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GENERIC DEFORMATIONS OF LAGRANGIAN AND LEGENDRIAN MAPS

SERGEY STANCHENKO
Moscow Aviation Institute
Volokolamskoe shosse, 4, 125871 Moscow, Russia
E-mail: stan@k804.mainet.msk.su

1. Introduction

a. Investigating ocean and atmosphere flows Nye and Thorndike [2] have studied typical bifurcations of three dimensional vector fields depending on time. One can describe such a field as a one-parameter family of maps from R^3 to R^3 or as a map from $R^1 \times R^3 = R^4$ to R^3.

To study them the authors of [2] consider sections of stable maps from R^4 to R^4. There is one family in their list of typical sections for which the set of critical values for an isolated value of the parameter is equal to the caustic of a Lagrangian D_4 map.

This leads to another problem: To study properties of Lagrangian and Legendrian maps included in generic families of maps with a suitable number of parameters. In this way V-versal deformations of Lagrangian D_k and Legendrian A_k maps are considered below.

b. As a Lagrangian map is the restriction to a Lagrangian submanifold of the projection that defines a Lagrangian fiber bundle, Lagrangian maps may be locally considered as maps from R^n to R^n. The normal form of Lagrangian stable maps is given by the corresponding classification theorem ([1]).

Example. The normal form of Lagrangian A_k maps coincides with that of stable Whitney maps A_k:

$$A_k : (R^n(x), 0) \to (R^n(y), 0),$$

$$y_1 = \pm(k+1)x_1^k + (k-1)x_2x_1^{k-2} + \ldots + 2x_{k-1}x_1, \tag{1}$$

$$y_i = x_i, \quad i = 2, \ldots, n, \quad k-1 \le n.$$

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It is obvious that Lagrangian A_k maps are stable in the class of general maps.

EXAMPLE. The Lagrangian $D_k^{\pm}$ maps have the following normal form:

$$
\begin{aligned}
&D_k^{\pm} : (R^n(x), 0) \to (R^n(y), 0),\\
&y_1 = x_2^2 \pm (k-1)x_1^{k-2} + (k-2)x_3x_1^{k-3} + \ldots + 2x_{k-1}x_1,\\
&y_2 = 2x_1x_2,\\
&y_i = x_i \quad i = 3, \ldots, n, \quad k-1 \leq n, \quad k \geq 4.
\end{aligned}
\tag{2}
$$

In this paper the properties of the maps (2) are described. The V-versal deformation preserving the origin is a one parameter deformation for the maps (2). For even k the Lagrangian D_k^+ maps fall into stable maps A_k (at two isolated points), the Lagrangian D_k^- maps decompose only into A_{k-1} , A_{k-2} etc. For odd k the Lagrangian D_k maps fall into A_k (at one isolated point).

c. A map from a Legendrian submanifold to the base of a Legendrian bundle may be locally considered as a map from R^n to R^{n+1}.

EXAMPLE. The normal form of Legendrian A_k maps is given by (see [1])

$$
\begin{aligned}
&LA_k : (R^n(y,x), 0) \to (R^{n+1}(q), 0),\\
&q_1 = \varphi_1(y,x),\\
&q_i = x_i, \quad i = 2, \ldots, n, \quad q_{n+1} = \varphi_2(y,x),
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
\varphi_1 &= (k+1)y^k + (k-1)x_2y^{k-2} + \ldots + 2x_{k-1}y,\\
\varphi_2 &= kx^{k+1} + (k-2)x_2x_1^{k-1} + \ldots + x_{k-1}y^2.
\end{aligned}
$$

In the Legendrian case the following results are obtained: The V-versal deformation of Lagrangian A_k maps preserving the origin is a $k-1$-parameter deformation. The bifurcational diagram for this family is constructed. Outside the bifurcational set the maps of this family are stable and at isolated points they are RL-equivalent to the trivial extension of the stable maps that has the image of a "Whitney umbrella". Legendrian A_k maps have infinite RL- and topological codimension.

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2. V-versal deformation of Lagrangian D_k maps

PROPOSITION 1. *The V-versal deformation $D_k^{\pm}(t,c)$ of the maps* (2) *is given by*

$$
\begin{aligned}
y_1 &= \varphi_1(x) + tx_2 + c_1,\\
y_2 &= \varphi_2(x) + c_2,\\
y_i &= x_i + c_i, \qquad i = 3, \ldots, n,
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
\varphi_1(x) &= x_2^2 \pm (k-1)x_1^{k-2} + (k-2)x_3x_1^{k-3} + \ldots + 2x_{k-1}x_1,\\
\varphi_2(x) &= 2x_1x_2,
\end{aligned}
$$

P r o o f. Direct calculations.

The main result for Lagrangian D_k **series.** Let $D_k(t)$ be the family (4) with $c = 0 : D_k^{\pm}(t) = D_k^{\pm}(t, 0)$, and let $k - 1 = n$.

THEOREM 1. *If k is even, $t \neq 0$, then $D_k^{+}(t)$ has 2 singular points at which it is RL-equivalent to A_k (1). These points have coordinates*

$$x_{10} = \pm s_1|t|^{2/(k-2)}, \quad x_{20} = -t/k, \quad x_{i0} = s_i x_{10}^{i-2}, \quad i = 3, \dots, n.$$

$D_k^{-}(t)$ has no A_k points.

If k is odd, $t \neq 0$, then $D_k(t)$ has one singular point at which it is RL-equivalent to A_k. This point has coordinates

$$x_{10} = s_1 t^{2/(k-2)}, \quad x_{20} = -t/k, \quad x_{i0} = s_i x_{10}^{i-2}, \quad i = 3, \dots, n,$$

for some s_1, $s_3, \dots, s_n$.

P r o o f. It is sufficient to prove two propositions:

A. If $t \neq 0$, then $D_k^{\pm}(t)$ has the corresponding number of singular points with Boardman type $\Sigma^{\frac{1\dots1}{k}}$.

B. $D_k^{\pm}(t)$ is stable at these points.

First we find all the points of $\Sigma^{\frac{1\dots1}{k}}$ Boardman type.

LEMMA 1. *Let $x_1^2 + x_2^2 \neq 0$. Then for $D_k^{\pm}(t)$*

$$\Sigma^{\frac{1\dots1}{i}} = \{x \in R^n \mid B_1(x) = 0, \dots B_{i-1}(x) = 0,\ B_i(x) \neq 0\},$$

where

$$B_i = b_{i1}x_1^{k-2} + b_{i2}x_3x_1^{k-3} + \dots + b_{ik-2}x_{k-1}x_1 + b_{ik-1}x_2 + b_{ik}x_2^2$$

and $(b_{ij}) = B$ is the $(k-1) \times k$-matrix:

(5)
$$\begin{pmatrix} \pm(k-1)(k-2) & (k-2)(k-3) & \dots & 2 & -t & -2 \\ \mp(k-1)(k-2)^2 & -(k-2)(k-3)^2 & \dots & -2 & -t & -4 \\ & & \dots & & & \\ (-1)^{k-2}(k-1)(k-2)^{k-1} & (-1)^{k-2}(k-2)(k-3)^{k-1} & \dots & (-1)^{k-2}2 & -t & -2^{k-1} \end{pmatrix}$$

P r o o f. Direct calculations.

Let $S = \{(x_1, \dots, x_n) \in R^n \mid x_1 = 0,\ x_2 = 0\}$.

LEMMA 2. *S contains no points of $\Sigma^{\frac{1\dots10}{k}}$ Boardman type, and it contains one point of $\Sigma^{\frac{1\dots1}{k-1}}$ Boardman type. This point is $(0\dots0)$.*

Thus to find the points of $\Sigma^{\frac{1\dots1}{k-1}}$ Boardman type we should solve the system of equations

$$B_1 = 0, \dots, B_{k-1} = 0$$

where B_i are as in lemma 1. This is a system of linear algebraic equations over the monomials x_1^{k-2}, $x_3x_1^{k-3}, \dots, x_{k-1}x_1$, x_2, x_2^2. It may be represented in the following way:

$$(\tilde{b}_{ij}) \cdot \begin{pmatrix} x_1^{k-2} \\ x_3x_1^{k-3} \\ \vdots \\ x_{k-1}x_1 \\ x_2 \end{pmatrix} = x_2^2 \cdot \begin{pmatrix} 2 \\ 4 \\ \vdots \\ 2^{k-1} \end{pmatrix}, \tag{6}$$

where $(\tilde{b}_{ij})$ is the matrix (b_{ij}) without the last column. If the linear system with $(k-1)\times(k-1)$ matrix $(\tilde{b}_{ij})$ is solvable, then for some values $s_1, \dots, s_{k-1}$,

$$x_1^{k-1} = s_1x_2^2, \quad tx_2 = s_2x_2^2, \quad x_ix_1^{k-i} = s_ix_2^2, \quad i = 3, \dots, k-1. \tag{7}$$

The equation $tx_2 = s_2x_2^2$ has 2 solutions: $x_{20} = t/s_2$ and $x_{20} = 0$. The second solution is non-proper by lemma 2. Then the number of solutions of the system (6) is equal to that of the equation $x_1^{k-2} = s_1x_2^2$.

If k is odd, then this equation has one real solution, and $D_k(t)$ has one point of $\Sigma^{\frac{1\dots1}{k-1}}$ type. To complete the proof of proposition A we need the following algebraic lemma:

Lemma 3. *Let $M = (m_{ij})$ be a $k \times n$ matrix $(n > k)$ with each column a geometric progression with ratio l_i, $l_i \neq l_j$. Then there exists a non-singular $k \times k$ matrix C such that $C \cdot M$ is as follows*:

$$\begin{pmatrix} * & & & \dots & \\ 0 & \ddots & & \dots & \\ & 0 & m_{kk}q_k & \dots & m_{kn}q_n \end{pmatrix},$$

where

$$q_i = (1 - l_1/l_i)\dots(1 - l_{k-1}/l_i).$$

In other words we can see the elements of the last row of M after reducing it to the triangle matrix.

Proof. Direct calculations.

Using Lemma 3 we may get the following results:

Corollary 1. *If $x_0 = (x_{01} \dots x_{0k-1})$ is the solution of the system* (6) *and $x_{02} \neq 0$ then*

$$\pm x_{01}^{k-2} = \frac{(-1)^{k-2}}{k-1} \cdot \left(\frac{t}{k}\right)^2, \qquad x_{02} = -\frac{t}{k}.$$

Corollary 2. *If k is even, then $D_k(t)$ has two points of $\Sigma^{\frac{1\dots1}{k-1}}$ type and for these points*

$$x_{01} = \frac{\pm 1}{1-k}\left(\frac{t}{k}\right)^{\frac{2}{k-2}}, \qquad x_{02} = -\frac{t}{k}.$$

If k is odd, then $D_k(t)$ has one point of $\Sigma^{\frac{1\dots1}{k-1}}$ type, and

$$x_{01} = \frac{1}{1-k}\left(\frac{t}{k}\right)^{\frac{2}{k-2}}, \qquad x_{02} = -\frac{t}{k}.$$

Corollary 3. *If x_0 is a point of $\Sigma^{\frac{1\dots1}{k-1}}$ Boardman type for $D_k(t)$, then $B_k(x_0) \neq 0$ (i.e. x_0 is a point of $\Sigma^{\frac{1\dots1}{k}}$ Boardman type for $D_k(t)$).*

That completes the proof of proposition A.

Corollary 4. 1) $B_1 \dots B_{k-1} \in \mathbf{m}(x_1 - x_{01}, \dots, x_{k-1} - x_{0k-1})$.
2) $\left|\dfrac{\partial(B_1, \dots, B_{k-2})}{\partial(x_2, \dots, x_{k-1})}\right|_{x=x_0} \neq 0.$

Now to prove the stability of $D_k(t)$ at x_0 we use the following construction: Let the germ of $D : (R^n, x_0) \to (R^n, y_0)$ have the Boardman type $\Sigma^{\frac{1\dots10}{k}}$ at x_0, and η be the germ of a smooth vector field whose direction coincides with the direction of the null-space of the derivative of the map D. Consider the functions $B_i(x)$ such that $B_1(x)$ is the Jacobian of D,

$$B_2(x) = dB_1(\eta), \dots, B_k = dB_{k-1}(\eta).$$

Obviously, $B_1(x_0) = \dots = B_{k-1}(x_0) = 0$.

Proposition 2. *If the differentials $dB_1 \dots dB_{k-2}$ are independent at x_0, then the germ D is RL-equivalent to the germ of a Whitney A_k map at x_0.*

By corollary 4 if $D_k(t)$ has the Boardman type $\Sigma^{\frac{1\dots10}{k}}$ at x_0, then all the conditions of proposition 2 are fulfilled. That completes the proof of the theorem.

3. Proof of proposition 2. This proof is based on two simple lemmas.

Lemma 4. *In some coordinates u and v the germ of D may be represented by*

$$D : (R^n(u), 0) \to (R^n(v), 0), \qquad v_1 = \varphi(u), \quad v_i = u_i, \quad i = 2, \dots, k-1, \tag{8}$$

where

$$\varphi(u) = u_1^k + \varphi_1(u_2 \dots u_n)u_1^{k-1} + \dots + \varphi_{k-1}(u_2 \dots u_n)u_1 + \varphi_k$$
$$\varphi_1, \dots, \varphi_{k-1} \in \mathbf{m}(u), \qquad n = k-1,$$

and in these coordinates $\eta = \partial/\partial u_1$.

Lemma 5. *The following conditions are equivalent*:

1) *The germ of map* (8) *is stable at* 0.

2)
$$\left|\frac{\partial(\varphi_2, \dots, \varphi_{k-1})}{\partial(u_2, \dots, u_{k-1})}\right|_{u=0} \neq 0.$$

3)
$$\left|\frac{\partial(\varphi', \dots, \varphi^{(k-2)})}{\partial(u_2, \dots, u_{k-1})}\right|_{u=0} \neq 0,$$

where $\varphi^{(i)} = \partial^i\varphi/\partial u_1^i$.

4)
$$\left|\frac{\partial(B_1, \dots, B_{k-2})}{\partial(x_2, \dots, x_{k-1})}\right|_{x=x_0} \neq 0,$$

where the basis vectors of the coordinates $x_2,\dots,x_{k-1}$ are transversal to the vector η at x_0.

Proof of lemma 5. a) 1)$\Leftrightarrow$2). This follows from the theorem on stability of expansion of genotype [1].

b) 2)$\Leftrightarrow$3). This follows from the rules of differentiation.

c) The functions $B_1,\dots,B_{k-2}$ are the sequential derivatives in the direction of η of the Jacobian $B_1 = |\partial y/\partial x|$. The functions $\varphi',\dots,\varphi^{(k-2)}$ are the sequential derivatives in the direction of η of the Jacobian $K = |\partial y/\partial x|$.

Thus the ideals generated by $B_1,\dots,B_{k-2}$ and by $\varphi',\dots,\varphi^{(k-2)}$ coincide. The basis vectors of the coordinates $x_2,\dots,x_{k-1}$ are transversal to the vector η, and the coordinates $u_2,\dots,u_{k-1}$ have the same property. Then 3) and 4) are equivalent.

4. V-versal deformations of Legendrian A_k maps

PROPOSITION 3. *The V-versal deformation of the map* (3) *is given by*

$$\begin{aligned} q_1 &= \varphi(y,x) + c_1,\\ q_i &= x_i + c_i, \qquad i = 2,\dots,n,\\ q_{n+1} &= \varphi_2(\lambda,y) + P(\lambda,y) + c_{n+1}, \end{aligned} \tag{9}$$

where $y\in R^1$, $x\in R^{n-1}$, $\lambda\in R^{k-1}$, $c\in R^{n+1}$ and

$$\begin{aligned} \varphi_1 &= (k+1)y^k + (k-1)x_2y^{k-1} + \dots + 2x_{k-1}y,\\ \varphi_2 &= ky^{k+1} + (k-2)x_2y^{k-1} + \dots + x_{k-1}y^2,\\ P(\lambda,y) &= \lambda_1y^{k-1} + \dots + \lambda_{k-2}y^2 + \lambda_{k-1}y. \end{aligned}$$

Let $\Sigma_P \subset R^{k-1}(\lambda)$ be the discriminant set for the polynomial $P'(\lambda,y) = \partial P/\partial y$.

THEOREM 2. *If $\lambda\in\Sigma_P$ then the maps* (9) *are nonstable for each c. If $\lambda\notin\Sigma_P$ then the maps* (9) *are stable. Their image is RL-equivalent to the trivial extension of the "Whitney umbrella". The preimage of the umbrellas set is a finite combination of planes of codimension* 2.

Proof. The Jacobi matrix is

$$\begin{pmatrix} \varphi_1' & * \\ 0 & E_{n-2} \\ y\varphi_1' + P' & ** \end{pmatrix}$$

where $\varphi' = \partial\varphi/\partial y$, $P' = \partial P/\partial y$.

The vector field $\eta = \partial/\partial y$ coincides with the direction of the null-space of M. The set $\Sigma^{\frac{1\dots1}{l}} = \Sigma^{1_l}$ is defined by the equations

$$\begin{matrix} \varphi_1'(y,x)=0, & \varphi_1''(y,x)=0, & \dots & \varphi_1^{(l)}(y,x)=0,\\ P'(\lambda,y)=0, & P''(\lambda,y)=0, & & P^{(l)}(\lambda,y)=0. \end{matrix}$$

If y_0 is a root of $P'(\lambda,y)=0$ with multiplicity l, then Σ^{1_l} is defined by $l+1$ equations

$$y = y_0, \quad \varphi'(y_0,x)=0,\ \dots,\ \varphi^{(l)}(y_0,x)=0.$$

Thus Σ^{1_l} is a plane of codimension $l+1$. According to the Boardman formula this codimension is $2l$. Then if $l>1$, we have nonstability. In case $l=1$ after some calculations, we may see the extension of the "Whitney umbrella".

COROLLARY 5. *The V-versal deformation of a Legendrian A_3 map is a 2-parameter deformation. It consists of maps equivalent to the "umbrella" at not more than one point.*

5. Generic deformations of Legendrian A_k maps.

Now we compare V- and RL-equivalence for deformations of Legandrian A_k maps. It is easy to prove

PROPOSITION 4. *A generic deformation of a Legendrian A_k map ($k \geq 3$) is RL-equivalent to the following deformation*:

$$q_1 = \varphi_1, \qquad q_i = x_i, \quad i = 2, \dots, n, \qquad q_{n+1} = \varphi_2 + h(x, y), \tag{10}$$

where φ_1 and φ_2 are the same as in (3), *and $h(x, y)$ is an arbitrary smooth function.*

As was shown in the preceding section the V-versal deformation of the Legendrian A_3 map has not more than one "umbrella" point. Another situation is for generic deformations:

Let O_ϵ be the ϵ-sphere in the space of all coefficients of the Taylor series of h at 0, Q_δ be the δ-sphere around the origin in $R^3(q)$ and let $n = 2$.

PROPOSITION 5. *For arbitrary $\epsilon > 0$, $\delta > 0$, and integer m there is a function h such that*

1) *All the Taylor coefficients of h are in O_ϵ.*

2) *The map* (10) *is equivalent to the "Whitney umbrella" at m points and all the preimages of these points are in Q_δ.*

Proof. All the points at which the image of the map (10) is equivalent to the "umbrella" can be defined from the system of equations

$$\varphi'_{1y} = 0, \qquad h'_y = 0.$$

We can take the polynomial $h = h_1 + h_2 y + \dots + h_m y^m$ with sufficiently small coefficients.

Thus Legendrian A_k maps have no finite RL-versal and finite topologically versal deformations.

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