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STATIONARY p-HARMONIC MAPS INTO SPHERES

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1. Introduction. In this lecture, we want to discuss some regularity properties of p-harmonic maps with values in euclidean spheres.

Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : \sum (x_i)^2 < 1\}$ denote the unit n-dimensional ball, and write S^{m-1} for the unit sphere in $\mathbb{R}^m$. Define the functional

$$I_p(u) = \int\limits_{\mathbb{B}^n} |\nabla u(x)|^p \, dx \qquad \text{for } u \in W^{1,p}(\mathbb{B}^n, S^{m-1}).$$

For exponents $p \in [2, n]$, we wish to investigate those maps which are critical points of I_p with respect both to variations in the range and in the parameter domain.

Definition. By a *stationary p-harmonic map* we mean here any u belonging to the Sobolev space

$$W^{1,p}(\mathbb{B}^n, S^{m-1}) \equiv \Big\{ f = (f_1, \ldots, f_m) : f_i \in W^{1,p}(\mathbb{B}^n) \text{ and } \sum_{i=1}^m (f_i(x))^2 = 1 \text{ a.e.} \Big\},$$

and satisfying the following two conditions:

$$\frac{d}{dt}\bigg|_{t=0} I_p\left(\frac{u + t\psi}{|u + t\psi|}\right) = 0 \qquad \text{for all } \psi = (\psi^1, \ldots, \psi^m) \in C_0^\infty(\mathbb{B}^n, \mathbb{R}^m), \tag{1}$$

$$\frac{d}{dt}\bigg|_{t=0} I_p(u(x + t\zeta(x))) = 0 \qquad \text{for all } \zeta = (\zeta^1, \ldots, \zeta^n) \in C_0^\infty(\mathbb{B}^n, \mathbb{B}^n). \tag{2}$$

If a map $u \in W^{1,p}(\mathbb{B}^n, S^{m-1})$ satisfies only the condition (1), we say that u is *weakly p-harmonic*.

Condition (1) is easily checked to be equivalent to the fact that $u \in W^{1,p}(\mathbb{B}^n, S^{m-1})$ is a weak solution to the Euler-Lagrange elliptic system

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$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p u. \tag{3}$$

More precisely, the integral identity

$$\int_{\mathbb{B}^n} |\nabla u|^{p-2}\nabla u^i \cdot \nabla\psi^i\,dx = \int_{\mathbb{B}^n} \psi^i u^i |\nabla u|^p\,dx \tag{4}$$

holds true for all $i = 1, \ldots, n$ and $\psi = (\psi^1, \ldots, \psi^n) \in C_0^\infty(B, \mathbb{R}^n)$. Here and everywhere below,

$$|\nabla u|^2 = \sum_{i=1}^m \sum_{j=1}^n \left(\frac{\partial u^i}{\partial x_j}\right)^2.$$

Another integral identity for stationary p-harmonic maps,

$$\int_{\mathbb{B}^n} |\nabla u|^p \mathrm{div}\,\zeta\,dx = p \sum_{\substack{1\le j\le m\\ 1\le k,\ell\le n}} \int_{\mathbb{B}^n} |\nabla u|^{p-2} \frac{\partial u^j}{\partial x_k}\frac{\partial u^j}{\partial x_\ell}\frac{\partial \zeta^\ell}{\partial x_k}\,dx, \tag{5}$$

which holds true for all $\zeta \in C_0^\infty(\mathbb{B}^n, \mathbb{B}^n)$, is a consequence of (2). Note that for smooth maps u formula (5) follows from (4) if we set $\psi^j := \zeta \cdot \nabla u^j$. By a suitable choice of the testing map ζ, one can obtain the so-called *monotonicity formula* for stationary p-harmonic maps,

$$r_1^p \fint_{\mathbb{B}^n(x,r_1)} |\nabla u(y)|^p\,dy \le r_2^p \fint_{\mathbb{B}^n(x,r_2)} |\nabla u(y)|^p\,dy, \qquad r_1 < r_2 \le \mathrm{dist}(x, \partial\mathbb{B}^n). \tag{6}$$

This fact was proved for Yang–Mills fields and stationary harmonic maps by Price [18]; Fuchs [8] observed that (6) is valid also for stationary p-harmonic maps.

It is also possible to define and consider stationary p-harmonic maps $u : M^m \to N^n$ between Riemannian manifolds. We shall not pursue that point further here.

It is well known that the regularity theory of p-harmonic maps is a delicate topic. The example of the map $u(x) = x/|x|$ from the unit ball $\mathbb{B}^n$ to its boundary $\partial\mathbb{B}^n \equiv S^{n-1}$, which is singular at 0 and weakly p-harmonic for all $p \in [1, n)$, shows that weakly p-harmonic maps do not have, in general, to be continuous. In fact, the state of affairs is even worse: T. Rivière [19] has recently given an example of a weakly harmonic map $u \in W^{1,2}(\mathbb{B}^3, S^2)$ which is discontinuous at every point of $\mathbb{B}^3$.

However, there are lots of results about regularity and partial regularity of weakly p-harmonic maps under various additional assumptions. Let us mention below just a few; the list is obviously far from being complete.

Hardt and Lin [12], Fuchs [9], and Luckhaus [16] proved independently a theorem stating that *minimizing* p-harmonic maps $u : M^m \to N^n$ are of class $C^{1,\alpha}$, $0 < \alpha < 1$, outside a set of Hausdorff dimension $m - [p] - 1$ (that was a generalization af an earlier result of Schoen and Uhlenbeck [20] concerning the case $p = 2$ of minimizing harmonic maps). Fuchs [8] was able to show that some partial regularity results are valid also for stationary p-harmonic maps with range contained in a small ball $B(r_0) \subset N^n$ of radius r_0 determined by the geometry of underlying manifolds.

There is also a series of recent developments which were obtained via applications of a theorem of Coifman, Lions, Meyer, and Semmes ([1])—this method allows proving regularity or partial regularity without assuming that u minimizes the Dirichlet integral. In his papers [13]–[15], F. Hélein proved that any weakly harmonic map $f : M \to N$ defined on a two-dimensional Riemannian manifold M is continuous; [13] contains the proof for $N = S^{n-1}$, [15] concerns the case when N is a compact manifold with a Lie group of isometries acting transitively, and [14] deals with the case of arbitrary compact Riemannian N. (By standard elliptic regularity methods, continuity of a weakly harmonic map implies its C^∞-smoothness.) Evans [4] and Bethuel [1] generalized Hélein's result to the case of stationary harmonic maps on n-dimensional manifolds, $n \geq 2$, proving their regularity outside a singular set of $(n-2)$-dimensional Hausdorff measure zero. Up to now, these recent developments hardly have any counterparts for $p \neq 2$.

Let us now state our main results.

THEOREM 1 (case $p = n$). *Any weakly n-harmonic map $u \in W^{1,n}(\mathbb{B}^n, S^{m-1})$ is locally Hölder continuous on $\mathbb{B}^n$.*

THEOREM 2. *Let $2 \leq p < n$ and assume that $u \in W^{1,p}(\mathbb{B}^n, S^{m-1})$ is a stationary p-harmonic map. Then the set $V \subset \mathbb{B}^n$ defined by*

$$V := \Big\{ x \in \mathbb{B}^n : r^p \fint_{\mathbb{B}^n(x,r)} |\nabla u(y)|^p \, dy \to 0 \text{ as } r \to 0 \Big\}$$

is open, $H^{n-p}(\mathbb{B}^n \setminus V) = 0$, and u is locally Hölder continuous on V.

Actually, M. Fuchs [7] proved these results independently and via different methods. A relatively short and direct proof of Theorem 1 can be found in [21]. Here, we would like to sketch the proof of Theorem 2.

Our proof combines earlier ideas due to Hélein and Evans with one simple observation from [21]. Namely, we note that the right-hand side of (3) is an element of the local Hardy space $\mathcal{H}^1_{\mathrm{loc}}$, a proper subspace of L^1 (for $p = 2$ this was noticed and exploited by Hélein [13]–[15]). Due to that fact we are able to model the main part of the argument on Evans [4], introducing some modifications in order to cope with nonlinearity and degeneracy of the p-Laplace operator $L_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We exploit the duality between $\mathcal{H}^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ to obtain, for the scaled energy

$$E(x,r) = r^{p-n} \int_{\mathbb{B}^n(x,r)} |\nabla u(y)|^p \, dy, \tag{7}$$

the following decay estimate.

THEOREM 3. *There exist constants (depending only on n, m, and p) $\varepsilon_0 \in (0,1)$ and $\theta \in (0,1)$ such that*

$$E(x,r) < \varepsilon_0 \Rightarrow E(x,\theta r) \leq \tfrac{1}{4} E(x,r) \tag{8}$$

for all $x \in \mathbb{B}^n$ and all positive $r < d(x, \partial\mathbb{B}^n)$.

([1]) The theorem states that the Jacobian of a map $u \in W^{1,n}(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$.

Then we apply the Dirichlet growth theorem [11] and a covering argument to deduce Hölder continuity of u on an open subset V of $\mathbb{B}^n$ with $H^{n-p}(\mathbb{B}^n \setminus V) = 0$.

2. Hardy space, BMO, and Fefferman–Stein duality theorem

Definition. A measurable function $f \in L^1(\mathbb{R}^n)$ belongs to the *Hardy space* $\mathcal{H}^1(\mathbb{R}^n)$ if and only if

$$f_* := \sup_{\varepsilon>0} |\varphi_\varepsilon * f| \in L^1(\mathbb{R}^n).$$

Here, $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$, and φ is a fixed function of class $C_0^\infty(B(0,1))$ with $\int \varphi(y)\,dy = 1$. The definition does not depend on the choice of φ (see [6]).

The interested reader will find other equivalent definitions of $\mathcal{H}^1(\mathbb{R}^n)$ and more details in [10] or [24]. Let us just mention here that $\mathcal{H}^1(\mathbb{R}^n)$ is a Banach space with the norm $\|f\|_{\mathcal{H}^1} = \|f\|_{L^1} + \|f_*\|_{L^1}$. Moreover, the condition $f \in \mathcal{H}^1(\mathbb{R}^n)$ implies $\int f(y)\,dy = 0$.

C. Fefferman [5], [6] proved that the dual of $\mathcal{H}^1(\mathbb{R}^n)$ is equal to the space of functions of bounded mean oscillation, $BMO(\mathbb{R}^n)$. More precisely, there exists a constant C such that

$$\int_{R^n} h(y)\psi(y)\,dy \le C\|h\|_{\mathcal{H}^1}\|\psi\|_{BMO} \tag{9}$$

for all functions $h \in \mathcal{H}^1(\mathbb{R}^n)$ and $\psi \in BMO(\mathbb{R}^n)$.

The interesting paper of S. Müller [17] inspired some of the research reported in [3], in particular the following remarkable theorem.

Theorem 4 (Coifman, Lions, Meyer, Semmes). *Assume that* $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, *and that* $H \in L^{p/(p-1)}(\mathbb{R}^n, \mathbb{R}^n)$ *satisfies the condition* $\operatorname{div} H = 0$ *in* $\mathcal{D}'(\mathbb{R}^n)$. *Then* $\nabla u \cdot H \in \mathcal{H}^1(\mathbb{R}^n)$, *and*

$$\|\nabla u \cdot H\|_{\mathcal{H}^1} \le C\|\nabla u\|_{L^p}\|H\|_{L^{p/(p-1)}} \tag{10}$$

for some constant C *depending only on* n *and* p.

The estimate (10) was not explicitly stated in [3], but follows from the proof presented there (cf. also [4, Section 2]). Let us now make explicit a corollary of the above theorem (more or less well known to specialists).

Corollary 5. *Let* Ω *be a ball in* $\mathbb{R}^n$. *Assume that* $u \in W^{1,p}(\Omega)$, $1 < p < \infty$, *and that* $H \in L^{p/(p-1)}(\Omega, \mathbb{R}^n)$ *satisfies the condition* $\operatorname{div} H = 0$ *in* $\mathcal{D}'(\Omega)$. *Then one can find a function* $h \in \mathcal{H}^1(\mathbb{R}^n)$ *such that*

$$h(x) = \nabla u(x) \cdot H(x), \qquad x \in \Omega,$$

and

$$\|h\|_{\mathcal{H}^1} \le C\|\nabla u\|_{L^p(\Omega)}\|H\|_{L^{p/(p-1)}(\Omega)}.$$

The constant C *does not depend on the size of* Ω.

Proof. See [21].

3. Energy decay estimates

Proof of Theorem 3. Assume that (8) is violated and for all positive $\theta \in (0,1)$ we can find a sequence of balls $B_k \equiv B(x_k, r_k) \subset \mathbb{B}^n$, $k = 1, 2, \ldots$, such that

$$E(x_k, r_k) = \lambda_k^p \overset{k\to\infty}{\longrightarrow} 0, \tag{11}$$

and at the same time

$$E(x_k, \theta r_k) > \tfrac{1}{4}\lambda_k^p. \tag{12}$$

We now change the variables,

$$\mathbb{B}^n \ni z \mapsto y = x_k + r_k z \in B_k, \qquad k = 1, 2, \ldots,$$

to rescale everything to the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$. Write

$$v_k(z) := \frac{u(x_k + r_k z) - a_k}{\lambda_k}, \qquad a_k := \fint_{B_k} u(y)\, dy.$$

Using the Poincaré inequality and the classical change of variables formula, we easily obtain the following three statements:

$$\sup_k \int_{\mathbb{B}^n} |v_k(z)|^p\, dz < +\infty, \tag{13}$$

$$\int_{\mathbb{B}^n} |\nabla v_k(z)|^p\, dz = 1 \quad \text{ for all } k \in \mathbb{N}, \tag{14}$$

$$\theta^{p-n} \int_{\mathbb{B}^n(0,\theta)} |\nabla v_k(z)|^p\, dz > 1/4 \quad \text{ for all } k \in \mathbb{N}. \tag{15}$$

Therefore, we may pass to a subsequence and assume without loss of generality that

$$v_k \to v \qquad \text{strongly in } L^p(\mathbb{B}^n, \mathbb{R}^m) \text{ and a.e.,} \tag{16}$$

$$\nabla v_k \rightharpoonup \nabla v \qquad \text{weakly in } L^p(\mathbb{B}^n, \mathbb{R}^{mn}). \tag{17}$$

What we now need to conclude the proof of Theorem 3 is the following.

MAIN LEMMA. $|\nabla v_k|^{(p-2)/2}\nabla v_k \overset{k\to\infty}{\longrightarrow} |\nabla v|^{(p-2)/2}\nabla v$ *in the* **strong** *topology of* $L^2(B(0,1/2))$. *Moreover, the limit function* v *satisfies the non-constrained* p*-harmonic equation, i.e.*

$$\int_{\mathbb{B}^n} |\nabla v|^{p-2}\nabla v \cdot \nabla \psi\, dy = 0 \tag{18}$$

for all $\psi = (\psi^1, \ldots, \psi^m) \in C_0^\infty(\mathbb{B}^n, \mathbb{R}^m)$

Let us first show that this fact implies Theorem 3.

Well-known results of N. Uraltseva [27] and K. Uhlenbeck [26] (see also [23]) assert that ∇v is locally Hölder continuous on $\mathbb{B}^n$. In particular, we have the estimate

$$\operatorname*{ess\,sup}_{y \in \mathbb{B}^n(0,1/2)} |\nabla v(y)| \le C(n,p)\Big(\int_{\mathbb{B}^n} |v(y)|^p\, dy\Big)^{1/p}, \tag{19}$$

which in view of (13) implies

$$\theta^{p-n} \int\limits_{\mathbb{B}^n(0,\theta)} |\nabla v(y)|^p \, dy \le C\theta^p < 1/8, \tag{20}$$

if only $\theta \in (0, 1/2)$ is small enough. On the other hand, the strong convergence of gradients, $|\nabla v_k|^{(p-2)/2}\nabla v_k \to |\nabla v|^{(p-2)/2}\nabla v$, combined with (15) forces

$$\theta^{p-n} \int\limits_{\mathbb{B}^n(0,\theta)} |\nabla v(z)|^p \, dz \ge 1/4,$$

a contradiction to (20). Therefore, the argument will be complete once we prove the Main Lemma. The proof given below is modelled on [4, Section 4], with some minor changes forced by nonlinearity of the p-Laplace operator.

Take a smooth function $\zeta : \mathbb{R}^n \to [0,1]$. To fix ideas for a moment, assume that $\operatorname{supp} \zeta \subset B(0, 9/16)$ and $\zeta \equiv 1$ on $B(0,1/2)$. Mimicking the arguments of [4] one can use the monotonicity formula to prove the following.

LEMMA 6. *For every $1 \le i \le m$, the double sequence $(\zeta(v_k^i - v_s^i))_{k,s\in\mathbb{N}}$ is bounded in $BMO(\mathbb{R}^n)$.*

Next, for $k \in \mathbb{N}$ and $1 \le i, j \le m$, define the vector field $b_k^{ij} \in L^{p/(p-1)}(\mathbb{B}^n, \mathbb{R}^n)$ with coordinates

$$b_{k,\ell}^{ij} = |\nabla v_k|^{p-2}\biggl((a_k^i + \lambda_k v_k^i)\frac{\partial v_k^j}{\partial x_\ell} - (a_k^j + \lambda_k v_k^j)\frac{\partial v_k^i}{\partial x_\ell}\biggr). \tag{21}$$

An easy straightforward calculation (see e.g. [21]) leads to

LEMMA 7. $\operatorname{div} b_k^{ij} = 0$ *in $\mathcal{D}'(\mathbb{B}^n)$; more precisely,*

$$\sum_{\ell=1}^n \int\limits_{\mathbb{B}^n} \frac{\partial \phi}{\partial x_\ell} b_{k,\ell}^{ij} \, dz = 0$$

for every function $\phi \in W^{1,p}(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n)$ with compact support.

Combining this lemma with Corollary 5 we obtain

LEMMA 8. *The sequence of functions*

$$\sum_{\ell=1}^n b_{k,\ell}^{ij} \frac{\partial(\zeta v_k^j)}{\partial x_\ell}, \qquad k = 1, 2, \ldots,$$

is bounded in $\mathcal{H}^1(\mathbb{R}^n)$ for each $1 \le i, j \le m$.

Proof of the Main Lemma. For sake of brevity, write $G(\xi) = |\xi|^{(p-2)/2}\xi$, and let $H(\xi) = |\xi|^{p-2}\xi$. Pick a bounded function $\psi \in W^{1,p}(\mathbb{B}^n, \mathbb{R}^m)$ with support contained in $\mathbb{B}^n$. Set $\psi_k(y) := \psi(z) = \psi((y - x_k)/r_k))$. Because $u \in W^{1,p}(\mathbb{B}^n, S^{m-1})$ is weakly p-harmonic, we have

$$\int\limits_{B_k} H(\nabla u) \cdot \nabla \psi_k \, dy = \int\limits_{B_k} |\nabla u|^p u \psi_k \, dy.$$

Changing integration variables from $y \in B_k$ to $z \in \mathbb{B}^n$ gives

$$\int_{\mathbb{B}^n} H(\nabla v_k) \cdot \nabla \psi \, dz = \lambda_k \int_{\mathbb{B}^n} |\nabla v_k|^p (a_k + \lambda_k v_k) \psi \, dz. \tag{22}$$

Write now the same identity with v_k (resp. a_k, λ_k) replaced by v_s (resp. a_s, λ_s), substitute $\psi = \zeta^2(v_k - v_s)$ in both of the resulting equalities and subtract one of them from the other to obtain

$$\begin{aligned} L_{k,s} \equiv & \int_{\mathbb{B}^n} \zeta^2 (H(\nabla v_k) - H(\nabla v_s)) \cdot (\nabla v_k - \nabla v_s) \, dz \\ & + 2 \int_{\mathbb{B}^n} \zeta (v_k - v_s) \cdot (H(\nabla v_k) - H(\nabla v_s)) \cdot \nabla \zeta \, dz \\ \leq & |R_{k,s}| + |R_{s,k}|, \end{aligned} \tag{23}$$

where, for $k, s \in \mathbb{N}$,

$$R_{k,s} = \lambda_k \sum_{\substack{1 \leq i,j \leq m \\ 1 \leq \ell \leq n}} \int_{\mathbb{B}^n} \zeta^2 |\nabla v_k|^{p-2} \frac{\partial v_k^j}{\partial x_\ell} \left(\frac{\partial v_k^j}{\partial x_\ell} (a_k^i + \lambda_k v_k^i)(v_k^i - v_s^i) \right) dz. \tag{24}$$

By the Hölder inequality, (16) and (17), the absolute value of the second integral on the left hand side of (23) does not exceed

$$C \|v_k - v_s\|_{L^p(\mathbb{B}^n)} \sup_{k \in \mathbb{N}} \|\nabla v_k\|_{L^p(\mathbb{B}^n)} = o(1) \quad \text{for } k, s \to \infty.$$

To estimate from below the first integral on the left hand side of (23), we apply the elementary inequality

$$(H(X) - H(Y)) \cdot (X - Y) \geq \frac{1}{p} |G(X) - G(Y)|^2,$$

valid for $p \geq 2$ and for vectors X, Y in any scalar product space. The calculations imply

$$L_{k,s} \geq \frac{1}{p} \int_{\mathbb{B}^n(0,1/2)} |G(\nabla v_k) - G(\nabla v_s)|^2 \, dz + o(1) \quad \text{for } k, s \to \infty. \tag{25}$$

To estimate the right hand side of (23), note that $|u| = 1$ implies

$$\sum_{j=1}^m (a_k^j + \lambda_k v_k^j) \frac{\partial v_k^j}{\partial x_i} = 0 \quad \text{for } i = 1, \dots, n \text{ and } k \in \mathbb{N}.$$

Therefore, the crucial trick of Hélein can be adapted to the case $p \neq 2$. We may use the b_k^{ij} to express the integral $R_{k,s}$, and then apply Fefferman's theorem. Here is the calculation:

$$\begin{aligned} R_{k,s} &= \lambda_k \sum_{\substack{1 \leq i,j \leq m \\ 1 \leq \ell \leq n}} \int_{\mathbb{B}^n} \zeta^2 |\nabla v_k|^{p-2} \frac{\partial v_k^j}{\partial x_\ell} \left(\frac{\partial v_k^j}{\partial x_\ell} (a_k^i + \lambda_k v_k^i) - \frac{\partial v_k^i}{\partial x_\ell} (a_k^j + \lambda_k v_k^j) \right) (v_k^i - v_s^i) \, dz \\ &= \lambda_k \sum_{\substack{1 \leq i,j \leq m \\ 1 \leq \ell \leq n}} \int_{\mathbb{B}^n} \zeta^2 \frac{\partial v_k^j}{\partial x_\ell} b_{k,\ell}^{ij} (v_k^i - v_s^i) \, dz \end{aligned}$$

$$= \lambda_k \sum_{1\le i,j\le m} \int_{\mathbb{R}^n} \Big(\sum_{\ell=1}^n b^{ij}_{k,\ell} \frac{\partial(\zeta v^j_k)}{\partial x_\ell}\Big)\zeta(v^i_k - v^i_s)\,dz$$
$$- \lambda_k \sum_{\substack{1\le i,j\le m\\ 1\le \ell\le n}} \int_{\mathbb{R}^n} v^j_k b^{ij}_{k,\ell}\zeta \frac{\partial \zeta}{\partial x_\ell}(v^i_k - v^i_s)\,dz \equiv \lambda_k(\varSigma^1_{k,s} - \varSigma^2_{k,s}).$$

The estimate of $\varSigma^1_{k,s}$ is provided by the Fefferman–Stein duality theorem, Corollary 6, and Lemma 8:

$$\sup_{k,s\in\mathbb{N}} |\varSigma^1_{k,s}| \le C \sum_{1\le i,j\le m} \sup_{k\in\mathbb{N}} \bigg(\bigg\|\sum_{\ell=1}^n b^{ij}_{k,\ell}\frac{\partial(\zeta v^j_k)}{\partial x_\ell}\bigg\|_{\mathcal{H}^1(\mathbb{R}^n)}\bigg) \sup_{k,s\in\mathbb{N}} \|\zeta(v^i_k - v^i_s)\|_{BMO(\mathbb{R}^n)} < \infty.$$

To get a bound for $\varSigma^2_{k,s}$, we employ the Hölder inequality with exponents $2p$, $2p$, $p/(p-1)$ to obtain

$$\sup_{k,s\in\mathbb{N}} |\varSigma^2_{k,s}| \le C < \infty,$$

since, by Lemma 6 and John–Nirenberg's inequality, the sequence v_k is bounded in $L^{2p}(B(0,15/16),\mathbb{R}^m)$, and $(b^{ij}_{k,\ell})_{k\in\mathbb{N}}$ is bounded in $L^{p/p-1}(B(0,15/16))$ by (14).

Hence, $|R_{k,s}| \le C\lambda_k$ for $k,s\to\infty$. Estimating $R_{s,k}$ in the same way, and using (23) and (25) we obtain the L^2-Cauchy condition for $G(\nabla v_k)$,

$$\int_{\mathbb{B}^n(0,1/2)} |G(\nabla v_k) - G(\nabla v_s)|^2\,dz \to 0 \quad \text{as } k,s\to\infty. \tag{26}$$

To identify the strong limit of $G(\nabla v_k)$, recall two other elementary inequalities:

$$|G(X) - G(Y)|^2 \ge 3^{-p}|X-Y|^p, \tag{27}$$
$$|H(X) - H(Y)| \le 2(p-1)(|X|^p + |Y|^p)^{\frac{p-2}{2p}}|G(X) - G(Y)|. \tag{27}$$

Combining (27) with (26) gives $\|\nabla v_k - \nabla v_s\|_{L^p} \xrightarrow{k,s} 0$. Therefore, (17) implies $\nabla v_k \to \nabla v$ in $L^p(B(0,1/2))$, and a subsequence is convergent pointwise a.e. Since the map $\xi \mapsto G(\xi)$ is a homeomorphism of $\mathbb{R}^{mn}$, the strong L^2 limit of $G(\nabla v_k)$ is equal to $G(\nabla v)$. This proves the first statement of the Main Lemma.

To prove that the limit function v satisfies $\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0$, note first that (28) and the Hölder inequality imply that

$$\int_{B(0,1/2)} |H(\nabla v_k) - H(\nabla v)|^{\frac{p}{p-1}}\,dz$$
$$\le C\Big(\int_{B(0,1/2)} |G(\nabla v_k) - G(\nabla v)|^2\,dz\Big)^{\frac{p}{2p-2}} \sup_{k\in\mathbb{N}}\Big(\int_{\mathbb{B}^n} |\nabla v_k(z)|^p\,dz\Big)^{\frac{p-2}{2p-2}}.$$

Therefore $H(\nabla v_k) \to H(\nabla v)$ strongly in $L^{p/(p-1)}(B(0,1/2),\mathbb{R}^{mn})$. Now, recall the identity (22), assume that $\operatorname{supp}\psi \subset B(0,1/2)$ and conclude upon letting $k\to\infty$ that

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0 \quad \text{in the interior of } \{x\in\mathbb{B}^n : \zeta\equiv 1\}.$$

Varying the initial choice of ζ, we obtain the second statement of the Main Lemma. ∎

Proof of Theorem 2. Define

$$\widetilde{V} = \{x \in \mathbb{B}^n : E(x,r) < \varepsilon_0 \text{ for some } r < d(x, \partial\mathbb{B}^n)\}. \tag{29}$$

Obviously, $\widetilde{V}$ is open, and moreover $H^{n-p}(\mathbb{B}^n \setminus \widetilde{V}) = 0$. To see this, check that the set V of those $x \in \mathbb{B}^n$ for which the normalized energy $E(x,r)$ tends to zero as $r \to 0$, is equal to $\widetilde{V}$. Therefore, by Frostman's lemma (see e.g. [27, Lemma 3.2.2 & Corollary 3.2.3]) we obtain immediately $0 = H^{n-p}(\mathbb{B}^n \setminus V) = H^{n-p}(\mathbb{B}^n \setminus \widetilde{V})$.

If $x \in V$, then by iterations of inequality (8), Theorem 3 implies that for some positive constants C and β we have $E(y,r) \le Cr^\beta$ for all y in a small neighbourhood of x and all sufficiently small radii r. Therefore, applying Morrey's imbedding theorem [11, pages 64–65], we conclude that u is uniformly Hölder continuous with exponent $\alpha = \beta/p$ on compact subsets of V. ∎

Remarks. Using a rather standard argument, one can use Gehring's lemma to prove that $|\nabla u| \in L^q_{\mathrm{loc}}(V)$ for some $q > p$. Global higher integrability of $|\nabla u|$ on the set V would obviously imply that the Hausdorff dimension of $\mathbb{B}^n \setminus V$ is strictly smaller than $n-p$. In spite of some efforts, we were not able to prove this fact.

Another interesting problem is to prove the following.

Conjecture 9. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\Omega \subset V$ is an open set with* $\operatorname{diam} \Omega \le \delta$, *then the inequality*

$$\fint_Q |\nabla u(y) - [\nabla u]_Q|\, dy \le \varepsilon \fint_{2Q} |\nabla u(y)|\, dy$$

is valid for any cube Q such that $2Q \subset \Omega$.

This result would imply that $|\nabla u| \in L^q_{\mathrm{loc}}(V)$ for *any* $q < \infty$, thus allowing for a new and relatively simple proof of the Hölder continuity of ∇u on V.

4. One more regularity theorem. As a byproduct of the proofs presented above, we are able to generalize a theorem of F. Hélein and F. Bethuel (asserting smoothness of weakly harmonic maps $u : M^m \to N^n$ with $|\nabla u| \in L^m_{\mathrm{loc}}$) to the p-harmonic case.

Theorem 10. *Assume that $u \in W^{1,p}(\mathbb{B}^n, S^{m-1})$ is a weakly p-harmonic map* (*not necessarily stationary*), *with $|\nabla u| \in L^n(\mathbb{B}^n)$. Then u is Hölder continuous on $\mathbb{B}^n$.*

We sketch below a proof of this result, without entering into technical details, which are either identical or very similar to those appearing in the proof of Theorem 3. The task of writing down a fully detailed proof is left to the interested readers as an easy but tedious exercise.

First, note that for u satisfying the assumptions of Theorem 10 we have, by Hölder inequality,

$$E(x,r) = r^{p-n} \int_{\mathbb{B}^n(x,r)} |\nabla u(y)|^p\, dy \le C\Big(\int_{\mathbb{B}^n(x,r)} |\nabla u(y)|^n\, dy\Big)^{p/n}.$$

Therefore, by the absolute continuity of the integral, the set V defined by (29) is equal to $\mathbb{B}^n$, no matter what value of $\varepsilon_0 > 0$ we choose. All that remains to be done is to prove inequality (8).

The first part of the proof of decay of $E(x, r)$ goes without any changes.

In the proof of Lemma 6, the monotonicity formula can be replaced by the imbedding $W^{1,n} \subset BMO$, and instead of applying John–Nirenberg's lemma we can use the classical Sobolev imbedding theorem (functions from $W^{1,n}_{\mathrm{loc}}$ are locally integrable with any power $q \in [1, \infty)$).

The rest of the proof also goes without changes.

Added in proof. A few months after having submitted this work to Banach Center Publications, the author learned that other mathematicians, among them Takeuchi, Toro, and Wang, have independently proved Theorems 1 and 2 (see [22], [25]).

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