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REMOVABLE SINGULARITIES OF SOLUTIONS OF NONLINEAR SINGULAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction. The study of singularities has been one of the main subjects of research in partial differential equations. In the case of linear equations the singularities are now pretty well understood; but in the nonlinear case there seems to be still very few studies.

In this paper I want to discuss the singularities of solutions of a class of nonlinear singular partial differential equations in the complex domain.

The class is only a model, but it helps one understand that the situation in the nonlinear case is more complicated than in the linear case.

2. Formulation. Consider the following type of nonlinear partial differential equation:

$$\text{(E)}\qquad \Big(t\frac{\partial}{\partial t}\Big)^m u = F\Big(t, x, \Big\{\Big(t\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u\Big\}_{\substack{j+|\alpha|\le m\\ j<m}}\Big),$$

where

$$m \in \{1,2,\ldots\},\quad t\in \mathbf{C},\quad x=(x_1,\ldots,x_n)\in \mathbf{C}^n,$$
$$\alpha=(\alpha_1,\ldots,\alpha_n)\in\{0,1,2,\ldots\}^n,\quad |\alpha|=\alpha_1+\ldots+\alpha_n,$$
$$\Big(\frac{\partial}{\partial x}\Big)^\alpha = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\cdots\Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n},$$

and for $F(t,x,Z)$ with $Z=\{Z_{j,\alpha}\}_{j+|\alpha|\le m,\, j<m}$ we assume the following conditions:

A_1) $F(t,x,Z)$ is holomorphic in (t,x,Z) near $(0,0,0)$;
A_2) $F(0,x,0)\equiv 0$ near $x=0$;
A_3) $(\partial F/\partial Z_{j,\alpha})(0,x,0)\equiv 0$ near $x=0$, if $|\alpha|>0$.

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R e m a r k 1. When $F(t,x,Z)$ is linear in Z, equation (E) is nothing but the Fuchsian type partial differential equation discussed in Tahara [5], [6]. When $m = 1$, it is the Briot-Bouquet type partial differential equation discussed in Gérard-Tahara [1].

Under A_1), A_2), A_3) we define

$$C(\varrho, x) = \varrho^m - \sum_{j<m} \frac{\partial F}{\partial Z_{j,0}}(0,x,0)\varrho^j$$

and denote by $\varrho_1(x), \ldots, \varrho_m(x)$ the roots of $C(\varrho, x) = 0$ in ϱ. We call these roots the *characteristic exponents* of (E). We have:

THEOREM 0 (Gérard-Tahara [2]). *If* $\varrho_i(0) \notin \{1, 2, \ldots\}$ *for any* $i = 1, 2, \ldots, m$, *then equation* (E) *has a unique solution* $u(t,x)$ *holomorphic in a neighborhood of* $(0,0)$ *and satisfying* $u(0,x) \equiv 0$.

3. Problem. By Theorem 0 we see that in the generic case equation (E) has one and only one holomorphic solution. This means that other solutions of (E) must be singular if they exist. Hence, if we want to characterize the equation by the property of solutions, we need to find the structure of all the singular solutions of (E). Thus the following problem naturally arises:

PROBLEM. *Determine all kinds of singularities which appear in the solutions of* (E).

If we could construct all the solutions of (E) explicitly, then the problem would be solved immediately.

But, in fact, it is very difficult and so it will be convenient to divide our problem into the following two parts:

(I) What kind of singularities are apparent?
(II) What kind of singularities can be admissible?

The problem (I) is the main theme of this paper. In Gérard-Tahara [2] some singular solutions are constructed explicitly, which contributes to the problem (II).

4. Result (1). We denote by:

- $\mathcal{R}(\mathbf{C} \setminus \{0\})$ the universal covering space of $\mathbf{C} \setminus \{0\}$;
- $S_\theta(\delta) = \{t \in \mathcal{R}(\mathbf{C} \setminus \{0\});\ 0 < |t| < \delta$ and $|\arg t| < \theta\}$;
- $D_r = \{x \in \mathbf{C}^n;\ |x| \le r\}$.

Put

$$\varrho^* = \max_{1 \le i \le m} \operatorname{Re} \varrho_i(0).$$

Then we have:

THEOREM 1. *If* $u(t,x)$ *is a solution of* (E) *holomorphic on* $S_\theta(\delta) \times D_r$ *for some* $\theta > 0$, $\delta > 0$, $r > 0$ *and satisfying*

$$\max_{|x|<r} |u(t,x)| = O(|t|^a) \quad (as\ t \to 0) \tag{4.1}$$

for some $a > \max\{\varrho^*, 0\}$, *then* $u(t,x)$ *is holomorphic in a full neighborhood of* $(0,0)$.

Sketch of proof. Assume that there exists a solution $u(t,x)$ of (E) satisfying (4.1) for some $a > \max\{\varrho^*, 0\}$. Put $\ell = \min\{q \in \mathbf{Z}\ ;\ q \geq a\}$. Then it is easy to see that the condition

$$F(t,x,0) = O(t^\ell) \quad (\text{as } t \to 0)$$

is satisfied and therefore equation (E) has a unique holomorphic solution $u_0(t,x)$ satisfying $u_0(t,x) = O(t^\ell)$ (as $t \to 0$). Since $a > 0$, $a > \varrho^*$ and $\ell > \varrho^*$, by applying Proposition 3 in Gérard-Tahara [2] we can obtain $u(t,x) \equiv u_0(t,x)$ near $(0,0)$. This proves Theorem 1. ∎

Remark 2. (1) From Gérard-Tahara [2, Théorème 3] we know the following. Let p be such that $\operatorname{Re} \varrho_p(0) = \varrho^*$. If

1) $\varrho_p(x)$ is holomorphic near $x = 0$,
2) $\operatorname{Re} \varrho_p(0) = \varrho^* > 0$,
3) $\varrho_1(0), \ldots, \varrho_m(0) \notin \{1, 2, \ldots\}$,

then we can construct a singular solution of (E) which is holomorphic on $S_\theta(\delta) \times D_r$ for some $\theta > 0$, $\delta > 0$, $r > 0$ and is of the form

$$u(t,x) = \sum_{i \geq 1} u_i(x) t^i + \sum_{\substack{i+2mj \geq k+2m \\ j \geq 1}} \phi_{i,j,k}(x) t^{i + j\varrho_p(x)} (\log t)^k$$

where $\phi_{0,1,0}(x)$ can be taken arbitrarily.

(2) The above singular solution $u(t,x)$ satisfies

$$u(t,0) = (\text{holomorphic part}) + O(|t|^{\varrho^*}) \quad (\text{as } t \to 0).$$

This implies that in the case $\varrho^* > 0$ the condition $a > \varrho^*$ is optimal in general.

Remark 3. If equation (E) is linear, the condition $a > \max\{\varrho^*, 0\}$ can be replaced by $a > \varrho^*$ (see Tahara [5]). But in the nonlinear case it is impossible as is seen in the following example.

Example 1. Let us consider

$$t\frac{\partial u}{\partial t} + \varrho u = tu\left(\frac{\partial u}{\partial x}\right)^k, \tag{e$_1$}$$

where $(t,x) \in \mathbf{C}^2$, $\varrho > 1/k$ and $k \geq 1$ (integer). Then (e_1) has a solution

$$u(t,x) = \left(\varrho - \frac{1}{k}\right)^{1/k} \frac{x+c}{t^{1/k}}, \qquad c \in \mathbf{C}.$$

In this case we have $\varrho^* = -\varrho < -1/k$, and

$$u(t,0) = \text{const. } t^{-1/k} \quad (\text{as } t \to 0).$$

5. Result (2). When $\varrho^* < 0$, for the singularity of the form (4.1) we have seen:

(1)(by Theorem 1) if $a > 0$, the singularity is removable;

(2)(by Example 1) if $a < 0$, the singularity is not removable in general.

The following theorem explains the situation near $a = 0$.

THEOREM 2. *If $\varrho^* < 0$ and if $u(t,x)$ is a solution of* (E) *holomorphic on $S_\theta(\delta) \times D_r$ for some $\theta > 0$, $\delta > 0$, $r > 0$ and satisfying*

$$\max_{|x|<r} |u(t,x)| = O\left(\frac{1}{(-\log|t|)^\varepsilon}\right) \quad (as\ \ t \to 0) \tag{5.1}$$

for some $\varepsilon > 0$, then $u(t,x)$ is holomorphic in a full neighborhood of $(0,0)$.

Sketch of proof. Let $u_0(t,x)$ be the unique holomorphic solution in Theorem 0 and let $u(t,x)$ be a solution of (E) satisfying (5.1) for some $\varepsilon > 0$. Since $\varrho^* < 0$, by the uniqueness theorem in Tahara [7] we can obtain $u(t,x) \equiv u_0(t,x)$ near $(0,0)$. This proves Theorem 2. ∎

Remark 4. In the case $\varrho^* = 0$, the singularity of the form (5.1) is not removable in general as is seen in the following example.

EXAMPLE 2. Let us consider

$$t\frac{\partial u}{\partial t} = Au\left(\frac{\partial u}{\partial x}\right)^k, \tag{e$_2$}$$

where $(t,x) \in \mathbf{C}^2$, $A \neq 0$ and $k \geq 1$ (integer). Then (e_2) has a solution

$$u(t,x) = \left(\frac{-1}{Ak}\right)^{1/k} \frac{x+c}{(\log t)^{1/k}}, \qquad c \in \mathbf{C}.$$

Remark 5. If $\operatorname{Re} \varrho_i(x) \leq 0$ (near $x = 0$) for any $i = 1, 2, \ldots, m$, then we can prove that the singularity of the form (5.1) with $\varepsilon > m$ is removable. Compare this with the result in Example 2.

6. Supplementary remark. In this paper I have always assumed the conditions A_1), A_2) and A_3). Among them, the condition A_3) is essential to the theorem on removable singularities of type of Theorem 1.

In this section, I will explain what happens if the condition A_3) is not satisfied.

Let us consider

$$\left(t\frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+|\alpha|\leq m \\ j<m}}\right) \tag{E$_*$}$$

and assume the condition A_1). Put

$$\begin{aligned}
\mathcal{J} &= \left\{(j,\alpha)\ ;\ \frac{\partial F}{\partial Z_{j,\alpha}}(0,x,0) \not\equiv 0 \text{ near } x = 0\right\}; \\
\mu &= \max\{|\alpha|\ ;\ (j,\alpha) \in \mathcal{J} \text{ for some } j\}; \\
\ell &= \max\{j\ ;\ (j,\alpha) \in \mathcal{J} \text{ for some } \alpha \text{ satisfying } |\alpha| = \mu\}.
\end{aligned}$$

Remark 6. The condition A_3) is equivalent to the condition $\mu = 0$.

In the case $\mu \geq 1$, we have:

PROPOSITION. *Assume* A_1) *and the following conditions*:

c_1) $F(t,x,0) \equiv 0$ *near* $(t,x) = (0,0)$;

c_2) $\mu \geq 1$;

c_3) $\ell + \mu = m$;
c_4) $(\partial F/\partial Z_{\ell,\alpha})(0,0,0) \neq 0$ *for some* α *satisfying* $|\alpha| = \mu$.

Then for any $\lambda > 0$ *sufficiently large we can find a singular solution* $u_\lambda(t,x)$ *of* (E_*) *which is holomorphic on* $S_\theta(\delta) \times D_r$ *for some* $\theta > 0$, $\delta > 0$, $r > 0$ *and is of the form*

$$u_\lambda(t,x) = t^\lambda(\phi(x) + w(t,x)),$$

where $\phi(x)$ *and* $w(t,x)$ *are holomorphic functions on* $S_\theta(\delta) \times D_r$ *satisfying*:

$$\phi(0) \neq 0; \quad \max_{|x|<r} |w(t,x)| = o(1) \quad (\textit{as } t \to 0).$$

Thus, if the condition A_3) is not satisfied, it seems difficult to get a theorem on removable singularities of type of Theorem 1.

Remark 7. When $F(t,x,Z)$ is algebraic in Z, similar singular solutions are constructed in Ishii [4]. We can prove the above proposition in the same way.

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