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WHITNEY STRATIFICATION OF SETS DEFINABLE IN THE STRUCTURE $\mathbb{R}_{\exp}$

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Abstract. The aim of this paper is to prove that every subset of $\mathbb{R}^n$ definable from addition, multiplication and exponentiation admits a stratification satisfying Whitney's conditions a) and b).

1. Preliminaries. Let $\mathcal{A}_n$ be the smallest ring of real-valued functions on $\mathbb{R}^n$ such that:

(a) $\mathcal{A}_n$ contains all polynomials, i.e. $\mathbb{R}[x_1, \ldots, x_n] \subset \mathcal{A}_n$.
(b) $\mathcal{A}_n$ is closed under taking exponentiation, i.e. if $f \in \mathcal{A}_n$, then $\exp f \in \mathcal{A}_n$.

1.1. Definition. Let $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ be the smallest class of subsets of Euclidean spaces $\mathbb{R}^n, n \in \mathbb{N}$, where $\mathcal{D}_n$ is a class of subsets of $\mathbb{R}^n$, satisfying the following properties for all n:

(D1) $\mathcal{D}_n$ contains all sets of the form $\{x \in \mathbb{R}^n : f(x) = 0\}$, where $f \in \mathcal{A}_n$.
(D2) If $S, T \in \mathcal{D}_n$, then $S \cup T$, $S \cap T$ and $S \setminus T \in \mathcal{D}_n$.
(D3) If $S \in \mathcal{D}_{n+1}$, then $\pi(S) \in \mathcal{D}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the natural projection.

A set S is called a *$\mathcal{D}_n$-set* iff $S \in \mathcal{D}_n$. A *$\mathcal{D}$-set* is a $\mathcal{D}_n$-set for some $n \in \mathbb{N}$. A function $f : S \to \mathbb{R}$ is called a *$\mathcal{D}$-function* iff its graph is a $\mathcal{D}$-set.

Remark. The class $\mathcal{D}$ contains all semi-algebraic sets. A $\mathcal{D}$-set, in general, is not subanalytic (e.g. $\{(x, y) : x > 0,\ y = \exp(-1/x)\}$). If f is a $\mathcal{D}$-function, then so is $\exp f$. If, in addition, $f > 0$, then $\log f$, f^α $(\alpha \in \mathbb{R})$ are $\mathcal{D}$-functions. The closure, the interior and the boundary in $\mathbb{R}^n$ of a $\mathcal{D}_n$-set are $\mathcal{D}_n$-sets.

The following theorem is due to Wilkie [9], [10], which is an essential result for the class $\mathcal{D}$.

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1.2. THEOREM (Wilkie). *Let $S \subset \mathbb{R}^n$ be a $\mathcal{D}$-set. Then there exists $f \in \mathcal{A}_{n+m}$, for some $m \in \mathbb{N}$, such that $S = \pi(f^{-1}(0))$, where $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the natural projection.*

Combining the theorem with a Khovanskiĭ result on fewnomials [4] it follows that every $\mathcal{D}$-set has only finitely many connected components.

1.3. DEFINITION (cf. [2]). (i) A map $f : S \to \mathbb{R}^m$ with $S \subset \mathbb{R}^n$ is called a *$\mathcal{D}$-map* if its graph belongs to $\mathcal{D}_{n+m}$. In this case it is called *$\mathcal{D}$-analytic* if there is an open neighborhood U of S in $\mathbb{R}^n$, $U \in \mathcal{D}_n$ and an analytic $\mathcal{D}$-map $F : U \to \mathbb{R}^m$ such that $F|_S = f$.

(ii) *$\mathcal{D}_n$-analytic cells* in $\mathbb{R}^n$ are defined by induction on n: $\mathcal{D}_1$-analytic cells are points $\{r\}$ or open intervals (a,b), $-\infty \le a < b \le +\infty$. If C is a $\mathcal{D}_n$-analytic cell and $f, g : C \to \mathbb{R}$ are $\mathcal{D}$-analytic such that $f < g$, then

$$\begin{aligned}
(f,g) &:= \{(x,r) \in C \times \mathbb{R} : f(x) < r < g(x)\},\\
(-\infty, f) &:= \{(x,r) \in C \times \mathbb{R} : r < f(x)\},\\
(g, +\infty) &:= \{(x,r) \in C \times \mathbb{R} : g(x) < r\},\\
\Gamma(f) &:= \operatorname{graph} f \text{ and } C \times \mathbb{R} \text{ are } \mathcal{D}_{n+1}\text{-analytic cells.}
\end{aligned}$$

(iii) A *$\mathcal{D}$-analytic decomposition* of $\mathbb{R}^n$ is defined by induction on n: A $\mathcal{D}$-analytic decomposition of $\mathbb{R}^1$ is a finite collection of intervals and points $\{(-\infty, a_1), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}$, where $a_1 < \ldots < a_k$, $k \in \mathbb{N}$. A $\mathcal{D}$-analytic decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into $\mathcal{D}_{n+1}$-analytic cells C such that the collection of all the projections $\pi(C)$ is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ (here $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the natural projection).

We say that a decomposition *partitions* S if S is a union of some cells of the decomposition.

1.4. THEOREM (van den Dries & Miller). (I_n) *For $S_1, \ldots, S_k \in \mathcal{D}_n$ there is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ partitioning $S_1, \ldots, S_k$.*

(II_n) *For every fuction $f : S \to \mathbb{R}$, $S \in \mathcal{D}_n$, there is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ partitioning S such that for each cell $C \subset S$ of the decomposition, the restriction $f|_C$ is $\mathcal{D}$-analytic.*

For the proof see [1], [2] or [6].

1.5. COROLLARY. *Let $S_1, \ldots, S_k$ be $\mathcal{D}_n$-sets. Then there is an analytic stratification of $\mathbb{R}^n$ compatible with $S_1, \ldots, S_k$. Precisely, there is a finite family $\{\Gamma^d_\alpha\}$ of subsets of $\mathbb{R}^n$ such that*:

(S1) *Γ^d_α are disjiont, $\mathbb{R}^n = \bigcup_{\alpha,d} \Gamma^d_\alpha$ and $S_i = \bigcup\{\Gamma^d_\alpha : \Gamma^d_\alpha \cap S_i \neq \emptyset\}$, $i = 1, \ldots, k$.*

(S2) *Each Γ^d_α is a $\mathcal{D}_n$-analytic cell of dimension d.*

(S3) *$\overline{\Gamma^d_\alpha} \setminus \Gamma^d_\alpha$ is a union of some cells Γ^e_β with $e < d$.*

1.6. COROLLARY. *Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{D}$-function. Then the limits $\lim_{+\infty} f$, $\lim_{-\infty} f$, $\lim_{c^+} f$ and $\lim_{c^-} f$ ($c \in \mathbb{R}$) exist in $\mathbb{R} \cup \{-\infty, +\infty\}$.*

1.7. COROLLARY (curve selecting lemma). *Let $S \subset \mathbb{R}^n$ be a $\mathcal{D}$-set and $a \in \overline{S}$ be a nonisolated point of S. Then there exists an analytic $\mathcal{D}$-map $\gamma : (0,1) \to S$ such that $\lim_{0^+} \gamma = a$.*

For the proof of the corollaries see [5], [6].

Remark. The class $\mathcal{D}$ shares many nice properties with those of semi-algebraic sets (see [1], [5], [7]).

2. Whitney stratification. In this section we prove the existence of the Whitney stratification of $\mathcal{D}$-sets. The proof is inspired by that of Łojasiewicz [8] for semianalytic sets.

Let $G_k(\mathbb{R}^n)$ denote the Grassmannian of k-dimensional vector subspaces of $\mathbb{R}^n$. Let $\langle , \rangle$ denote the scalar product with respect to the canonical base of $\mathbb{R}^n$. Then $G_k(\mathbb{R}^n)$ can be identified with the set of all $n \times n$ matrices $A \in \operatorname{Mat}(n,n)$ with $A^2 = A$, ${}^tA = A$ and $\operatorname{trace} A = k$. Therefore $G_k(\mathbb{R}^n)$ is an algebraic subset of $\mathbb{R}^{n^2}$. So it is a $\mathcal{D}$-set.

2.1. Proposition. *Let X be an analytic submanifold of $\mathbb{R}^n$ which is also a $\mathcal{D}$-set. Suppose that $\phi_1, \ldots, \phi_k : X \to \mathbb{R}^n$ are analytic $\mathcal{D}$-maps such that for all x in X the vectors $\phi_1(x), \ldots, \phi_k(x)$ generate a k-dimensional vector subspace $\Phi(x)$ of $\mathbb{R}^n$. Then the map $\Phi : X \to G_k(\mathbb{R}^n)$ is a $\mathcal{D}$-map. Consequently, if X is of dimension k, then the map*

$$\mathcal{T}_X : X \to G_k(\mathbb{R}^n) \quad \textit{defined by} \quad \mathcal{T}_X(x) = T_{X,x}$$

(*where $T_{X,x}$ denotes the tangent space of X at x*) *is a $\mathcal{D}$-map.*

Proof. Since $\Phi(x)$ is identified with the orthogonal projection of $\mathbb{R}^n$ onto $\Phi(x)$,

$$\Phi(x) \cdot h = \sum_{i=1}^{k} a_i(x,h)\phi_i(x),$$

where

$$\begin{pmatrix} a_1(x,h) \\ \vdots \\ a_k(x,h) \end{pmatrix} = A^{-1}(x) \begin{pmatrix} \langle h, \phi_1(x) \rangle \\ \vdots \\ \langle h, \phi_k(x) \rangle \end{pmatrix},$$

with $A(x)$ being the $k \times k$ matrix $(\langle \phi_i(x), \phi_j(x) \rangle)$. So the coefficients of Φ are $\mathcal{D}$-functions. This implies that Φ is a $\mathcal{D}$-map.

Let $C \subset \mathbb{R}^n$ be a $\mathcal{D}$-analytic cell of dimension k. Then, by Definition 1.3, C can be parametrized by an analytic $\mathcal{D}$-map $\phi : U \to \mathbb{R}^n$, where U is an open $\mathcal{D}$-set of $\mathbb{R}^k$. Put $\phi_i(x) = (\partial\phi/\partial y_i)(\phi^{-1}(x))$, $x \in C$, $i = 1, \ldots, k$. By the first part of the proposition, $\mathcal{T}_C$ is a $\mathcal{D}$-map. If X is of dimension k, then, by Theorem 1.4, X can be partitioned into finitely many cells C_j. It is easy to see that

$$\operatorname{graph} \mathcal{T}_X = \{(x,T) : x \in X, T = T_{X,x}\} = X \times G_k(\mathbb{R}^n) \cap \Big(\bigcup_{j:\dim C_j = k} \overline{\operatorname{graph} \mathcal{T}_{C_j}} \Big)$$

Thus $\mathcal{T}_X$ is a $\mathcal{D}$-map. ∎

2.2. Definition. Let X, Y be analytic submanifolds of $\mathbb{R}^n$ of dimensions k and l respectively. Suppose that $X \cap Y = \emptyset$ and $Y \subset \overline{X}$. Let $y \in Y$. We say that (X,Y) satisfies *Whitney's condition* a) *at* y if the following condition is satisfied:

a) For any sequence $(x_\nu)_{\nu \in \mathbb{N}}$ of points of X with $\lim x_\nu = y$, if $\lim T_{X,x_\nu} = \tau$ in $G_k(\mathbb{R}^n)$, then $\tau \supset T_{Y,y}$.

We say that (X,Y) satisfies *Whitney's condition* b) *at* y iff

b) For any pair of sequences $(x_\nu)_{\nu\in\mathbb{N}}$, $x_\nu\in X$, and $(y_\nu)_{\nu\in\mathbb{N}}$, $y_\nu\in Y$, with $\lim x_\nu = \lim y_\nu = y$, if $\lim T_{X,x_\nu}=\tau$ and the sequence of lines $\mathbb{R}(x_\nu-y_\nu)$ has a limit λ in $G_1(\mathbb{R}^n)$, then $\tau\supset\lambda$.

2.3. Remark. Let $\delta : G_l(\mathbb{R}^n)\times G_k(\mathbb{R}^n)\to\mathbb{R}$ be the function defined by

$$\delta(E,F)=\sup\{d(x,F): x\in E, \|x\|=1\},\quad E\in G_l(\mathbb{R}^n),\ F\in G_k(\mathbb{R}^n)\quad (l\le k).$$

Then δ is semialgebraic (so it is a $\mathcal{D}$-function) and $\delta(E,F)=0\Leftrightarrow E\subset F$. If $K\subset F$ is a vector subspace, then $\delta(E,F)\le\delta(E,K)$. If $E=\operatorname{graph}\eta$ and $F=\operatorname{graph}\theta$, where $\eta,\theta:\mathbb{R}^p\to\mathbb{R}^q$ are linear maps $(p+q=n)$, then $\delta(E,F)\le\|\theta-\eta\|$.

2.4. Proposition. *Under the notation of Def.* 2.2, *let* $W_{\mathrm{a}}(X,Y)$ (*resp.* $W_{\mathrm{b}}(X,Y)$) *be the set of points of* Y *at which* (X,Y) *satisfies Whitney's condition* a) (*resp.* b)). *Then* $W_{\mathrm{a}}(X,Y)$ *and* $W_{\mathrm{b}}(X,Y)$ *are* $\mathcal{D}$*-sets.*

Proof. We have

$$\begin{aligned} W_{\mathrm{a}}(X,Y)&=\{y\in Y:\forall\tau\in G_k(\mathbb{R}^n),(y,\tau)\in\overline{\operatorname{graph}\mathcal{T}_X}\Rightarrow\tau\supset T_{Y,y}\}\\ &=\{y\in Y:\forall\tau\in G_k(\mathbb{R}^n),(y,\tau)\in\overline{\operatorname{graph}\mathcal{T}_X}\Rightarrow\delta(T_{Y,y},\tau)=0\}.\end{aligned}$$

By Proposition 2.1, Remark 2.3 and Definition 1.1, $W_{\mathrm{a}}(X,Y)$ is a $\mathcal{D}$-set.

Similarly, let $V=\{(x,T,y,d)\in\mathcal{T}_X\times Y\times G_1(\mathbb{R}^n): d=\mathbb{R}(x-y)\}$. By Proposition 2.1 the map $X\times Y\ni(x,y)\mapsto\mathbb{R}(x-y)\in G_1(\mathbb{R}^n)$ is a $\mathcal{D}$-map. So V is a $\mathcal{D}$-set. Then

$$W_{\mathrm{b}}(X,Y)=\{y\in Y:\forall\tau\in G_k(\mathbb{R}^n),\forall\lambda\in G_1(\mathbb{R}^n),(y,\tau,y,\lambda)\in\overline{V}\Rightarrow\delta(\lambda,\tau)=0\}$$

is also a $\mathcal{D}$-set. ∎

2.5. Theorem. *Let* X,Y *be analytic submanifolds of* $\mathbb{R}^n$ *which are* $\mathcal{D}$*-sets. Suppose that* $X\cap Y=\emptyset$ *and* $Y\subset\overline{X}$. *Then*

$$\dim(Y\setminus W_{\mathrm{a}}(X,Y))<\dim Y\quad\textit{and}\quad\dim(Y\setminus W_{\mathrm{b}}(X,Y))<\dim Y.$$

To prove this theorem we prepare some lemmas.

2.6. Lemma (definable selection). *Let* $S\subset\mathbb{R}^p\times\mathbb{R}^m$ *be a* $\mathcal{D}$*-set and let* $\pi:\mathbb{R}^p\times\mathbb{R}^m\to\mathbb{R}^p$ *be the natural projection. Then there exists a* $\mathcal{D}$*-map* $\varrho:\pi(S)\to\mathbb{R}^p\times\mathbb{R}^m$ *such that* $\pi(\varrho(x))=x$ *for all* $x\in\pi(S)$.

Proof (cf. [1, Ch. 8, Prop. (1.2)]). Clearly, it is sufficient to prove the lemma for $m=1$. Moreover, by Theorem 1.4, we may assume that S is a cell. We define ϱ as follows:

If $S=(f,g)$, where $f,g:\pi(S)\to\mathbb{R}$ are $\mathcal{D}$-functions, let $\varrho(x)=(x,\frac{1}{2}(f(x)+g(x)))$.

If $S=(-\infty,g)$, where $g:\pi(S)\to\mathbb{R}$ is a $\mathcal{D}$-function, let $\varrho(x)=(x,g(x)-1)$.

If $S=(f,+\infty)$, where $f:\pi(S)\to\mathbb{R}$ is a $\mathcal{D}$-function, let $\varrho(x)=(x,f(x)+1)$.

If $S=\Gamma(f)$, where $f:\pi(S)\to\mathbb{R}$ is a $\mathcal{D}$-function, let $\varrho(x)=(x,f(x))$.

If $S=\pi(S)\times\mathbb{R}$, let $\varrho(x)=(x,0)$. ∎

2.7. Lemma (half wing). *Let* $S,V\subset\mathbb{R}^p\times\mathbb{R}^q$ *be* $\mathcal{D}$*-sets,* $S\cap V=\emptyset$ *and* $V\subset\overline{S}$. *Suppose that* V *is open in* $\mathbb{R}^p$, *where* $\mathbb{R}^p\equiv\mathbb{R}^p\times O\subset\mathbb{R}^p\times\mathbb{R}^q$. *Let* $\pi:\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^p$ *be the natural projection. Then there are an open* $\mathcal{D}$*-set* U *in* V, $r>0$ *and an analytic* $\mathcal{D}$*-map* $\overline{\theta}:(0,r)\times U\to S$ *such that* $\overline{\theta}(t,y)=(y,\theta(t,y))$ *and* $\|\theta(t,y)\|\le t$ *for all* $(t,y)\in(0,r)\times U$.

Proof. Let
$$A = \{(t, y, x) : 0 < t < 1, y \in V, x \in S, \|x - y\| \le t, \pi(x) = y\}.$$
Then A is a $\mathcal{D}$-set. If π_1 is the projection defined by $\pi_1(t, y, x) = y$, then $\pi_1(A)$ is dense in V (so is of dimension p). Indeed, let $y_0 \in V$ and $\delta > 0$. Then there is $\delta', 0 < \delta' < \min(\frac{1}{2}, \delta)$ such that $B(y_0, \delta') \cap \mathbb{R}^p \times O \subset V$. Since $y_0 \in \overline{S} \setminus S$, there is $x \in B(y_0, \delta') \cap S$. Then $y = \pi(x) \in B(y_0, \delta') \cap \pi_1(A)$.

Now, let π_2 be the projection defined by $\pi_2(t, y, x) = (t, y)$. Put
$$\pi_2(A)_y = \{t \in (0, 1) : (t, y) \in \pi_2(A)\}, \quad \varepsilon(y) = \inf \pi_2(A)_y, \quad y \in \pi_1(A).$$
Then ε is a $\mathcal{D}$-function and $\varepsilon(y) > 0 \Rightarrow (0, \varepsilon(y)) \cap \pi_2(A)_y = \emptyset$.

Claim 1. $\dim\{y \in \pi_1(A) : \varepsilon(y) > 0\} < p$.

Conversely, suppose that the dimension equals p. Then, by Theorem 1.4, there is an open $\mathcal{D}$-set W in $\mathbb{R}^p$, $W \subset \pi_1(A)$ on which ε is analytic and $\varepsilon > c$ for some $c > 0$. Let $y_0 \in W$ and $\delta \in \mathbb{R}, 0 < \delta < c$, such that $B(y_0, \delta) \cap \mathbb{R}^p \times O \subset W$. Then $\|\pi(x) - y\| > c > \delta$, $\forall y \in B(y_0, \delta) \cap \mathbb{R}^p \times O$, $\forall x \in S$ with $\pi(x) = y$. This contradicts the argument above. The claim is verified.

Claim 2. *If* $y \in \pi_1(A)$, $\varepsilon(y) = 0$, *then* $\pi_2(A)_y \supset (0, 1)$.

Since $\pi_2(A)_y$ is a nonempty $\mathcal{D}$-set and $0 \in \overline{\pi_2(A)_y} \setminus \pi_2(A)_y$, there is $\delta > 0$ such that $(0, \delta) \subset \pi_2(A)_y$, i.e. there is x in S, $\pi(x) = y$ and $\|x - y\| \le \delta$. So for every $t \in (0, 1), t \ge \delta$, $\|x - y\| \le t$, i.e. $t \in \pi_2(A)_y$. The claim follows.

Let $V_1 = \{y \in \pi_1(A) : \varepsilon(y) = 0\}$. Then, from Claim 1, $\dim V_1 = p$ and, from Claim 2, $\pi_2(A) \supset (0, 1) \times V_1$. By the definable selection lemma there is a $\mathcal{D}$-map $\varrho : (0, 1) \times V_1 \to A$ such that $\varrho(t, y) = (t, y, \bar{\theta}(t, y))$. That means $\bar{\theta} : (0, 1) \times V_1 \to S$ satisfies $\pi(\bar{\theta}(t, y)) = y$ and $\|\bar{\theta}(t, y) - y\| \le t$, i.e. $\bar{\theta}(t, y) = (y, \theta(t, y))$ and $\|\theta(t, y)\| \le t$.

By Theorem 1.4, with t regarded as the last coordinate, $(0, 1) \times V_1$ can be partitioned into cells such that the restriction of θ to each of the cells is analytic. Let C be a cell of the partition with $\dim \pi_3(C) = p$ (here $\pi_3(t, y) = y$). By the definition of cells, there is an analytic $\mathcal{D}$-function $f : \pi_3(C) \to \mathbb{R}$, $f > 0$, such that $(0, f)$ is a cell of the partition. This implies that there are an open $\mathcal{D}$-set U in $\pi_3(C)$ and $r > 0$ such that $f > r$ on U. Therefore, θ is analytic on $(0, r) \times U$. This finishes the proof of the lemma. ∎

2.8. Lemma. *Under the notation of the above lemma, for every* $c > 0$ *there is* $(t_c, y_c) \in (0, r) \times U$ *such that* $\|d_y\theta(t_c, y_c)\| < c$.

Proof. Let $\theta = (\theta_1, \ldots, \theta_q)$. For each $i \in \{1, \ldots, q\}$, let
$$A_i = \{(t, y) \in (0, r) \times U : \|d_y\theta_i(t, y)\| < c/\sqrt{q}\}.$$
Then $A_1, \ldots, A_q$ are open $\mathcal{D}$-sets.

Claim: $0 \times U \subset \overline{A}_i \setminus A_i$, *for all* $i \in \{1, \ldots, q\}$.

Let $y_0 \in U$, $\delta \in \mathbb{R}$, $0 < \delta < \min(d(y_0, {}^cU), c/(2\sqrt{q}))$, and $t \in (0, r), 0 < t < \delta^2/2$. For each i consider the function
$$\psi_i : U \ni y \mapsto \theta_i(t, y) + \|y - y_0\|^2 - t \in \mathbb{R}.$$

By Lemma 2.6, $-t \le \theta_i(t,y) \le t$ for all $y \in U$. We have

$$\psi_i(y) \ge -2t + \delta^2 > 0 \quad \forall y \in U, \|y - y_0\| = \delta,$$
$$\psi_i(y_0) = \theta_i(t, y_0) - t \le 0.$$

Therefore ψ_i has a critical point in $B(y_0,\delta)$, i.e. there is a $\overline{y} \in B(y_0,\delta)$ such that

$$d_y\theta_i(t,\overline{y}) + 2(\overline{y} - y_0) = 0.$$

This implies $\|d_y\theta_i(t,\overline{y})\| < 2\delta < c/\sqrt{q}$. The claim is verified.

Now, let $(A_i)_y = \{t \in (0,r) : (t,y) \in A_i\}$. For each $i \in \{1,\dots,q\}$ define

$$\varepsilon_i(y) = \begin{cases} \inf(\overline{(A_i)}_y \setminus (A_i)_y) \cap (0,r) & \text{if } 0 \in \overline{(A_i)}_y, \\ 0 & \text{if } 0 \notin \overline{(A_i)}_y. \end{cases}$$

Then ε_i is a $\mathcal{D}$-function on U and $\varepsilon_i(y) \ne 0 \Leftrightarrow (A_i)_y \supset (0, \varepsilon_i(y))$.

Since $0 \times U \subset \overline{A}_i \setminus A_i$, $\dim\{y \in U : \varepsilon_i(y) = 0\} < p$. (If not, then there is an open $\mathcal{D}$-set U_i in U on which $\varepsilon_i \equiv 0$. Let $\alpha_i : U_i \to \mathbb{R}$ be defined by $\alpha_i(y) = \inf(A_i)_y$. Then α_i is a $\mathcal{D}$-function, $\alpha_i > 0$ and $(0,\alpha_i(y)) \cap (A_i)_y = \emptyset$. By Theorem 1.4 there are an open cell $V_i \subset U_i$ and $M > 0$ such that $\alpha_i|_{V_i} \ge M$. This implies $O \times V_i \not\subset \overline{A}_i \setminus A_i$, a contradiction.) So $U \setminus \bigcup_{i=1}^q \{y \in U : \varepsilon_i(y) = 0\}$ is of dimension p. For each y in this set, $\varepsilon(y) := \min_{1\le i\le q} \varepsilon_i(y) > 0$. Thus $(\varepsilon(y)/2, y) \in A_i, \forall i \in \{1,\dots,q\}$, i.e. this point satisfies the demand of the lemma. ∎

2.9. Lemma. *Let X, Y be analytic submanifolds of $\mathbb{R}^n$ of dimensions k and p respectively. Suppose that X, Y are $\mathcal{D}$-sets, $X \cap Y = \emptyset$, $Y \subset \overline{X}$ and Y is open in $\mathbb{R}^p \equiv \mathbb{R}^p \times O$. Let $\pi : \mathbb{R}^n \equiv \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ be the natural projection. Let $W_{\mathrm{b}'}(X,Y)$ be the set of points y of Y where (X,Y) satisfies the following condition*:

b′) *For any sequence $(x_\nu)_{\nu\in\mathbb{N}}$ of points of X with $\lim x_\nu = y$, if $\lim T_{X,x_\nu} = \tau \in G_k(\mathbb{R}^n)$ and $\lim \mathbb{R}(x_\nu - \pi(x_\nu)) = \lambda \in G_1(\mathbb{R}^n)$, then $\tau \supset \lambda$.*

Then $W_{\mathrm{b}'}(X,Y)$ is a $\mathcal{D}$-set and $W_{\mathrm{a}}(X,Y) \cap W_{\mathrm{b}'}(X,Y) \subset W_{\mathrm{b}}(X,Y)$.

Proof. Similarly to the proof of Proposition 2.4 it is easy to prove that $W_{\mathrm{b}'}(X,Y)$ is a $\mathcal{D}$-set. We prove the second part of the lemma.

Let $y \in W_{\mathrm{a}}(X,Y) \cap W_{\mathrm{b}'}(X,Y)$. Define

$$F\{(x,T,y,d) : x \in X, T = T_{X,x}, y \in Y, d = \mathbb{R}(x-y)\}.$$

Let $(y,\tau,y,\lambda) \in \overline{F}$. It suffices to prove that $\lambda \subset \tau$.

By Corollary 1.7 there is a continuous $\mathcal{D}$-map

$$\gamma : [0,1] \to \mathbb{R}^n \times G_k(\mathbb{R}^n) \times \mathbb{R}^n \times G_1(\mathbb{R}^n)$$

such that γ is analytic on $(0,1)$, $\gamma(0) = (y,\tau,y,\lambda)$ and for all t in $(0,1]$, $\gamma(t) = (\gamma_1(t), T_{X,\gamma_1(t)}, \gamma_2(t), \mathbb{R}(\gamma_1(t) - \gamma_2(t)))$ with $\gamma_1(t) \in X$, $\gamma_2(t) \in Y$.

Since (X,Y) satisfies condition a) at y, $\tau \supset T_{Y,y}$.

Since (X,Y) satisfies condition b′) at y, $\tau \supset \lim_{0^+} \mathbb{R}(\gamma_1(t) - \pi(\gamma_1(t)))$ (this limit exists by Corollary 1.6). This implies $\lambda = \lim_{0^+} \mathbb{R}(\gamma_1(t) - \gamma_2(t)) \subset \tau$ because $\mathbb{R}(\gamma_1(t) - \gamma_2(t))$ is contained in the vector subspace spanned by $\mathbb{R}(\gamma_1(t) - \pi(\gamma_1(t)))$ and $\mathbb{R}(\gamma_2(t) - \pi(\gamma_1(t))) \subset T_{Y,y} = \mathbb{R}^p \times O$. ∎

2.10. LEMMA. *Let X be a $\mathcal{D}$-set which is also an analytic submanifold of $\mathbb{R}^n$. Let $y \in \overline{X} \setminus X$ and $\gamma : (0, \varepsilon) \to X$ $(\varepsilon > 0)$ be an analytic $\mathcal{D}$-function with* $\lim_{0^+} \gamma = y$. *Then* $\lim_{t \to 0^+} \delta(\mathbb{R}(\gamma(t) - y), T_{X,\gamma(t)}) = 0$.

Proof. Since $y \notin X$, $\gamma(t) \not\equiv \mathrm{const}$. So $\gamma' \not\equiv 0$. Moreover, γ' is a $\mathcal{D}$-map, and reducing ε, we can assume that $\gamma' \neq 0$. By Corollary 1.6 the limit $\lim_{0^+} \gamma'/\|\gamma'\|$ exists. This implies the curve $C = \{y\} \cup \gamma(0, \varepsilon)$ is a C^1 curve. So $\lim_{0^+} \delta(\mathbb{R}(\gamma(t) - y), \mathbb{R}\gamma'(t)) = 0$. But $\gamma'(t) \in T_{X,\gamma(t)}$, $\forall t \in (0, \varepsilon)$. Thus $\lim_{0^+} \delta(\mathbb{R}(\gamma(t) - y), T_{X,\gamma(t)}) = 0$. ∎

Proof of Theorem 2.5. Let $p = \dim Y$. By Theorem 1.4, Y can be partitioned into finitely many cells C_j. It is therefore sufficient to prove that for any j with $\dim C_j = p$, both $\dim(C_j \setminus W_{\mathrm{a}}(X, Y))$ and $\dim(C_j \setminus W_{\mathrm{b}}(X, Y))$ are smaller than p.

Moreover, the Whitney conditions are of a local nature and invariant under analytic isomorphisms, and from the definition of cells, we may assume that Y is an open $\mathcal{D}$-set in $\mathbb{R}^p \equiv \mathbb{R}^p \times O \subset \mathbb{R}^p \times \mathbb{R}^q$ $(p + q = n)$.

Proof of $\dim(Y \setminus W_{\mathrm{a}}(X, Y)) < p$. Define

$$\phi(y, t) = \sup\{\delta(\mathbb{R}^p, T_{X,x}) : x \in X, \|x - y\| \leq t\}, \qquad y \in Y, t > 0.$$

Then ϕ is a $\mathcal{D}$-function. For each $y \in Y$, $\phi(y, \cdot)$ is a bounded $\mathcal{D}$-function with respect to t. Then, by Corollary 1.6, there exists $\lim_{t \to 0^+} \phi(y, t) = f(y) \in \mathbb{R}, \forall y \in Y$. Note that $f : Y \to \mathbb{R}$ is a $\mathcal{D}$-function and $f(y) \neq 0 \Leftrightarrow y \in W_{\mathrm{a}}(X, Y)$.

Suppose, contrary to our assertion, that $\dim(Y \setminus W_{\mathrm{a}}(X, Y)) = p$. Then, from Theorem 1.4, there are an open $\mathcal{D}$-set V in Y and $c > 0$ such that $f > c$ on V. Let $S = \{x \in X : \delta(\mathbb{R}^p, T_{X,x}) \geq c\}$. Then $V \subset \overline{S} \setminus S$. By Lemma 2.7, there are an open $\mathcal{D}$-set $U \subset V$, $r > 0$ and an analytic $\mathcal{D}$-map $\bar{\theta} : (0, r) \times U \to S$ such that $\bar{\theta}(t, y) = (y, \theta(t, y))$ and $\|\theta(t, y)\| \leq t$, for all $(t, y) \in (0, r) \times U$.

From Lemma 2.8, there exists $(t_c, y_c) \in (0, r) \times U$ such that $\|d_y\theta(t_c, y_c)\| < c$. But $T_{X,\bar{\theta}(t_c,y_c)} \supset \operatorname{Im} d_y\bar{\theta}(t_c, y_c) = \operatorname{graph} d_y\theta(t_c, y_c)$, and from Remark 2.3 we have

$$\delta(\mathbb{R}^p, T_{X,\bar{\theta}(t_c,y_c)}) \leq \|d_y\theta(t_c, y_c)\| < c.$$

This is a contradiction.

Proof of $\dim(Y \setminus W_{\mathrm{b}}(X, Y)) < p$. By Lemma 2.9 it suffices to prove that $\dim(Y \setminus W_{\mathrm{b}'}(X, Y)) < p$. Define

$$\psi(y, t) = \sup\{\delta(\mathbb{R}(x - \pi(x)), T_{X,x}) : x \in X, \|x - y\| \leq t\}, \qquad y \in Y, t > 0.$$

Then ψ is a $\mathcal{D}$-function and there exists $\lim_{t \to 0^+} \psi(y, t) = g(y) \in \mathbb{R}$ for each $y \in Y$. Note that $g : Y \to \mathbb{R}$ is a $\mathcal{D}$-function and $g(y) \neq 0 \Leftrightarrow y \in W_{\mathrm{b}'}(X, Y)$.

If $\dim(Y \setminus W_{\mathrm{b}'}(X, Y)) = p$, then, by Theorem 1.4, there is an open $\mathcal{D}$-set V' in Y such that $g > c'$ on V' for some $c' > 0$.

Let $S' = \{x \in X : \delta(\mathbb{R}(x - \pi(x)), T_{X,x}) \geq c'\}$. Then $V' \subset \overline{S'} \setminus S'$. So, by Lemma 2.7, there are an open set $U' \subset V'$, $\varepsilon > 0$ and an analytic $\mathcal{D}$-map $\widetilde{\theta} : (0, \varepsilon) \times U' \to S'$ such that $\pi \circ \widetilde{\theta}(t, y) = y$. Fix $y \in U'$, define $\gamma(t) = \widetilde{\theta}(t, y)$. Then $\gamma(t) \in X$ and $\pi(\gamma(t)) = y$ for all $t \in (0, \varepsilon)$. Applying Lemma 2.10 we have

$$\lim_{0^+} \delta(\mathbb{R}(\gamma(t) - \pi(\gamma(t))), T_{X,\gamma(t)}) = 0,$$

a contradiction. ■

2.11. THEOREM (Whitney stratification). *Let $S_1, \ldots, S_k$ be $\mathcal{D}$-sets in $\mathbb{R}^n$. Then there exists a finite family $\mathcal{W} = \{\Gamma_\alpha\}$ of subsets of $\mathbb{R}^n$ satisfying* (S1)–(S3) *of Corollary* 1.5 *which has the following property*:

(W) *If $\Gamma_\alpha, \Gamma_\beta \in \mathcal{W}$, $\Gamma_\beta \subset \overline{\Gamma}_\alpha \setminus \Gamma_\alpha$, then $(\Gamma_\alpha, \Gamma_\beta)$ satisfies Whitney's conditions* a) *and* b) *at all points of Γ_β.*

Proof. We construct the families $\mathcal{W}^d$, $d = 0, \ldots, n$, by decreasing induction on d such that $\mathcal{W}^d$ has the following property:

$(*_d)$ $\quad \forall \Gamma_i, \Gamma_j \in \mathcal{W}^d, \Gamma_j \subset \overline{\Gamma}_i \setminus \Gamma_i, \dim \Gamma_j \geq d \Rightarrow W_{\mathrm{a}}(\Gamma_i, \Gamma_j) = W_{\mathrm{b}}(\Gamma_i, \Gamma_j) = \Gamma_j.$

Let $\mathcal{W}^n$ be a stratification of $\mathbb{R}^n$ compatible with $S_1, \ldots, S_k$ as in Corollary 1.5. Suppose that $\mathcal{W}^d$ is constructed ($d \geq 1$). For every $\Gamma_j \in \mathcal{W}^d$ with $\dim \Gamma_j = d - 1$ define

$$T_j = \Big(\bigcup \{\Gamma_j \setminus (W_{\mathrm{a}}(\Gamma_i, \Gamma_j) \cap W_{\mathrm{b}}(\Gamma_i, \Gamma_j)) : \Gamma_i \in \mathcal{W}^d, \Gamma_j \subset \overline{\Gamma}_i \setminus \Gamma_i\}\Big) \cap \Gamma_j.$$

Note that $\dim T_j < \dim \Gamma_j$ by Theorem 2.5.

Let $\mathcal{V}^d$ be a stratification of $\mathbb{R}^n$ into cells which is compatible with $\Gamma_j \setminus T_j$, T_j ($\Gamma_j \in \mathcal{W}^d$, $\dim \Gamma_j = d - 1$) and Γ_l ($\Gamma_l \in \mathcal{W}^d, \dim \Gamma_l < d - 1$) (such a stratification exists by Corollary 1.5). Define $\mathcal{W}^{d-1} = \{\Gamma \in \mathcal{W}^d : \dim \Gamma \geq d\} \cup \{\Gamma \in \mathcal{V}^d : \dim \Gamma \leq d - 1\}$. Then $\mathcal{W}^{d-1}$ satisfies $(*_{d-1})$. The family of cells $\mathcal{W} = \mathcal{W}^0$ is the desired stratification. ■

Since Whitney stratified spaces can be triangulated (see, for example, [3]), Theorem 2.11 implies

2.12. COROLLARY (triangulation). *Let $S \subset \mathbb{R}^n$ be a $\mathcal{D}$-set and $S_1, \ldots, S_k$ be $\mathcal{D}$-sets contained in S. Then S admits a triangulation compatible with $S_1, \ldots, S_k$, i.e. there exist a simplicial complex K and a homeomorphism $h : |K| \to S$ such that each S_i, $i = 1, \ldots, k$, is a union of some elements of $\{h(\sigma) : \sigma \in K\}$.*

Note that in [1] van den Dries proved that h can be taken to be a $\mathcal{D}$-function.

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References

[1] L. van den Dries, *Tame topology and O-minimal structures*, mimeographed notes (1991).

[2] L. van den Dries and C. Miller, *The field of reals with restricted analytic functions and unrestricted exponentiation*, Israel J. Math. (1991).

[3] R. M. Goresky, *Triangulation of stratified objects*, Proc. Amer. Math. Soc. 72 (1978), 193–200.

[4] A. G. Khovanskiĭ, *Fewnomials*, Transl. Math. Monographs 88, Amer. Math. Soc., 1991.

[5] T. L. Loi, thesis, Jagiellonian University, Kraków 1993.

[6] T. L. Loi, *Analytic cell decomposition of sets definable in the structure* $\mathbb{R}_{\exp}$, Ann. Polon. Math. 59 (1994), 255–266.

[7] —, *On the global Łojasiewicz inequalities for the class of analytic logarithmico-exponential functions*, C. R. Acad. Sci. Paris Sér. I 318 (1994), 543–548.

[8] S. Łojasiewicz, *Ensembles Semi-Analytiques*, I.H.E.S., Bures-sur-Yvette, 1965.

[9] A. J. Wilkie, *Some model completeness results for expansions of the ordered field of real numbers by Pfaffian functions*, preprint, 1991.

[10] —, *Model completeness results for expansions of the real field II: The exponential function*, manuscript, 1991.