_N is an isolated singularity and the Euler characteristics of F and F_N are related by*

$$\chi(F) = \chi(F_N) - N \sum_{i\in I} d_i \mu_i,$$

where μ_i is the Milnor number of the transversal singularity at some point of $\Sigma_i \setminus \{0\}$ and $d_i := \mathrm{mult}_0 \Sigma_i$.

The geometric proof given by Lê D. T. in [Lê-1] provides more information than just the Euler number formula above. Lê proves that F_N is made up from F by attaching $N \sum_{i\in I} d_i \mu_i$ cells of dimension $n - 1$. Starting from Lê's approach, we prove in a general setting a more refined attaching formula, at the homotopy type level.

1991 *Mathematics Subject Classification*: 32S55, 32S25.

The paper is in final form and no version of it will be published elsewhere.

Results related to the Iomdin-Lê theorem were obtained e.g. in [Si-1], [Si-2], [Va], [Ti-1]. The main result proved here shows that one can control (up to a certain degree) the attaching of cells and thus obtain new information in homotopy. Our proof is based on the construction developed in [Ti-2].

We first give the statement, then show some interesting consequences.

Let $(\mathbf{X}, x)$ be a complex analytic space germ of dimension n and let $f : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ be an analytic function germ. We fix on $\mathbf{X}$ some Whitney stratification $\mathcal{S} := \{\mathcal{S}_j\}_{j\in\Lambda}$. There is, however, a canonical one, as proved by J. Mather.

Definition. Let Σ_{f_j} be the (possibly void) germ of the closure of the singular locus of the restriction $f_j := f_{|\mathcal{S}_j}$. Define the *singular locus of f* by

$$\operatorname{Sing} f := \bigcup_{j\in\Lambda} \Sigma_{f_j}.$$

This definition depends on $\mathcal{S}$, but one may take as $\mathcal{S}$ the canonical stratification and then $\operatorname{Sing} f$ becomes intrinsic.

One easily shows that $\operatorname{Sing} f \subset f^{-1}(0)$.

Suppose from now on that f has 1-dimensional singular locus, i.e. $\dim \operatorname{Sing} f = 1$. Let then $\operatorname{Sing} f = \bigcup_{i\in I} \Sigma_i$ be the decomposition into irreducible curves. We denote by d_i the multiplicity of Σ_i at $\{x\}$.

The following well-known fact can be proved for instance by methods developed in §2.

Fact. *Take a transversal slice H to Σ_i at some point $a \in \Sigma_i \setminus \{0\}$. Then the Milnor fibre F_i of the restriction $f_{|H} : (H, a) \to (\mathbf{C}, 0)$ does not depend on H or a, up to homeomorphism.*

We denote the Milnor number of F_i by μ_i, the cone over F_i by $C(F_i)$ and the suspension over F_i by $S(F_i)$. It will be shown later that F_i is naturally embedded in F.

Let Ω_f be a Zariski open dense subset of "general" linear functions $l : (\mathbf{X}, x) \to (\mathbf{C}, 0)$, in a sense to be precised later. Let $f_N := f + l^N$, for $N \in \mathbf{N}$. Our main result is the following.

Theorem. *Let $f : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ have a 1-dimensional singular locus. Then, for $N \gg 0$, f_N is an isolated singularity (i.e. $\operatorname{Sing} f_N = \{x\}$) and one has the following homotopy equivalence:*

$$F_N \overset{\text{ht}}{\simeq} (F \cup E) \bigvee_{i\in I} \bigvee_{\# M_i} S(F_i),$$

where $\# M_i = N d_i - 1$, $E := \bigcup_{i\in I} C(F_i)$ and $F \cup E$ is the result of attaching to F the cones $C(F_i)$ over $F_i \subset F$, $\forall i \in I$.

The proof will be given later, we now discuss some consequences.

Grothendieck introduced the notion of *rectified homotopical depth*. Lê shows in [Lê-4] the close relation between two properties of a space $(\mathbf{X}, x)$: having maximal rectified homotopical depth, i.e. $\operatorname{rhd}(\mathbf{X}, x) = n$, and the constant sheaf $\mathbf{C}^\bullet_{\mathbf{X}}$ being *perverse* (middle perversity). In particular, the former implies the latter.

Corollary 1. *If* $\operatorname{rhd}(\mathbf{X}, x) = n$, *then*

$$F_N \overset{\text{ht}}{\simeq} (F \cup E) \bigvee_{\#I} \bigvee_{\#M_i} S^{n-1}$$

and $F \cup E$ *is homotopy equivalent to a* (*possibly trivial*) *bouquet of spheres* S^{n-1}.

Corollary 2. *If* $\mathbf{C}^\bullet_{\mathbf{X}}$ *is perverse* (*e.g. if* $(\mathbf{X}, x)$ *is a complete intersection*), *then*

$$b_{n-1}(F) \le \sum_{i \in I} \mu_i.$$

Proof. If $\operatorname{rhd}(\mathbf{X}, x) = n$, then for each $i \in I$, F_i is a bouquet of spheres of dimension $n-2$ and F_N is a bouquet of spheres of dimension $n-1$, by [Lê-4], hence $F \cup E$ must itself be a bouquet of such spheres. Corollary 2 is a direct consequence of the first one. ∎

Corollary 1 is much stronger than the attaching result obtained by Vannier [Va] in the smooth case $\mathbf{X} = \mathbf{C}^n$. Our improvement is in quality: the presence of the bouquet of spheres was not known before. In homology, the inequality in Corollary 2 is not surprising, since more recently D. Siersma proved, still on a smooth underlying space, an even sharper bound [Si-2]:

$$b_{n-1}(F) \le \sum_{i \in I} \operatorname{Ker}(h_i - \mathbf{I}),$$

where h_i is the monodromy of the isolated singularity with Milnor fibre F_i, when turning around $\{x\}$ along a simple loop contained in $\Sigma_i \setminus \{x\}$.

2. Polar curves and geometric monodromy. Let $g : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ be any function. One regards $(\mathbf{X}, x)$ as embedded in $(\mathbf{C}^m, 0)$, for some sufficiently large $m \in \mathbf{N}$.

By [Lê-3], there is a topological fibration $g_{|\mathbf{X} \cap B} : \mathbf{X} \cap B \setminus g^{-1}(0) \to D \setminus \{0\}$ induced by g, where B is a small ball at $0 \in \mathbf{C}^m$ and D is a small enough disc centred at 0. One calls it the *Milnor fibration* of g. Let $l : (\mathbf{X}, x) \to (\mathbf{C}, 0)$ be a linear function. Let $\operatorname{Crt}\Phi$ be the critical locus, with respect to the fixed Whitney stratification $\mathcal{S}$, of the map

$$\Phi := (l, g) : \mathbf{X} \to \mathbf{C}^2.$$

We denote by $\Gamma(l, g)$ the closure of the set $\operatorname{Crt}\Phi \setminus \operatorname{Sing} g$.

By [Lê-3], there is an open dense subset of linear forms l such that $\Gamma(l, g)$ is a curve or it is void and that $l^{-1}(0) \cap \operatorname{Sing} g = \{x\}$. We denote such a set by $\hat{\Omega}_g$. The curve $\Gamma(l, g)$ is called the *polar curve* of g with respect to l, relative to $\mathcal{S}$. For our previous function f with 1-dimensional singular locus, we can prove the following:

Lemma. *If* $l \in \widehat{\Omega}_f$ *then*, *for* $N \gg 0$, *the function* f_N *has an isolated singularity.*

Proof. We prove that the restriction $f_{N|\mathcal{S}_i}$ is nonsingular, for any stratum $\mathcal{S}_i \in \mathcal{S}$ of dimension ≥ 1.

By a local change of coordinates at $p \in \mathcal{S}_i$, one may assume that l is the first coordinate x_1. If $p \notin \Gamma(l, f) \cup \operatorname{Sing} f$ then the germ $f_{N|\mathcal{S}_i}$ is clearly nonsingular, hence we only have to prove the assertion at some point $p \in \Gamma(l, f) \cup \operatorname{Sing} f$, $p \ne x$, arbitrarily close to x. Denote by $\partial(f_N)$ the Jacobian ideal of f_N in the chosen coordinates. If $p \in \operatorname{Sing} f$ then

$V(\partial(f_N)) \cap \operatorname{Sing} f = V(l) \cap \operatorname{Sing} f = \{x\}$, by the definition of $\hat{\Omega}_f$. Since $p \neq x$, we get a contradiction.

If now $p \in \Gamma(l, f)$ then $(\partial f_N/\partial x_1)(p) = (\partial f/\partial x_1 + N x_1^{N-1})(p)$ can be equal to 0 for at most one value of N. (If not, then $x_1(p) = 0$, which would again contradict the definition of $\widehat{\Omega}_f$.) This N depends on the point p, hence it is locally constant, thus constant on each component of $\Gamma(l, f)$. It follows that $V(\partial f_N) \cap \Gamma(l, f) = \{x\}$, for all N except a finite number of values. ∎

From now on, we shall only consider the subset $\Omega_g \subset \hat{\Omega}_g$ of linear forms with the property that $l^{-1}(0)$ is transversal to all Thom strata in a fixed (a_g)-stratification of $g^{-1}(0)$.

We resume Lê's carrousel construction, following [Lê-2]. Let $l \in \Omega_g$. The curve germ (with reduced structure) $\Delta(l, g) := \Phi(\Gamma(l, g))$ is called the *Cerf diagram* (of g, with respect to l, relative to $\mathcal{S}$). We use the same notation $\Gamma(l, f)$, respectively $\Delta(l, g)$ for suitable representatives of these germs. Let (u, λ) be local coordinates at $0 \in \mathbf{C}^2$.

There is a fundamental system of "privileged" open polydiscs in $\mathbf{C}^m$, centred at 0, of the form $(D_\alpha \times P_\alpha)_{\alpha \in A}$ and a corresponding fundamental system $(D_\alpha \times D'_\alpha)_{\alpha \in A}$ of 2-discs at 0 in $\mathbf{C}^2$, such that Φ induces, for any $\alpha \in A$, a mapping

$$\Phi_\alpha : \mathbf{X} \cap (D_\alpha \times P_\alpha) \to D_\alpha \times D'_\alpha$$

which is a topological fibration over $D_\alpha \times D'_\alpha \setminus (\Delta(l, g) \cup \{\lambda = 0\})$.

Moreover, g induces a topological fibration

$$g_\alpha : \mathbf{X} \cap (D_\alpha \times P_\alpha) \cap g^{-1}(D'_\alpha \setminus \{0\}) \to D'_\alpha \setminus \{0\},$$

respectively

$$g'_\alpha : \mathbf{X} \cap (\{0\} \times P_\alpha) \cap g^{-1}(D'_\alpha \setminus \{0\}) \to D'_\alpha \setminus \{0\},$$

which is fibre homeomorphic to the Milnor fibration of g, respectively to the Milnor fibration of $g_{|\{l=0\}}$. The disc D'_α has been chosen small enough such that $\Delta(l, g) \cap \partial\overline{D_\alpha} \times D'_\alpha = \emptyset$.

One builds an integrable smooth vector field on $D_\alpha \times S'_\alpha$, where $S'_\alpha := \partial\overline{D'_\alpha}$, tangent to $\Delta(l, g) \cap (D_\alpha \times S'_\alpha)$ and lifting the unit vector field of S'_α by the projection $D_\alpha \times S'_\alpha \to S'_\alpha$. The vector field on $D_\alpha \times S'_\alpha$ can be lifted by Φ_α, and this lift—which is tangent to the polar curve $\Gamma(l, g) \cap \Phi^{-1}(D_\alpha \times S'_\alpha)$—can be integrated to get a characteristic homeomorphism of the fibration induced by g_α over S'_α, hence a geometric monodromy $\mathbf{h}$ of the Milnor fibre F_g of g. We call it the (*geometric*) *carrousel monodromy*.

We fix some $\eta \in S'_\alpha$ and denote $\mathbf{D} = \mathbf{D}(l, g) := D_\alpha \times \{\eta\}$. Let

$$l_\alpha : \mathbf{X} \cap \Phi_\alpha^{-1}(\mathbf{D}) \to \mathbf{D}$$

be the restriction of Φ_α and notice that F_g is homeomorphic to $l_\alpha^{-1}(\mathbf{D})$.

The integration of the vector field on $D_\alpha \times S'_\alpha$ produces a homeomorphism $h : \mathbf{D} \to \mathbf{D}$ which we call the *carrousel* of the disc $\mathbf{D}$: the trajectory inside $D_\alpha \times S'_\alpha$ of some point $a \in \mathbf{D}$ is such that after one turn around the circle S'_α we get another point $a' := h(a) \in \mathbf{D}$. By construction, the vector field restricted to $\{0\} \times S'_\alpha$ is the unit vector field of S'_α, hence the centre $(0, \eta)$ of the carrousel disc $\mathbf{D}$ is indeed fixed; the circle $\partial\overline{\mathbf{D}}$ is also pointwise fixed.

The distinguished points $\Delta(l, g) \cap \mathbf{D}$ of the disc have a complex motion around $(0, \eta)$, depending on the Puiseux parametrizations of the branches of Δ. Let Δ_i be such a branch and consider a Puiseux parametrization of it, in coordinates (u, λ): $u = \sum_{j \geq m_i} c_{i,j} t^j$, $\lambda = t^{n_i}$, where

$$m_i := \mathrm{mult}_0 \Delta_i, \qquad n_i := \mathrm{mult}_0(\Delta_i, \{\lambda = 0\}).$$

Let $\rho_i := m_i / n_i$ be the *Puiseux ratio* of Δ_i and notice that $\rho_i \leq 1$, since l is general.

Lê D. T. defines the *polar filtration* of the disc $\mathbf{D}$ as follows. Assume that the Puiseux ratios are decreasingly ordered: $\rho_1 \geq \rho_2 \geq \ldots$ Then there is a corresponding sequence of open discs $D_1 \subseteq D_2 \subseteq \ldots \subset \mathbf{D}$ centred at $(0, \eta)$, where $D_i = D_{i+1}$ if and only if $\rho_i = \rho_{i+1}$, such that $\Delta_{i+1} \cap \mathbf{D} \subset D_{i+1} \setminus \overline{D_i}$, for $i > 1$ and $\Delta_1 \cap \mathbf{D} \subset D_1$.

In each annulus $A_i := D_i \setminus \overline{D_{i-1}}$, the carrousel is not that easy to describe (we refer to [Lê-2], [Ti-1], [Ti-2] for details), but at the "first approximation", each point is rotated by $2\pi\rho_i$. There must be a continuous transition between successive annuli: within a thin enough annulus containing the circle $\overline{A_i} \cap \overline{A_{i+1}}$ each point will have a carrousel movement which is exactly a rotation by $2\pi\rho(r)$, where r is the radius to that point and $\rho(r)$ is a continuous decreasing real function with values in the interval $[\rho_{i+1}, \rho_i]$. Then Lê proves the following result, see [Si-1] for more details:

PROPOSITION. *Let $l \in \Omega_f$ and let $\{\rho_i, i \in K\}$ be the set of polar ratios of $\Delta(l, f)$, where $\Delta(l, f) = \bigcup_{i \in K} \Delta_i$ is the decomposition into irreducible curves. If $N > 1/\rho_i$, $\forall i \in K$, then the polar filtrations of (l, f_N) and (l, f) are the same, except that (l, f_N) has one disc more. Consequently, the Milnor fibre F is naturally embedded in the Milnor fibre F_N.*

Proof (sketch). The technique is due to Lê. One notices that $\Delta(l, f_N)$ has all the polar quotients of $\Delta(l, f)$ and additionally one, namely $1/N$. This is because the singular locus of f becomes a component of the polar curve $\Gamma(l, f_N)$, hence its image by the map (l, f_N) is the supplementary component of $\Delta(l, f_N)$.

Replace g by f in the definition of the map Φ. We look at the image of Φ in coordinates (u, λ). Then $F = \Phi^{-1}(\mathbf{D}) = \Phi^{-1}\{\lambda = \eta\}$ and $F_N = \Phi^{-1}(\lambda + u^N = \eta)$. Using the 1-parameter deformation $\lambda + \varepsilon u^N = \eta$, where $\varepsilon \in [0, 1]$, one constructs a nonsingular vector field, tangent to Δ and such that, by integrating it, one defines an embedding of the carrousel disc $\mathbf{D}$ of (l, f) into $\{\lambda + u^N = \eta\}$, see Figure 1. The image of this embedding does not intersect $\Phi(\mathrm{Sing}\, f)$. Therefore one can identify the carrousel disc $\mathbf{D}(l, f)$ with the disc just before $\mathbf{D}(l, f_N)$ in the increasing polar filtration of the carrousel disc $\mathbf{D}(l, f_N)$. The vector field on $\mathbf{C}^2$ can be lifted, then integrated to give an embedding $F \subset F_N$, which is actually the lift of the embedding $\mathbf{D}(l, f) \subset \mathbf{D}(l, f_N)$.

We intend to use the constructions developped in [Ti-2] in order to prove that F_N is obtained from F by a controlled attaching of cells. We need some more notation.

Let $\Phi_N := (l, f_N) : (\mathbf{X}, x) \to (\mathbf{C}^2, 0)$ be the map obtained by replacing g by f_N in the definition of Φ. We saw that $F \subset F_N$ and that $F_N \setminus F = \Phi_N^{-1}(\mathbf{D}(l, f_N) \setminus \mathbf{D}(l, f))$, where $\mathbf{D}(l, f)$ is identified with the disc just before $\mathbf{D}(l, f_N)$ in the increasing polar filtration of $\mathbf{D}(l, f_N)$. Denote by A the annulus $\mathbf{D}(l, f_N) \setminus \mathbf{D}(l, f)$ and by Δ_Σ the component of $\Delta(l, f_N)$ which comes from the singular locus of f; this has the following parametrization: $u = t$, $\lambda = t^N$. Then $A \cap \Delta(l, f_N) = A \cap \Delta_\Sigma$ is a set of N points equally distributed on a circle included in A.

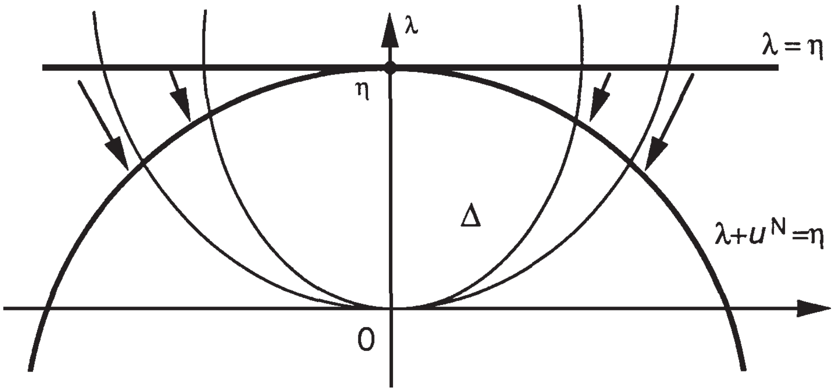

Fig. 1

R e m a r k. The carrousel movement of any point in A is a rotation by $2\pi\frac{1}{N}$.

We next define a "good" system of paths, see Figure 2. Let $\delta_1, \ldots, \delta_N$ be small enough equal discs included in A, centred at the N points of intersection $A \cap \Delta_S$. Let $b \in \partial\overline{\mathbf{D}(l, f_N)}$ and define a path from b to $a \in \partial\overline{\mathbf{D}(l, f)}$ along a radius. Assume that, when rotating this path counterclockwise, it intersects the discs $\delta_1, \ldots, \delta_N$ in this order. Then define a path γ_1 from b to some $b_1 \in \partial\overline{\delta_1}$, say a segment.

By [Ti-2], the set of paths $\{\gamma_{i+1} := h^i(\gamma_1) \mid i \in \{0, N-1\}\}$ is a non-selfintersecting system of paths from b to $b_{i+1} := h^i(b_1)$, respectively.

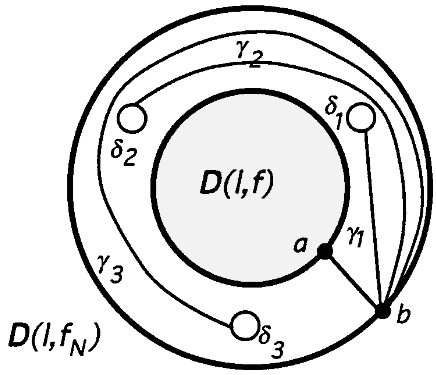

Fig. 2

Definition. For $i \in I$, $k \in \{1, \ldots, d_i\}$ and $j \in \{1, \ldots, N\}$, let $B_{i,j,k}$ be a small Milnor ball centred at the point $b_{i,j,k} \in \Gamma(l, f_N) \cap \Phi_N^{-1}(\delta_j)$. Let $F_{i,j,k} := B_{i,j,k} \cap \Phi_N^{-1}(b_j)$ be the local Milnor fibre of the germ at $b_{i,j,k}$ of the function $l : \Phi_N^{-1}(A) \to A$. Since l is trivial over γ_i, we may fix a trivialization. Then the union of the cone $C(F_{i,j,k})$ over $F_{i,j,k}$ and the mapping cylinder $F_{i,j,k} \times \gamma_i$ along the just named trivialization is called the *generalized thimble* on $\Phi_N^{-1}(b)$ associated to $F_{i,j,k}$.

It is now clear that the Milnor fibre F_N is obtained, up to homotopy, as follows: add to the fibre $\Phi_N^{-1}(b)$ all the generalized thimbles, then transport this fibre along the path from b to a (using some trivialization). Hence F_N is built up from F by attaching all those generalized thimbles. Moreover, we have proved in [Ti-2] the following result. (Notice that $F_{i,j,k}$ is homeomorphic to F_i, for any j and k.)

PROPOSITION [Ti-2]. *There is the following homotopy equivalence*:

$$F_N \overset{\text{ht}}{\simeq} (F \cup_{i\in I} \cup_{d_i} C(F_i)) \bigvee_{i\in I} \bigvee_{d_i(N-1)} S(F_i),$$

where each union $F \cup_{d_i} C(F_i)$ is the result of attaching the d_i generalized thimbles associated to $F_{i,1,k}$, for $k \in \{1, \ldots, d_i\}$, as described above.

P r o o f (main lines). We have to prove that, for each $i \in I$, a number of $d_i(N-1)$ thimbles are attached over something contractible, and therefore become suspensions over the corresponding fibres $F_{i,j,k}$.

We have a good control over the attaching, given by the use of the carrousel h. One may assume that the restriction of the geometric monodromy $\mathbf{h}$ on $\Phi_N^{-1}(b)$ is the identity (this comes from the fact that f_N is an isolated singularity). Furthermore, one may also assume that, by definition, the thimble associated to $F_{i,j+1,k}$ is obtained as the transformation by $\mathbf{h}$ of the thimble associated to $F_{i,j,k}$.

Now fix i and k and attach the cones $C(F_{i,j,k})$, through their thimbles, in the order given by the index j. It follows that the attaching of $C(F_{i,j+1,k})$ is made over the already attached cone $C(F_{i,j,k})$. A cone is contractible, thus the conclusion follows. ∎

3. Proof of the Theorem. By the preceding Proposition, it remains to prove that, for each $i \in I$, the attaching to F of the d_i cones $C(F_{i,1,k})$, $k \in \{1, \ldots, d_i\}$, gives a homotopy equivalence:

$$F \cup \bigcup_{k\in\{1,\ldots,d_i\}} C(F_{i,1,k}) \overset{\text{ht}}{\simeq} (F \cup C(F_{i,1,1})) \bigvee_{k\in\{2,\ldots,d_i\}} S(F_{i,1,k}).$$

Recall that the carrousel h, when iterated N times, will fix the small discs δ_j in the annulus A. In particular $\mathbf{h}^N$ is an automorphism of $\Phi_N^{-1}(b_j)$, $\forall j \in \{1, \ldots, N\}$.

The crucial observation is that we may take, by definition,

$$b_{i,1,k+1} := \mathbf{h}^N(b_{i,1,k}) \quad \text{and} \quad F_{i,1,k+1} := \mathbf{h}^N(F_{i,1,k}),$$

and that the fibres $F_{i,1,k}$, $k \in \{1, \ldots, d_i\}$, are pairwise disjoint. This is due to the geometric monodromy $\mathbf{h}$, obtained by integrating a vector field tangent to the trace of the polar curve

$$\Gamma(l, f_N) \cap \Phi_N^{-1}(\mathbf{D}(l, f_N) \times S'_\eta).$$

Therefore the cones $C(F_{i,j,k})$ are also related by

$$C(F_{i,j,k+1}) = \mathbf{h}^N(C(F_{i,j,k})), \qquad \forall k \in \{1, \ldots, d_i - 1\},\ \forall j \in \{1, \ldots, N\}.$$

Let us denote by $\Gamma_\Sigma := \Gamma(l, f_N) \cap \Phi_N^{-1}(\Delta_\Sigma)$ the component of the polar curve which comes from the singular locus of f and let $\Gamma_\Sigma = \bigcup_{i\in I} \Gamma_i$ be its decomposition into branches. Such a branch Γ_i comes from the branch Σ_i of $\operatorname{Sing} f$. Recall that the restriction $\Phi_{N|\Gamma_i} : \Gamma_i \to \Delta_\Sigma$ is a covering of degree d_i ramified over $\{0\}$.

We have to be more precise in the definition of the vector field on $\mathbf{D}(l, f_N)$, hence on the carrousel movement. We may assume that the circle $\partial\overline{\mathbf{D}(l,f)}$ is pointwise fixed by the carrousel, say its radius is r_0. Then consider a sufficiently thin closed annulus $A_{[r_0,r_0+\varepsilon]}$ (i.e. its interior circle is of radius r_0 and the exterior one of radius $r_0+\varepsilon$), where $\varepsilon > 0$ is very small, such that $A_{[r_0,r_0+\varepsilon]} \cap \Delta_\Sigma = \emptyset$. One requires that on the exterior circle of $A_{[r_0,r_0+\varepsilon]}$ the carrousel movement is a rotation by $2\pi\frac{1}{N}$, see the previous Remark.

The annulus $A_{[r_0,r_0+\varepsilon]}$ is the "transition zone" where one defines a continuous transition between the carrousel speed 0 (on the interior circle) and the carrousel speed $2\pi\frac{1}{N}$ (on the exterior one).

We fix a path from the point b_1 to the centre of the disc $\mathbf{D}(l, f_N)$, say along a radius. We call α the piece of this path from b_1 to the intersection c_1 with the exterior circle of $A_{[r_0,r_0+\varepsilon]}$ and β the piece from c_1 to the intersection d_1 with $\partial\overline{\mathbf{D}(l,f)}$. We then fix a trivialisation τ of Φ_N along the composed path $\beta\circ\alpha$ and transport by τ the fibre $\Phi_N^{-1}(b_1)$ along $\beta\circ\alpha$. Denote by $F'_{i,1,k}$ (resp. $F''_{i,1,k}$) the image by τ of $F_{i,1,k}$ into $\Phi_N^{-1}(c_1)$ (resp. $\Phi_N^{-1}(d_1)$).

Notice that $F'_{i,1,k}$ (resp. $F''_{i,1,k}$), $k \in \{1,\dots,d_i\}$, are also pairwise disjoint.

Now the Milnor fibres $F'_{i,1,k}$ are cyclically permuted by the iterated carrousel monodromy $\mathbf{h}^N$, but for the fibres $F''_{i,1,k} \subset \Phi_N^{-1}(d_1)$ we have the following

LEMMA. *The action of* $\mathbf{h}$ *on* $\Phi_N^{-1}(d_1)$ *is the identity on* $\Phi_N^{-1}(d_1) \setminus \bigcup_k F''_{i,1,k}$ *and an automorphism of each* $F''_{i,1,k}$, $k \in \{1,\dots,d_i\}$.

Proof. Let $\{d'_1\}\times D'_\alpha$ be the disc in $D_\alpha \times D'_\alpha \subset \mathbf{C}^2$ which contains d_1 on its boundary. By the fact that the carrousel fixes the point d_1, the action of $\mathbf{h}$ on $\Phi_N^{-1}(d_1)$ is exactly the monodromy of the fibration on a circle $\Phi_N^{-1}(\{d'_1\}\times\partial\overline{D'_\alpha}) \to \{d'_1\}\times\partial\overline{D'_\alpha}$.

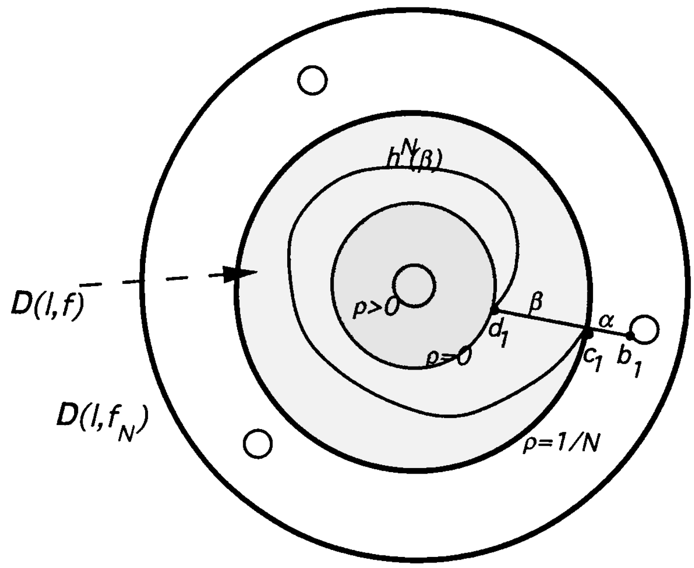

Fig. 3

Since $\Delta \cap \{d'_1\} \times D'_\alpha = \Delta_\Sigma \cap \{d'_1\} \times D'_\alpha$ is a single point, say q, the only singular fibre in $\Phi_N^{-1}(\{d'_1\} \times D'_\alpha)$ is $\Phi_N^{-1}(q)$. Therefore, the monodromy of the fibration $\Phi_N^{-1}(\{d'_1\} \times \partial \overline{D'_\alpha})$ is isotopic to the monodromy of the fibration over a small circle $\partial\overline{\delta}$ in $\{d'_1\} \times D'_\alpha$, centred at q.

The singularities of the fibre $\Phi_N^{-1}(q)$ are isolated, namely they are the set $\Gamma_i \cap \Phi_N^{-1}(q)$. In turn, this latter monodromy splits into local monodromies of the Milnor fibres of the local singularities $\Gamma_i \cap \Phi_N^{-1}(q)$, since the restriction of $\mathbf{h}$ to $\Phi_N^{-1}(\delta) \setminus \bigcup_{k\in\{1,\ldots,d_1\}} B_k$ is a trivial bundle over δ, where B_k is a Milnor ball at a point in $\Gamma_i \cap \Phi_N^{-1}(q)$. ∎

We apply the iterated monodromy $\mathbf{h}^N$ to the trivialisation τ over the path β and get a trivialisation $\mathbf{h}^N(\tau)$ over the path $\mathbf{h}^N(\beta)$, see Figure 3.

In particular, we have a mapping cylindre from $F'_{i,1,k+1}$ to $\mathbf{h}^N(F''_{i,1,k}) = F''_{i,1,k}$, by combining with the above Lemma.

Finally, let us notice that the fibre F can be identified with $\Phi_N^{-1}(\mathbf{D}(l,f) \cup A_{[r_0,r_0+\varepsilon]})$, since $\Phi_N^{-1}(\mathbf{D}(l,f) \cup A_{[r_0,r_0+\varepsilon]})$ retracts to $\Phi_N^{-1}(\mathbf{D}(l,f))$ homeomorphically.

The conclusion becomes now visible: the attaching to F of the cone $C(F'_{i,1,k+1})$ is, by using the trivialisation $\mathbf{h}^N(\tau)$ over the path $\mathbf{h}^N(\beta)$, the attaching of a cone over $F''_{i,1,k}$. But one has already attached, at one step before, a cone $C(F''_{i,1,k})$. Thus the new attaching is over something contractible (i.e. over the base of a cone) and this concludes our proof.

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