1. Introduction. In this lecture, we want to discuss some regularity properties of $p$-harmonic maps with values in euclidean spheres.

Let $B^n = \{ x \in \mathbb{R}^n : \sum (x_i)^2 < 1 \}$ denote the unit $n$-dimensional ball, and write $S^{m-1}$ for the unit sphere in $\mathbb{R}^m$. Define the functional
\[
I_p(u) = \int_{\mathbb{B}^n} |\nabla u(x)|^p \, dx \quad \text{for } u \in W^{1,p}(\mathbb{B}^n, S^{m-1}).
\]

For exponents $p \in [2, n]$, we wish to investigate those maps which are critical points of $I_p$ with respect both to variations in the range and in the parameter domain.

Definition. By a stationary $p$-harmonic map we mean here any $u$ belonging to the Sobolev space
\[
W^{1,p}(\mathbb{B}^n, S^{m-1}) \equiv \left\{ f = (f_1, \ldots, f_m) : f_i \in W^{1,p}(\mathbb{B}^n) \text{ and } \sum_{i=1}^m (f_i(x))^2 = 1 \text{ a.e.} \right\},
\]
and satisfying the following two conditions:
\begin{align*}
(1) \quad & \left. \frac{d}{dt} \right|_{t=0} I_p \left( \frac{u + t\psi}{|u + t\psi|} \right) = 0 \quad \text{for all } \psi = (\psi^1, \ldots, \psi^m) \in C_0^\infty(\mathbb{B}^n, \mathbb{R}^m), \\
(2) \quad & \left. \frac{d}{dt} \right|_{t=0} I_p(u(x + t\zeta(x))) = 0 \quad \text{for all } \zeta = (\zeta^1, \ldots, \zeta^n) \in C_0^\infty(\mathbb{B}^n, \mathbb{B}^n).
\end{align*}

If a map $u \in W^{1,p}(\mathbb{B}^n, S^{m-1})$ satisfies only the condition (1), we say that $u$ is weakly $p$-harmonic.

Condition (1) is easily checked to be equivalent to the fact that $u \in W^{1,p}(\mathbb{B}^n, S^{m-1})$ is a weak solution to the Euler-Lagrange elliptic system
\[
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\]

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More precisely, the integral identity
\[
\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u^i \cdot \nabla \psi^i \, dx = \int_{\mathbb{R}^n} \psi^i u^i |\nabla u|^p \, dx
\]
holds true for all \(i = 1, \ldots, n\) and \(\psi = (\psi^1, \ldots, \psi^n) \in C_0^\infty(B, \mathbb{R}^n)\). Here and everywhere below,
\[
|\nabla u|^2 = \sum_{i=1}^m \sum_{j=1}^n \left( \frac{\partial u^i}{\partial x_j} \right)^2.
\]

Another integral identity for stationary \(p\)-harmonic maps,
\[
\int_{\mathbb{R}^n} |\nabla u|^p \text{div} \zeta \, dx = p \sum_{1 \leq j \leq m} \int_{\mathbb{R}^n} |\nabla u|^{p-2} \frac{\partial u^j}{\partial x_k} \frac{\partial u^j}{\partial x_k} \, dx,
\]
which holds true for all \(\zeta \in C_0^\infty(\mathbb{B}^n, \mathbb{B}^n)\), is a consequence of (2). Note that for smooth maps \(u\) formula (5) follows from (4) if we set \(\psi^j := \zeta \cdot \nabla u^j\). By a suitable choice of the testing map \(\zeta\), one can obtain the so-called monotonicity formula for stationary \(p\)-harmonic maps,
\[
\int_{\mathbb{R}^n} |\nabla u(y)|^p \, dy \leq \int_{\mathbb{R}^n} |\nabla u(y)|^p \, dy, \quad r_1 < r_2 \leq \text{dist}(x, \partial \mathbb{B}^n).
\]
This fact was proved for Yang–Mills fields and stationary harmonic maps by Price [18]; Fuchs [8] observed that (6) is valid also for stationary \(p\)-harmonic maps.

It is also possible to define and consider stationary \(p\)-harmonic maps \(u : M^m \to N^n\) between Riemannian manifolds. We shall not pursue that point further here.

It is well known that the regularity theory of \(p\)-harmonic maps is a delicate topic. The example of the map \(u(x) = x/|x|\) from the unit ball \(\mathbb{B}^n\) to its boundary \(\partial \mathbb{B}^n \equiv S^{n-1}\), which is singular at 0 and weakly \(p\)-harmonic for all \(p \in [1, n)\), shows that weakly \(p\)-harmonic maps do not, in general, to be continuous. In fact, the state of affairs is even worse: T. Rivi`ere [19] has recently given an example of a weakly harmonic map \(u \in W^{1,2}(\mathbb{B}^3, S^2)\) which is discontinuous at every point of \(\mathbb{B}^3\).

However, there are lots of results about regularity and partial regularity of weakly \(p\)-harmonic maps under various additional assumptions. Let us mention below just a few; the list is obviously far from being complete.

Hardt and Lin [12], Fuchs [9], and Luckhaus [16] proved independently a theorem stating that minimizing \(p\)-harmonic maps \(u : M^m \to N^n\) are of class \(C^{1,\alpha}\), \(0 < \alpha < 1\), outside a set of Hausdorff dimension \(m - \lfloor p \rfloor - 1\) (that was a generalization of an earlier result of Schoen and Uhlenbeck [20] concerning the case \(p = 2\) of minimizing harmonic maps). Fuchs [8] was able to show that some partial regularity results are valid also for stationary \(p\)-harmonic maps with range contained in a small ball \(B(r_0) \subset N^n\) of radius \(r_0\) determined by the geometry of underlying manifolds.
There is also a series of recent developments which were obtained via applications of a theorem of Coifman, Lions, Meyer, and Semmes (1)—this method allows proving regularity or partial regularity without assuming that \( u \) minimizes the Dirichlet integral. In his papers [13]–[15], F. Hélein proved that any weakly harmonic map \( f : M \to N \) defined on a two-dimensional Riemannian manifold \( M \) is continuous; [13] contains the proof for \( N = S^{n-1} \), [15] concerns the case when \( N \) is a compact manifold with a Lie group of isometries acting transitively, and [14] deals with the case of arbitrary compact Riemannian \( N \). (By standard elliptic regularity methods, continuity of a weakly harmonic map implies its \( C^\infty \)-smoothness.) Evans [4] and Bethuel [1] generalized Hélein’s result to the case of stationary harmonic maps on \( n \)-dimensional manifolds, \( n \geq 2 \), proving their regularity outside a singular set of \((n-2)\)-dimensional Hausdorff measure zero. Up to now, these recent developments hardly have any counterparts for \( p \neq 2 \).

Let us now state our main results.

**Theorem 1** (case \( p = n \)). Any weakly \( n \)-harmonic map \( u \in W^{1,n}(\mathbb{B}^n, S^{m-1}) \) is locally Hölder continuous on \( \mathbb{B}^n \).

**Theorem 2.** Let \( 2 \leq p < n \) and assume that \( u \in W^{1,p}(\mathbb{B}^n, S^{m-1}) \) is a stationary \( p \)-harmonic map. Then the set \( V \subset \mathbb{B}^n \) defined by

\[
V := \left\{ x \in \mathbb{B}^n : r^p \int_{\mathbb{B}^n(x,r)} |\nabla u(y)|^p \, dy \to 0 \text{ as } r \to 0 \right\}
\]

is open, \( H^{n-p}(\mathbb{B}^n \setminus V) = 0 \), and \( u \) is locally Hölder continuous on \( V \).

Actually, M. Fuchs [7] proved these results independently and via different methods. A relatively short and direct proof of Theorem 1 can be found in [21]. Here, we would like to sketch the proof of Theorem 2.

Our proof combines earlier ideas due to Hélein and Evans with one simple observation from [21]. Namely, we note that the right-hand side of (3) is an element of the local Hardy space \( H^1_{loc} \), a proper subspace of \( L^1 \) (for \( p = 2 \) this was noticed and exploited by Hélein [13]–[15]). Due to that fact we are able to model the main part of the argument on Evans [4], introducing some modifications in order to cope with nonlinearity and degeneracy of the \( p \)-Laplace operator \( L_p(u) := \text{div}(|\nabla u|^{p-2}\nabla u) \). We exploit the duality between \( H^1(\mathbb{R}^n) \) and \( BMO(\mathbb{R}^n) \) to obtain, for the scaled energy

\[
E(x,r) = r^{p-n} \int_{\mathbb{B}^n(x,r)} |\nabla u(y)|^p \, dy,
\]

the following decay estimate.

**Theorem 3.** There exist constants (depending only on \( n \), \( m \), and \( p \)) \( \varepsilon_0 \in (0,1) \) and \( \theta \in (0,1) \) such that

\[
E(x,r) < \varepsilon_0 \Rightarrow E(x,\theta r) \leq \frac{1}{4} E(x,r)
\]

for all \( x \in \mathbb{B}^n \) and all positive \( r < d(x,\partial \mathbb{B}^n) \).

\(^{(1)}\) The theorem states that the Jacobian of a map \( u \in W^{1,n}(\mathbb{R}^n) \) belongs to the Hardy space \( H^1(\mathbb{R}^n) \).
Then we apply the Dirichlet growth theorem [11] and a covering argument to deduce Hölder continuity of \( u \) on an open subset \( V \) of \( \mathbb{B}^n \) with \( H^{n-p}(\mathbb{B}^n \setminus V) = 0 \).

### 2. Hardy space, BMO, and Fefferman–Stein duality theorem

**Definition.** A measurable function \( f \in L^1(\mathbb{R}^n) \) belongs to the Hardy space \( \mathcal{H}^1(\mathbb{R}^n) \) if and only if

\[
 f_* := \sup_{\epsilon > 0} |\varphi_\epsilon * f| \in L^1(\mathbb{R}^n).
\]

Here, \( \varphi_\epsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon) \), and \( \varphi \) is a fixed function of class \( C_0^\infty(B(0,1)) \) with \( \int \varphi(y) \, dy = 1 \). The definition does not depend on the choice of \( \varphi \) (see [6]).

The interested reader will find other equivalent definitions of \( \mathcal{H}^1(\mathbb{R}^n) \) and more details in [10] or [24]. Let us just mention here that \( \mathcal{H}^1(\mathbb{R}^n) \) is a Banach space with the norm \( \| f \|_{\mathcal{H}^1} = \| f \|_{L^1} + \| f_* \|_{L^1} \). Moreover, the condition \( f \in \mathcal{H}^1(\mathbb{R}^n) \) implies \( \int f(y) \, dy = 0 \).

C. Fefferman [5], [6] proved that the dual of \( \mathcal{H}^1(\mathbb{R}^n) \) is equal to the space of functions of bounded mean oscillation, \( BMO(\mathbb{R}^n) \). More precisely, there exists a constant \( C \) such that

\[
 (9) \quad \int_{\mathbb{R}^n} h(y)\psi(y) \, dy \leq C \| h \|_{\mathcal{H}^1} \| \psi \|_{BMO}
\]

for all functions \( h \in \mathcal{H}^1(\mathbb{R}^n) \) and \( \psi \in BMO(\mathbb{R}^n) \).

The interesting paper of S. Müller [17] inspired some of the research reported in [3], in particular the following remarkable theorem.

**Theorem 4 (Coifman, Lions, Meyer, Semmes).** Assume that \( u \in W^{1,p}(\mathbb{R}^n) \), \( 1 < p < \infty \), and that \( H \in L^{p/(p-1)}(\mathbb{R}^n, \mathbb{R}^n) \) satisfies the condition \( \text{div} \, H = 0 \) in \( \mathcal{D}'(\mathbb{R}^n) \). Then \( \nabla u \cdot H \in \mathcal{H}^1(\mathbb{R}^n) \), and

\[
 (10) \quad \| \nabla u \cdot H \|_{\mathcal{H}^1} \leq C \| \nabla u \|_{L^p} \| H \|_{L^{p/(p-1)}}
\]

for some constant \( C \) depending only on \( n \) and \( p \).

The estimate (10) was not explicitly stated in [3], but follows from the proof presented there (cf. also [4, Section 2]). Let us now make explicit a corollary of the above theorem (more or less well known to specialists).

**Corollary 5.** Let \( \Omega \) be a ball in \( \mathbb{R}^n \). Assume that \( u \in W^{1,p}(\Omega) \), \( 1 < p < \infty \), and that \( H \in L^{p/(p-1)}(\Omega, \mathbb{R}^n) \) satisfies the condition \( \text{div} \, H = 0 \) in \( \mathcal{D}'(\Omega) \). Then one can find a function \( h \in \mathcal{H}^1(\mathbb{R}^n) \) such that

\[
 h(x) = \nabla u(x) \cdot H(x), \quad x \in \Omega,
\]

and

\[
 \| h \|_{\mathcal{H}^1} \leq C \| \nabla u \|_{L^p(\Omega)} \| H \|_{L^{p/(p-1)}(\Omega)}.
\]

The constant \( C \) does not depend on the size of \( \Omega \).

**Proof.** See [21].
3. Energy decay estimates

Proof of Theorem 3. Assume that (8) is violated and for all positive \( \theta \in (0,1) \) we can find a sequence of balls \( B_k \equiv B(x_k, r_k) \subset \mathbb{B}^n \), \( k = 1, 2, \ldots \), such that

\[
E(x_k, r_k) = \lambda_k \xrightarrow{k \to \infty} 0,
\]

and at the same time

\[
E(x_k, \theta r_k) > \frac{1}{4} \lambda_k^p.
\]

We now change the variables,

\[
\mathbb{B}^n \ni z \mapsto y = x_k + r_k z \in B_k, \quad k = 1, 2, \ldots,
\]

to rescale everything to the unit ball \( \mathbb{B}^n \subset \mathbb{R}^n \). Write

\[
v_k(z) := \frac{u(x_k + r_k z) - a_k}{\lambda_k}, \quad a_k := \int_{B_k} u(y) \, dy.
\]

Using the Poincaré inequality and the classical change of variables formula, we easily obtain the following three statements:

\[
\sup_{B^n} \int_{\mathbb{B}^n} |v_k(z)|^p \, dz < +\infty,
\]

\[
\int_{\mathbb{B}^n} |\nabla v_k(z)|^p \, dz = 1 \quad \text{for all} \quad k \in \mathbb{N},
\]

\[
\theta^{p-n} \int_{\mathbb{B}^n(0, \theta)} |\nabla v_k(z)|^p \, dz > 1/4 \quad \text{for all} \quad k \in \mathbb{N}.
\]

Therefore, we may pass to a subsequence and assume without loss of generality that

\[
v_k \to v \quad \text{strongly in} \quad L^p(\mathbb{B}^n, \mathbb{R}^m) \quad \text{and a.e.,}
\]

\[
\nabla v_k \rightharpoonup \nabla v \quad \text{weakly in} \quad L^p(\mathbb{B}^n, \mathbb{R}^{mn}).
\]

What we now need to conclude the proof of Theorem 3 is the following.

Main Lemma. \(|\nabla v_k|^{(p-2)/2} \nabla v_k \xrightarrow{k \to \infty} |\nabla v|^{(p-2)/2} \nabla v| in the strong topology of \( L^2(\mathbb{B}(0,1/2)) \). Moreover, the limit function \( v \) satisfies the non-constrained \( p \)-harmonic equation, i.e.

\[
\int_{\mathbb{B}^n} |\nabla v|^{p-2} \nabla v \cdot \nabla \psi \, dy = 0
\]

for all \( \psi = (\psi^1, \ldots, \psi^m) \in C_0^\infty(\mathbb{B}^n, \mathbb{R}^m) \).

Let us first show that this fact implies Theorem 3.

Well-known results of N. Uraltseva [27] and K. Uhlenbeck [26] (see also [23]) assert that \( \nabla v \) is locally Hölder continuous on \( \mathbb{B}^n \). In particular, we have the estimate

\[
\text{ess sup}_{y \in \mathbb{B}^n(0,1/2)} |\nabla v(y)| \leq C(n, p) \left( \int_{\mathbb{B}^n} |v(y)|^p \, dy \right)^{1/p},
\]
which in view of (13) implies
\[ \theta^{p-n} \int_{\mathbb{R}^n(0,\theta)} |\nabla v(y)|^p \, dy \leq C\theta^p < 1/8, \] if only \( \theta \in (0,1/2) \) is small enough. On the other hand, the strong convergence of gradients, \( |\nabla u_k|^{(p-2)/2} \nabla u_k \to |\nabla u|^{(p-2)/2} \nabla u \), combined with (15) forces
\[ \theta^{p-n} \int_{\mathbb{R}^n(0,\theta)} |\nabla v(z)|^p \, dz \geq 1/4, \] a contradiction to (20). Therefore, the argument will be complete once we prove the Main Lemma. The proof given below is modelled on [4, Section 4], with some minor changes forced by nonlinearity of the \( p \)-Laplace operator.

Take a smooth function \( \zeta : \mathbb{R}^n \to [0,1] \). To fix ideas for a moment, assume that \( \text{supp} \zeta \subset B(0,9/16) \) and \( \zeta \equiv 1 \) on \( B(0,1/2) \). Mimicking the arguments of [4] one can use the monotonicity formula to prove the following.

**Lemma 6.** For every \( 1 \leq i \leq m \), the double sequence \( (\zeta(v^k_i - v^s_i))_{k,s \in \mathbb{N}} \) is bounded in \( \text{BMO}(\mathbb{R}^n) \).

Next, for \( k \in \mathbb{N} \) and \( 1 \leq i,j \leq m \), define the vector field \( b^i_{k,j} \in L^{p/(p-1)}(\mathbb{R}^n,\mathbb{R}^n) \) with coordinates
\[ b^i_{k,j} = |\nabla v_k|^{p-2} \left( (a^i_k + \lambda_k v_k^i) \frac{\partial v_j}{\partial x^i} - (a^j_k + \lambda_k v_k^j) \frac{\partial v_i}{\partial x^j} \right). \]
An easy straightforward calculation (see e.g. [21]) leads to

**Lemma 7.** \( \text{div} b^i_k = 0 \) in \( \mathcal{D}'(\mathbb{R}^n) \); more precisely,
\[ \sum_{\ell=1}^n \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x^\ell} b^i_{k,\ell} \, dz = 0 \]
for every function \( \phi \in W^{1,p}(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n) \) with compact support.

Combining this lemma with Corollary 5 we obtain

**Lemma 8.** The sequence of functions
\[ \sum_{\ell=1}^n b^i_{k,\ell} \frac{\partial (\zeta v^j_k)}{\partial x^\ell}, \quad k = 1, 2, \ldots, \]
is bounded in \( \mathcal{H}^1(\mathbb{R}^n) \) for each \( 1 \leq i,j \leq m \).

**Proof of the Main Lemma.** For sake of brevity, write \( G(\xi) = |\xi|^{(p-2)/2} \xi \), and let \( H(\xi) = |\xi|^{p-2} \xi \). Pick a bounded function \( \psi \in W^{1,p}(\mathbb{B}^n,\mathbb{R}^m) \) with support contained in \( \mathbb{B}^n \). Set \( \psi_k(y) := \psi((y-x_k)/r_k) \). Because \( u \in W^{1,p}(\mathbb{B}^n, S^{m-1}) \) is weakly \( p \)-harmonic, we have
\[ \int_{B_k} H(\nabla u) : \nabla \psi_k \, dy = \int_{B_k} |\nabla u|^p \psi_k \, dy. \]
Changing integration variables from $y \in B_k$ to $z \in \mathbb{B}^n$ gives
\begin{equation}
\int_{\mathbb{B}^n} H(\nabla v_k) \cdot \nabla \psi \, dz = \lambda_k \int_{\mathbb{B}^n} |\nabla v_k|^p (a_k + \lambda_k v_k) \psi \, dz.
\end{equation}

Write now the same identity with $v_k$ (resp. $a_k, \lambda_k$) replaced by $v_s$ (resp. $a_s, \lambda_s$), substitute $\psi = \zeta^2(v_k - v_s)$ in both of the resulting equalities and subtract one of them from the other to obtain
\begin{equation}
L_{k,s} \equiv \int_{\mathbb{B}^n} \zeta^2 (H(\nabla v_k) - H(\nabla v_s)) \cdot (\nabla v_k - \nabla v_s) \, dz \\
+ 2 \int_{\mathbb{B}^n} \zeta (v_k - v_s) \cdot (H(\nabla v_k) - H(\nabla v_s)) \cdot \nabla \zeta \, dz
\leq |R_{k,s}| + |R_{s,k}|,
\end{equation}
where, for $k, s \in \mathbb{N}$,
\begin{equation}
R_{k,s} = \lambda_k \sum_{1 \leq i, j, m \leq n} \int_{\mathbb{B}^n} \zeta^2 |\nabla v_k|^{p-2} \frac{\partial v_i^j}{\partial x_k} (\frac{\partial v_i^j}{\partial x_k} (a_k^i + \lambda_k v_k^i) (v_k^i - v_s^i)) \, dz.
\end{equation}

By the H"older inequality, (16) and (17), the absolute value of the second integral on the left hand side of (23) does not exceed
\[ C \|v_k - v_s\|_{L^p(\mathbb{B}^n)} \sup_{k \in \mathbb{N}} \|\nabla v_k\|_{L^p(\mathbb{B}^n)} = o(1) \quad \text{for} \ k, s \to \infty. \]

To estimate from below the first integral on the left hand side of (23), we apply the elementary inequality
\[ (H(X) - H(Y)) \cdot (X - Y) \geq \frac{1}{p} |G(X) - G(Y)|^2, \]
valid for $p \geq 2$ and for vectors $X, Y$ in any scalar product space. The calculations imply
\begin{equation}
L_{k,s} \geq \frac{1}{p} \int_{\mathbb{B}^n(0, 1/2)} |G(\nabla v_k) - G(\nabla v_s)|^2 \, dz + o(1) \quad \text{for} \ k, s \to \infty.
\end{equation}

To estimate the right hand side of (23), note that $|u| = 1$ implies
\[ \sum_{j=1}^m (a_k^j + \lambda_k v_k^j) \frac{\partial v_i^j}{\partial x_k} = 0 \quad \text{for} \ i = 1, \ldots, n \text{ and } k \in \mathbb{N}. \]

Therefore, the crucial trick of Hélein can be adapted to the case $p \neq 2$. We may use the $b_k^{ij}$ to express the integral $R_{k,s}$, and then apply Fefferman’s theorem. Here is the calculation:
\begin{align*}
R_{k,s} &= \lambda_k \sum_{1 \leq i, j, m \leq n} \int_{\mathbb{B}^n} \zeta^2 |\nabla v_k|^{p-2} \frac{\partial v_i^j}{\partial x_k} (\frac{\partial v_i^j}{\partial x_k} (a_k^i + \lambda_k v_k^i) - \frac{\partial v_i^j}{\partial x_k} (a_s^i + \lambda_s v_s^i)) (v_k^i - v_s^i) \, dz \\
&= \lambda_k \sum_{1 \leq i, j, m \leq n} \int_{\mathbb{B}^n} \zeta^2 \frac{\partial v_i^j}{\partial x_k} b_k^{ij} (v_k^i - v_s^i) \, dz.
\end{align*}
\[ = \lambda_k \sum_{1 \leq i, j \leq m} \int \left( \sum_{l=1}^n b_{k,l}^i \frac{\partial (\zeta v_k^j)}{\partial x_l} \right) (v_k^i - v_j^i) \, dz \]
\[- \lambda_k \sum_{1 \leq i, j \leq m} \int b_{k,l}^i \zeta (v_k^i - v_j^i) \, dz \equiv \lambda_k (\Sigma_{k,s}^1 - \Sigma_{k,s}^2).\]

The estimate of \(\Sigma_{k,s}^1\) is provided by the Fefferman–Stein duality theorem, Corollary 6, and Lemma 8:

\[ \sup_{k,s \in \mathbb{N}} |\Sigma_{k,s}^1| \leq C \sum_{1 \leq i, j \leq m} \sup_{k \in \mathbb{N}} \left( \left\| \sum_{l=1}^n b_{k,l}^i \frac{\partial (\zeta v_k^j)}{\partial x_l} \right\|_{\mathcal{H}^1(\mathbb{R}^n)} \right) \sup_{k,s \in \mathbb{N}} \|v_k^i - v_j^i\|_{BMO(\mathbb{R}^n)} < \infty.\]

To get a bound for \(\Sigma_{k,s}^2\), we employ the Hölder inequality with exponents \(2p, 2p, p/(p-1)\) to obtain

\[ \sup_{k,s \in \mathbb{N}} |\Sigma_{k,s}^2| \leq C < \infty,\]

since, by Lemma 6 and John–Nirenberg’s inequality, the sequence \(v_k\) is bounded in \(L^{3p}(B(0, 15/16), \mathbb{R}^m)\), and \((v_k^j)_{k \in \mathbb{N}}\) is bounded in \(L^{p/(p-1)}(B(0, 15/16))\) by (14).

Hence, \(|R_{k,s}| \leq C \lambda_k\) for \(k, s \to \infty\). Estimating \(R_{k,s}\) in the same way, and using (23) and (25) we obtain the \(L^2\)-Cauchy condition for \(G(\nabla v_k)\),

\[ \int_{\mathbb{R}^n(0,1/2)} |G(\nabla v_k) - G(\nabla v_s)|^2 \, dz \to 0 \quad \text{as} \quad k, s \to \infty.\]

To identify the strong limit of \(G(\nabla v_k)\), recall two other elementary inequalities:

\[ (27) \quad |G(X) - G(Y)|^2 \geq 3^{-p}|X - Y|^p, \]
\[ (28) \quad |H(X) - H(Y)| \leq 2(p-1)(|X|^p + |Y|^p)^{p-2} \frac{2}{p} |G(X) - G(Y)|.\]

Combining (27) with (26) gives \(\|\nabla v_k - \nabla v_s\|_{L^p} \overset{k,s}{\longrightarrow} 0\). Therefore, (17) implies \(\nabla v_k \rightharpoonup \nabla v\) in \(L^p(B(0, 1/2))\), and a subsequence is convergent pointwise a.e. Since the map \(\xi \mapsto G(\xi)\) is a homeomorphism of \(\mathbb{R}^m\), the strong \(L^2\) limit of \(G(\nabla v_k)\) is equal to \(G(\nabla v)\). This proves the first statement of the Main Lemma.

To prove that the limit function \(v\) satisfies \(\text{div} (|\nabla v|^{p-2} \nabla v) = 0\), note first that (28) and the Hölder inequality imply that

\[ \int_{B(0,1/2)} |H(\nabla v_k) - H(\nabla v)|^\frac{p}{p-1} \, dz \leq C \left( \int_{B(0,1/2)} |G(\nabla v_k) - G(\nabla v)|^2 \, dz \right)^{\frac{p}{2}} \sup_{k \in \mathbb{N}} \left( \int_{\mathbb{R}^n} |\nabla v_k(z)|^p \, dz \right)^{\frac{p-2}{2}}. \]

Therefore \(H(\nabla v_k) \rightharpoonup H(\nabla v)\) strongly in \(L^{p/(p-1)}(B(0, 1/2), \mathbb{R}^m)\). Now, recall the identity (22), assume that \(\text{supp} \psi \subset B(0, 1/2)\) and conclude upon letting \(k \to \infty\) that

\[ \text{div} (|\nabla v|^{p-2} \nabla v) = 0 \quad \text{in the interior of} \quad \{x \in \mathbb{R}^n : \zeta \equiv 1\}. \]

Varying the initial choice of \(\zeta\), we obtain the second statement of the Main Lemma. \(\blacksquare\)
Proof of Theorem 2. Define
\[ \tilde{V} = \{ x \in \mathbb{B}^n : E(x, r) < \varepsilon_0 \text{ for some } r < d(x, \partial \mathbb{B}^n) \}. \]

Obviously, \( \tilde{V} \) is open, and moreover \( H^{n-p}(\mathbb{B}^n \setminus \tilde{V}) = 0 \). To see this, check that the set \( V \) of those \( x \in \mathbb{B}^n \) for which the normalized energy \( E(x, r) \) tends to zero as \( r \to 0 \), is equal to \( \tilde{V} \). Therefore, by Frostman’s lemma (see e.g. [27, Lemma 3.2.2 & Corollary 3.2.3]) we obtain immediately \( 0 = H^{n-p}(\mathbb{B}^n \setminus V) = H^{n-p}(\mathbb{B}^n \setminus \tilde{V}) \).

If \( x \in V \), then by iterations of inequality (8), Theorem 3 implies that for some positive constants \( C \) and \( \beta \) we have \( E(y, r) \leq Cr^{\beta} \) for all \( y \) in a small neighbourhood of \( x \) and all sufficiently small radii \( r \). Therefore, applying Morrey’s imbedding theorem [11, pages 64–65], we conclude that \( u \) is uniformly Hölder continuous with exponent \( \alpha = \beta/p \) on compact subsets of \( V \).

Remarks. Using a rather standard argument, one can use Gehring’s lemma to prove that \( |\nabla u| \in L^q_{\text{loc}}(V) \) for some \( q > p \). Global higher integrability of \( |\nabla u| \) on the set \( V \) would obviously imply that the Hausdorff dimension of \( \mathbb{B}^n \setminus V \) is strictly smaller than \( n - p \). In spite of some efforts, we were not able to prove this fact.

Another interesting problem is to prove the following.

Conjecture 9. For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( \Omega \subset V \) is an open set with \( \text{diam} \Omega \leq \delta \), then the inequality
\[ \int_Q |\nabla u(y) - [\nabla u]_Q| \, dy \leq \varepsilon \int_{2Q} |\nabla u(y)| \, dy \]
is valid for any cube \( Q \) such that \( 2Q \subset \Omega \).

This result would imply that \( |\nabla u| \in L^q_{\text{loc}}(V) \) for any \( q < \infty \), thus allowing for a new and relatively simple proof of the Hölder continuity of \( \nabla u \) on \( V \).

4. One more regularity theorem. As a byproduct of the proofs presented above, we are able to generalize a theorem of F. Hélein and F. Bethuel (asserting smoothness of weakly harmonic maps \( u : M^m \to N^n \) with \( |\nabla u| \in L^m_{\text{loc}} \)) to the \( p \)-harmonic case.

Theorem 10. Assume that \( u \in W^{1,p}(\mathbb{B}^n, S^{m-1}) \) is a weakly \( p \)-harmonic map (not necessarily stationary), with \( |\nabla u| \in L^n(\mathbb{B}^n) \). Then \( u \) is Hölder continuous on \( \mathbb{B}^n \).

We sketch below a proof of this result, without entering into technical details, which are either identical or very similar to those appearing in the proof of Theorem 3. The task of writing down a fully detailed proof is left to the interested readers as an easy but tedious exercise.

First, note that for \( u \) satisfying the assumptions of Theorem 10 we have, by Hölder inequality,
\[ E(x, r) = r^{p-n} \int_{\mathbb{B}^n(x, r)} |\nabla u(y)|^p \, dy \leq C \left( \int_{\mathbb{B}^n(x, r)} |\nabla u(y)|^n \, dy \right)^{p/n}. \]

Therefore, by the absolute continuity of the integral, the set \( V \) defined by (29) is equal to \( \mathbb{B}^n \), no matter what value of \( \varepsilon_0 > 0 \) we choose. All that remains to be done is to prove inequality (8).
The first part of the proof of decay of $E(x, r)$ goes without any changes.

In the proof of Lemma 6, the monotonicity formula can be replaced by the imbedding $W^{1,n} \subset BMO$, and instead of applying John–Nirenberg’s lemma we can use the classical Sobolev imbedding theorem (functions from $W^{1,n}_{loc}$ are locally integrable with any power $q \in [1, \infty)$).

The rest of the proof also goes without changes.

**Added in proof.** A few months after having submitted this work to Banach Center Publications, the author learned that other mathematicians, among them Takeuchi, Toro, and Wang, have independently proved Theorems 1 and 2 (see [22], [25]).

**References**


