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SINGULARITIES OF WAVE FRONTS AT THE BOUNDARY BETWEEN TWO MEDIA

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1. Introduction. The subject of this talk is the study of Legendre singularities which arise from the problem of wave front refraction at a common boundary of two media.

Example. Let a wave front propagate in a plane divided by a line in two media with the propagation velocities v_1 and v_2. It is well known that the formula of Snellius holds: $\sin\varphi_1/\sin\varphi_2 = v_1/v_2$, where φ_1 and φ_2 are the incidence angles of incoming and refracted rays. If $v_1 < v_2$, then there exists a value $\bar\varphi_1$ such that the refracted ray is tangent to the boundary. This point of the boundary is called the point of complete reflection. It is not difficult to see that at the point of complete reflection an envelope of the refracted rays appears and the refracted front becomes singular.

In terms of contact geometry this situation may be formulated as follows:

Let M be a smooth manifold of dimension $n+1$, $J^1(M,\mathbf{R})$ be the contact manifold of 1-jets of functions on M. For any hypersurface $H\subset J^1(M,\mathbf{R})$ and an integral submanifold $\partial L\subset H$ of dimension n (for the contact hyperplanes distribution (see [2], Chapter 3)), we call a Legendre submanifold L such that $\partial L\subset L\subset H$ *the solution of the Cauchy problem for H with the initial value ∂L*. If L is a 1-graph of some function on M, then this function is the solution of the corresponding first order partial differential equation ([1]).

The *front* of a Legendre manifold L is the image of L under the natural projection of forgetting derivatives, $\pi : J^1(M,\mathbf{R})\to M\times\mathbf{R}$. A family of hypersurfaces $F(x,q,t) = 0, x\in X, q\in M, t\in\mathbf{R}$, where X is a space of additional parameters, defining the Legendre submanifold $L = \{(t,q,p)\in J^1(M,\mathbf{R}) \mid \exists x,\ F = F'_x = 0, p = F'_q\}$ with the front $\{(t,q)\in\mathbf{R}\times M \mid \exists x,\ F = F'_x = 0\}$ is called a *generating family of hypersurfaces* of L ([2]).

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Let $\partial M \subset M$ be a smooth hypersurface dividing M in two domains M_1 and M_2; H_1, $H_2 \subset J^1(M, \mathbf{R})$ be smooth hypersurfaces of first order partial differential equations. We are interested in the discontinuous equation $H = (H_1 \cap J^1(M_1, \mathbf{R})) \cup (H_2 \cap J^1(M_2, \mathbf{R}))$.

Let us consider the Cauchy problem for H_1 with an initial value $\partial L_0 \subset J^1(M_1, \mathbf{R})$. Let L_1 be the solution of this problem. Let $\partial J^1(M, \mathbf{R})$ denote the preimage of $\partial M \times \mathbf{R}$ under $\pi : J^1(M, \mathbf{R}) \to M \times \mathbf{R}$. The intersection $\partial L_1 = L_1 \cap \partial J^1(M, \mathbf{R})$ defines the initial value for H_2 as follows ([3]):

1. Let $\rho : \partial J^1(M, \mathbf{R}) \to J^1(\partial M, \mathbf{R})$ be the natural projection along the characteristics of $\partial J^1(M, \mathbf{R})$ and $\bar{\partial} L = \rho(\partial L_1)$. In general ($L_1$ is transversal to $\partial J^1(M, \mathbf{R})$), $\bar{\partial} L$ is a Legendre *submanifold* of $J^1(\partial M, \mathbf{R})$.

2. Let ∂L_2 denote the preimage of $\bar{\partial} L$ in $H_2 \cap \partial J^1(M, \mathbf{R})$ under ρ. We consider this integral (for the contact hyperplanes distribution) variety (smooth if $\bar{\partial} L$ is transversal to the critical values set of ρ restricted to $H_2 \cap \partial J^1(M, \mathbf{R})$) as the initial value for H_2.

Definition. The solution L_2 of the Cauchy problem for H_2 with initial value ∂L_2 is called *refracted*.

We construct the solution of the Cauchy problem for discontinuous H with the initial value ∂L_0 as $L = (L_1 \cap J^1(M_1, \mathbf{R})) \cup (L_2 \cap J^1(M_2, \mathbf{R}))$.

Remark. The construction of ∂L_2 naturally arises in theoretical mechanics: trajectories of a Hamiltonian system minimize the action functional. Studying Hamiltonian systems with discontinuous hamiltonians $h : T^*M \to \mathbf{R}$, $h(q,p) = h_1(q,p)$ if $q \in M_1$, $h(q,p) = h_2(q,p)$ if $q \in M_2$, where h_i are smooth, and minimizing the corresponding action functional one gets the construction described above.

At a point $x \in \partial J^1(M, \mathbf{R}) \cap H_2$ let the restriction of ρ to $H_2 \cap \partial J^1(M, \mathbf{R})$ be equivalent to the A_i-singularity (i.e. to the projection of a hypersurface $\{(z, \lambda) \mid z^{i+1} + \lambda_1 z^{i-1} + \ldots + \lambda_i = 0\}$ in the total space of a fibration $(z, \lambda) \mapsto \lambda$ to the base space $\{\lambda\}$), $\rho(x) \in \bar{\partial} L$ and $y = \pi_\partial \circ \rho(x)$ where $\pi_\partial : J^1(\partial M, \mathbf{R}) \to \partial M \times \mathbf{R}$ is the derivatives forgetting projection.

Definition. The point y is called an *A_i-point of complete reflection*.

Example. An optical wave front propagating in $\mathbf{R}^n$ divided by a smooth $\partial \mathbf{R}^n$ in two isotropic and homogeneous media, in the natural coordinates (t, q, p) on $J^1(\mathbf{R}^n, \mathbf{R})$ (here q are some coordinates on $\mathbf{R}^n$, t is a coordinate on $\mathbf{R}$, $p = t'_q$) is the front of an H's solution for $H_i = \{(t, q, p) \mid v_i^2(p_1^2 + \ldots + p_n^2) = 1\}$, $i = 1, 2$, where v_i are the propagation velocities. One can see that the notion of complete reflection point defined above coincides with that of geometrical optics.

In the following, saying that an object is in general position (or simply generic) we mean that it belongs to an open and everywhere dense set in the Whitney topology.

2. Results

2.1. *Wave front refraction*

Theorem 1. *If the front of $\bar{\partial} L$ at an A_i-point of complete reflection is smooth, then for a generic refracted solution its front in some neighbourhood of this complete reflection* *point is given by a generating family of hypersurfaces*

$$yx^{i+1} + y^2\varphi(x, y, q_3, \dots, q_{n+1}) + yx^{i-1}q_{i+1} + \dots + yxq_3 + yq_2 + xq_0 + q_1 = 0$$

for some i such that $i < n + 1$, where φ is some smooth function, and $(q_0, \dots, q_{n+1})$ are coordinates on $M \times \mathbf{R}$ such that $\partial M \times \mathbf{R} = \{q_0 = 0\}$.

Corollary 1. *If the front of $\bar{\partial}L$ at an A_1-point of complete reflection is smooth, then for a generic refracted solution its front in some neighbourhood of this complete reflection point is given by the generating family of hypersurfaces*

$$yx^2 + y^{m-1} + y^{m-2}q_{m-1} + \dots + yq_2 + xq_1 + q_0 = 0$$

for some m such that $2 < m < n + 3$.

Corollary 2. *Let $\varphi(0) \neq 0$, then for a generic refracted solution its front in some neighbourhood of the considered A_i-point of complete reflection, in some coordinates $(q_0, \dots, q_{n+1})$ on $M \times \mathbf{R}$, is given by a generating family of hypersurfaces*

$$x^{2i+2} + x^{2i}q_i + \dots + x^{i+1}q_2 + x^i\varphi_1(q) + \dots + x^2\varphi_{i-1}(q) + xq_0 + q_1 = 0,$$

where $i < n+1$ and φ_j are some smooth functions such that the rank of $\frac{\partial(\varphi_1,\dots,\varphi_{i-1})}{\partial(q_{i+1},\dots,q_{n+1})}|_{q=0}$ is equal to zero.

In what follows we assume $H_2 = \{h(t, q, p) = 0\}$, the function h is smooth, *quadratic* and convex in p. This condition guarantees that only A_1-points of complete reflection may occur. Note that the Hamilton-Jacobi equations of classical mechanics' natural systems belong to this class.

Theorem 2. *If the restriction of the projection π_∂ to $\bar{\partial}L$ at the point in $J^1(\partial M, \mathbf{R})$ over an A_1-point of complete reflection is equivalent to the A_{n-k+1}-singularity, then for a generic refracted solution in some neighbourhood of this point of complete reflection, in suitable coordinates $(q_0, \dots, q_{n+1})$ on $M \times \mathbf{R}$, its front is given by one of the following generating families*:

1. $n = 1$: $F(x, y, q) = x^2y + y^2\varphi(x, y) + yq_1 + xq_0 + q_2 = 0$,
2. $n > 1$: $F(x, y, z, q) = z^{n+2-k} + y(x^2 + z^m) + y^2\varphi(x, y, z, q_2, \dots, q_{n-1}) + z^{n-k}q_{k+1} + \dots + zq_n + y(z^{m-1}q_m + \dots + zq_2 + q_1) + xq_0 + q_{n+1} = 0$, *where $0 < m \leq k \leq n$, $m < 2+n-k$, φ is some smooth function.*

This theorem provides typical singularities which may occur at points of complete reflection in $J^1(M, \mathbf{R})$ for $\dim M = 2, 3, 4$. Namely:

Corollary 3. *Generic generating families* 1. *and* 2. *are V-equivalent* (*i.e. can be transferred to each other by a diffeomorphism of the form*

$$(x, y, z) \mapsto (x'(x, y, z, q), y'(x, y, z, q), z'(x, y, z, q), q'(q));$$

such families define Legendre equivalent germs) *to the following generating families*:

$$n = k = 1 : y^2 + x^4 + x^2q_1 + xq_2 + q_3 = 0,$$
$$n = k = 2 : z^2 + y^2 + x^4 + x^2q_1 + xq_2 + q_3 = 0,$$
$$z^2 + y^3 \pm yx^2 + y^2q_1 + yq_2 + xq_3 + q_4 = 0,$$
$$n = 2, k = m = 1 : z^2 + y^2 + x^6 + x^4q_1 + x^3\varphi(q) + x^2q_2 + xq_3 + q_4 = 0,$$

$$\begin{aligned}
n=k=3: &\; z^2+y^2+x^4+x^2q_1+xq_2+q_3=0,\\
&\; z^2+y^3\pm yx^2+y^2q_1+yq_2+xq_3+q_4=0,\\
&\; z^2+y^4+yx^2+y^3q_1+y^2q_2+yq_3+xq_4+q_5=0,\\
n=3, k=m=2: &\; z^2+x^3+y^4+y^2xq_1+y^2q_2+xy\varphi(q)+yq_3\\
&\; +xq_4+q_5=0,\\
n=3, k=2, m=1: &\; z^2+y^2+x^6+x^4q_1+x^3q_2+x^2q_3+xq_4+q_5=0,\\
n=3, k=m=1: &\; z^2+y^2+x^8+x^6q_1+x^5\varphi_1(q)+x^4q_2=0,\\
&\; +x^3\varphi_2(q)+x^2q_3+xq_4+q_5=0.
\end{aligned}$$

Here φ and φ_i are functional moduli.

2.2. *Normal forms of smooth functions on Legendre submanifolds in the spaces of Legendre fibrations.* For the proof of Theorem 2 we need to reduce to the normal form a generic smooth function on $\bar{\partial}L$ by a Legendre equivalence (i.e. a contactomorphism of the total space of a Legendre fibration preserving the fibers).

Let $PT^*\mathbf{R}^{n+1}$ be the projectivization of the cotangent bundle of $\mathbf{R}^{n+1}$, $i:(L,x)\hookrightarrow(PT^*\mathbf{R}^{n+1},y)$ be a Legendre submanifold germ, $\Phi:(PT^*\mathbf{R}^{n+1},y)\to(\mathbf{R},0)$ be a smooth function germ, $(q_1,\dots,q_{n+1};p_1:\dots:p_{n+1})$ be the standard coordinates on $PT^*\mathbf{R}^{n+1}$ ($q_1,\dots,q_{n+1}$ being some coordinates on $\mathbf{R}^{n+1}$, and $p_1:\dots:p_{n+1}$ the homogeneous coordinates on the fibers of the projectivized cotangent bundle induced from the canonical coordinates on the fibers of the cotangent bundle of $\mathbf{R}^{n+1}$) in some neighbourhood of y. Let the derivative of $\pi\circ i$ at x , where $\pi:PT^*\mathbf{R}^{n+1}\to\mathbf{R}^{n+1}$ is the cotangent bundle projectivization, have one-dimensional kernel.

THEOREM 3. *For a generic pair $(i(L),\Phi|_{i(L)})$ its germ at the point $i(x)$ such that $(\pi\circ i)_{*,x}$ has one-dimensional kernel and $\Phi(i(x))=0$, is Legendre equivalent to the germ at $(q=0,\ p_1=\ldots=p_n=0)$ of the pair (L_0,Φ_0) defined by*:

1. $L_0=\{(q;p)\mid q_{n+1}=f(q_{k+1},\dots,q_{n-1},p_n)+p_nq_n; p_i=\partial f/\partial q_i$ *for* $i=1,\dots,n-1; q_n=-\partial f/\partial p_n; p_{n+1}=1\}$, *where* $f=p_n^{n+2-k}+p_n^{n-k}q_{k+1}+\ldots+p_n^2q_{n-1}$,
2. $\Phi_0(q_1,\dots,q_{n-1},p_n)=p_n^m+q_mp_n^{m-1}+\ldots+q_1$

for some k,m such that $0<m\le k\le n$, $m<2+n-k$.

3. Proofs

3.1. *Preliminary results.* Since we are interested in local situation, we may replace the Legendre fibration $J^1(M,\mathbf{R})\to M\times\mathbf{R}$ of derivatives forgetting by the Legendre fibration $PT^*(M\times\mathbf{R})\to M\times\mathbf{R}$ (the projectivized cotangent bundle) because all Legendre fibrations of equal dimensions are locally isomorphic (see [2], Chapter 3).

Let us consider the action on $\partial PT^*(M\times\mathbf{R})$ of a Legendre equivalence of $PT^*(M\times\mathbf{R})$ preserving $\partial PT^*(M\times\mathbf{R})$. In the canonical coordinates $(q;p)=(q_0,q';p_0:p')$ induced from the coordinates q on $M\times\mathbf{R}$ such that $\partial M\times\mathbf{R}=\{q_0=0\}$, any Legendre equivalence of $PT^*(M\times\mathbf{R})$ preserving $\partial PT^*(M\times\mathbf{R})$ is generated by a diffeomorphism of $M\times\mathbf{R}$

of the form

$$(q_0, q') \mapsto (q_0\xi_0(q), \tilde{q}'(q)), \qquad \xi_0(0) \neq 0.$$

Writing this Legendre equivalence explicitly one gets the following:

LEMMA 1. *In the chosen coordinates the restriction to $\partial PT^*(M\times\mathbf{R})$ of any Legendre equivalence of $PT^*(M\times\mathbf{R})$ preserving $\partial PT^*(M\times\mathbf{R})$ is the superposition of the following two*:

1. *Legendre equivalence of* $PT^*(\partial M\times\mathbf{R})$,
2. $\tilde{p}_0 = \xi_0(q')p_0 + \ldots + \xi_{n+1}(q')p_{n+1}$, $\xi_0(0) \neq 0$.

Conversely for any composition of 1. *and* 2. *there exists a Legendre equivalence of* $PT^*(M\times\mathbf{R})$ *preserving* $\partial PT^*(M\times\mathbf{R})$ *and inducing this transformation.*

Let $H_2 = \{(q;p) \mid h(q,p) = 0\}$, with h smooth and quadratic in p. In the canonical coordinates $(q_0,\ldots,q_{n+1};p_0 : \ldots : p_{n+1}) = (q_0,q';p_0 : p')$ on $PT^*(M\times\mathbf{R})$ such that $\partial M\times\mathbf{R} = \{q_0 = 0\}$ we have

$$h(0,q',p_0,p') = (\xi_0(q')p_0 + \ldots + \xi_{n+1}(q')p_{n+1})^2 + \Phi(q',p')$$

for some smooth ξ_i , $\xi_0(0) \neq 0$ and $\Phi : PT^*(\partial M\times\mathbf{R}) \to \mathbf{R}$.

Applying Lemma 1 we get:

LEMMA 2. *The restriction of the canonical projection ρ to H_2 is reducible to the form $\{(0,q';p_0 : p') \mid p_0^2 + \Phi(q',p') = 0\} \mapsto (q';p')$ by a Legendre equivalence preserving $\partial PT^*(M\times\mathbf{R})$.*

Notice once again that H_2 is the zero set of some smooth and *quadratic* (in p) function.

3.2. *Proof of Theorem 1.* Let us choose a system of coordinates $(q_0,\ldots,q_{n+1})$ in some neighbourhood of the considered A_i-point of complete reflection $y = \pi_\partial\circ\rho(x)$ such that $\partial M\times\mathbf{R} = \{q_0 = 0\}$ and $\pi_\partial(\bar{\partial}L) = \{q_0 = q_1 = 0\}$. This is possible because $\pi_\partial(\bar{\partial}L)$ is smooth. These coordinates induce the canonical coordinates $(q;p)$ $(= (q_0,q';p_0 : p') = (q_0,q_1,q'';p_0 : p_1 : p''))$ in some neighbourhood of x. Under the assumption of the theorem at the point x the projection $\rho : \partial PT^*(M\times\mathbf{R})\cap H_2 \to PT^*(\partial M\times\mathbf{R})$ is equivalent to the A_i-singularity, hence $\partial PT^*(M\times\mathbf{R})\cap H_2 = \{(q;p) \mid q_0 = 0, p_0^{i+1} + f_0(q',p')p_0^i + \ldots + f_i(q',p') = 0\}$.

We have $\bar{\partial}L = \{(q;p) \mid q_0 = q_1 = p_0 = p'' = 0\}$, hence $\partial L_2 = \rho^{-1}(\bar{\partial}L)\cap H_2$ is of the form

$$\partial L_2 = \{(q;p) \mid p_0^{i+1} + \varphi_0(q'')p_0^i + \ldots + \varphi_i(q'') = q_0 = q_1 = p'' = 0, p_1 = 1\},$$

where $\varphi_j(q'') = f_j(0,q_2,\ldots,q_{n+1},1,0,\ldots,0)$. We can see that in general $i \leq n$.

A diffeomorphism of $M\times\mathbf{R}$ such that its inverse is of the form $q_0 = \tilde{q}_0$, $q_1 = \tilde{q}_1 - \tilde{q}_0\varphi_0(\tilde{q}')/(i+1)$, $q'' = \tilde{q}''$, induces a Legendre equivalence of $PT^*(M\times\mathbf{R})$ reducing ∂L_2 (forgetting tilde): $\partial L_2 = \{(q;p) \mid p_0^{i+1} + \psi_1(q'')p_0^{i-1} + \ldots + \psi_i(q'') = q_0 = q_1 = p'' = 0, p_1 = 1\}$.

In general the rank of $\frac{\partial(\psi_1,\ldots,\psi_i)}{\partial(q_2,\ldots,q_{n+1})}$ is maximal , say $\det \frac{\partial(\psi_1,\ldots,\psi_i)}{\partial(q_2,\ldots,q_{i+1})}\,|_{q=0}\neq 0$. A Legendre equivalence of $PT^*(M\times\mathbf{R})$ induced by $\tilde{q}_0 = q_0$, $\tilde{q}_1 = q_1$, $\tilde{q}_2 = \psi_i(q'')$, $\ldots$, $\tilde{q}_{i+1} =$

$\psi_1(q'')$, $\tilde{q}_{i+2} = q_{i+2}$, $\tilde{q}_{n+1} = q_{n+1}$ preserves $\partial PT^*(M \times \mathbf{R})$ and brings ∂L_2 to the form

$$\partial L_2 = \{(q;p) \mid p_0^{i+1} + q_{i+1}p_0^{i-1} + \ldots + q_2 = q_0 = q_1 = p'' = 0, p_1 = 1\}.$$

Assume that L_2 is a Legendre submanifold containing ∂L_2. Locally it is given by some generating function $S(p_J, q_I)(I \cup J = \{0, 2, \ldots, n+1\},\ 0, 2 \in J) : L_2 = \{(q;p) \mid q_1 = S, q_J = -S'_{p_J}, p_I = S'_{q_I}, p_1 = 1\}$, and we know the values of S and its derivatives at $p_2 = 0$:

$$\begin{aligned} S(p_0, 0, q_3, \ldots, q_{n+1}) &= S'_{p_0}(p_0, 0, q_3, \ldots, q_{n+1}) = 0, \\ S'_{q_i}(p_0, 0, q_3, \ldots, q_{n+1}) &= 0, \qquad i = 3, \ldots, n+1, \\ S'_{p_2}(p_0, 0, q_3, \ldots, q_{n+1}) &= p_0^{i+1} + q_{i+1}p_0^{i-1} + \ldots + q_3 p_0. \end{aligned}$$

Hence $S = p_0^{i+1}p_2 + p_2^2\varphi(p_0, p_2, q_3, \ldots, q_{n+1}) + p_0^{i-1}p_2 q_{i+1} + \ldots + p_0 p_2 q_3$, where φ is some smooth function. Hence L_2 can be given by the generating family from the statement of the theorem.

3.3. *Proof of Theorem 2.* The proof of the case $n = 1$ is exactly the same as in Theorem 1 (here $i = 1$ and in the generic case a point of complete reflection does not belong to the caustic in the first medium, hence the front of $\bar{\partial} L$ is smooth at this point).

Let $n > 1$. In the generic case $\bar{\partial} L$ is smooth. If the projection of $\bar{\partial} L$ to $\partial M \times \mathbf{R}$ is equivalent to the A_{n-k+1}-singularity, then from Theorem 3 and Lemma 2 we get that $\partial L_2 = \rho^{-1}(\bar{\partial} L) \cap H_2$ is of the form

$$\begin{aligned} & q_{n+1} = p_n^{n+2-k} + p_n^{n-k} q_{k+1} + \ldots + p_n^2 q_{n-1} + p_n q_n. \\ & q_0 = 0. \\ & q_1 = -p_0^2 - p_n^m - p_n^{m-1} q_m - \ldots - p_n q_2. \\ & q_n = -(n+2-k)p_n^{n+1-k} - (n-k)p_n^{n-k-1} q_{k+1} - \ldots - 2p_n q_{n-1}. \\ & p_1 = \ldots = p_k = 0. \\ & p_{k+1} = p_n^{n-k},\ \ldots,\ p_{n-1} = p_n^2. \\ & p_{n+1} = 1. \end{aligned}$$

Hence for any Legendre manifold L_2 such that

$$\partial L_2 = L_2 \cap \partial PT^*(M \times \mathbf{R}),$$

a generating function S is of the form

$$\begin{aligned} S(p_0, p_1, p_n, q_2, \ldots, q_{n-1}) = p_n^{n+2-k} &+ p_n^{n-k} q_{k+1} + \ldots \\ &+ p_n^2 q_{n-1} + p_1(p_0^2 + p_n^m + p_n^{m-1} q_m + \ldots + p_n q_2) \\ &+ p_1^2 \varphi(p_0, p_1, p_n, q_2, \ldots, q_{n-1}). \end{aligned}$$

This proves the theorem.

3.4. *Proof of Theorem 3.* The technique used here was developed in [5].

First of all we can reduce the germ of a generic Legendre submanifold $i(L) \subset PT^*\mathbf{R}^{n+1}$ at a point $i(x)$, where $(\pi \circ i)_*$ has one-dimensional kernel, by a Legendre equivalence of $PT^*\mathbf{R}^{n+1}$ to the germ at the point $q = p_1 = \ldots = p_n = 0$ of the Legendre submanifold $L_0 \subset PT^*\mathbf{R}^{n+1}$ given by the generating function $f(q_{k+1}, \ldots, q_{n-1}, p_n) =$

$p_n^{n+2-k} + p_n^{n-k} q_{k+1} + \ldots + p_n^2 q_{n-1}$, for some $k < n$, as in the theorem (see [2], Chapter 3). Now we should reduce a generic smooth function $\Phi|_{L_0} = \Phi(q_1, \ldots, q_{n-1}, p_n)$ (functions $q_1, \ldots, q_{n-1}, p_n$ form a coordinate system on L_0) such that $\Phi(0) = 0$ to the normal form by a Legendre equivalence of $PT^*\mathbf{R}^{n+1}$ preserving L_0.

Let us consider a subsidiary fibration $\kappa : \mathbf{R}^{n+2} \to \mathbf{R}^{n+1}$, $\kappa : (x, q) \mapsto q$, and the Legendre submanifold $N \subset PT^*\mathbf{R}^{n+2}$ having the smooth front $\{(x, q) \in \mathbf{R}^{n+2} \mid F(x, q) = f(q_{k+1}, \ldots, q_{n-1}, x) + xq_n + q_{n+1} = 0\}$. In $PT^*\mathbf{R}^{n+2}$ consider an hypersurface PA formed by hyperplanes in spaces tangent to $\mathbf{R}^{n+2}$, which contain lines tangent to fibers of κ. Then $L_0 = \kappa^*\pi(N \cap PA)$, where $\kappa^*\pi : PA \to PT^*\mathbf{R}^{n+1}$ is the fibration induced from κ by π, $\kappa^*\pi : (x, q; 0 : p) \mapsto (q; p)$ (see [2], Chapter 3). The function Φ on L_0 can be lifted by $\kappa^*\pi$ to $N \cap PA$ and can be extended to a function (also denoted by Φ) on N. The manifold N is diffeomorphic to its front $\{F(x, q) = x^{n-k+2} + x^{n-k} q_{k+1} + \ldots + q_{n+1} = 0\}$. This diffeomorphism sends Φ on N to Φ on the front of N, $\{F = 0\}$. In the chosen coordinates it can be expressed much more simply: $\Phi(q_1, \ldots, q_{n-1}, p_n)$ on L_0 goes to $\Phi(q_1, \ldots, q_{n-1}, x)$ on $\{F = 0\} \subset \mathbf{R}^{n+2}$. In these terms the problem of reducing a function $\Phi(q_1, \ldots, q_{n-1}, p_n)$ to the normal form by a Legendre equivalence of $PT^*\mathbf{R}^{n+1}$ preserving L_0 is equivalent to reducing the function $\Phi(q_1, \ldots, q_{n-1}, x)$ to the normal form on the hypersurface $\{F(x, q) = 0\} \subset \mathbf{R}^{n+2}$ (i.e. on the generating family of hypersurfaces for L_0) by a diffeomorphism of $\mathbf{R}^{n+2}$ preserving $\{F = 0\}$ and respecting κ. Indeed, preserving κ's fibers we get a generating family defining a Legendre submanifold which is Legendre equivalent to L_0 ([2], Chapter 3). Preserving $\{F = 0\}$ we get that in fact this new Legendre submanifold of $PT^*\mathbf{R}^{n+1}$ coincides with L_0.

Definition. A diffeomorphism $g : \mathbf{R}^{n+2} \to \mathbf{R}^{n+2}$ preserving $\{F = 0\}$ and respecting κ is called *admissible*.

Now we are going to describe vector fields which are the velocities of one-parameter families of analytic admissible diffeomorphisms. Let $\mathcal{E}_{n+2}$ and $\mathcal{E}_{n+1}$ denote the rings of analytic functions' germs at the points $0 \in \mathbf{R}^{n+2}$ and $0 \in \mathbf{R}^{n+1}$, and let V denote the real vector space of germs at 0 of vector fields on $\mathbf{R}^{n+2}$ with pointwise addition and multiplication by scalars. In V we consider a subspace $\tilde{V} = V_1 \oplus V_2$, where V_1 is the $\mathcal{E}_{n+2}$-module generated by $v^0 = F\frac{\partial}{\partial x}$, V_2 is the $\mathcal{E}_{n+1}$-module generated by v^i, where $v^i = \frac{\partial}{\partial q_i}$ for $i = 1, \ldots, k$ and $v^i = v_x^i(x, q)\frac{\partial}{\partial x} + v_{q_{k+1}}^i(q)\frac{\partial}{\partial q_{k+1}} + \ldots + v_{q_{n+1}}^i(q)\frac{\partial}{\partial q_{n+1}}$ for $i = k+1, \ldots, n+1$ are the solutions of the equations

$$v^i \bullet F = FF'_{q_i} (= Fx^{n+1-i}), \qquad i = k+1, \ldots, n+1.$$

Here the symbol $\bullet$ stands for the differentiation along the corresponding vector field.

Lemma 3. *The space $\tilde{V}$ is the tangent space to the space of germs at the origin of analytic admissible diffeomorphisms.*

Proof. Let us consider a one-parameter family of admissible diffeomorphisms g_t:

$$g_t(x, q) = (h_t(x, q), \varphi_{1,t}(q), \ldots, \varphi_{n+1,t}(q)),$$
$$H_t(g_t(x, q))F(g_t(x, q)) = F(x, q), \tag{1}$$

where H_t is some one-parameter family of analytic functions such that $H_t(0) \neq 0$.

Differentiating (1) with respect to t, for any t' we get the equation

$$v_x F'_x + v_{q_{k+1}} F'_{q_{k+1}} + \ldots + v_{q_{n+1}} F'_{q_{n+1}} = MF, \tag{2}$$

where $v_x, M \in \mathcal{E}_{n+2}$, $v_{q_i} \in \mathcal{E}_{n+1}$, $i = k+1, \ldots, n+1$, and

$$v = v_x \frac{\partial}{\partial x} + v_{q_{k+1}} \frac{\partial}{\partial q_{k+1}} + \ldots + v_{q_{n+1}} \frac{\partial}{\partial q_{n+1}} = \frac{d}{dt}\bigg|_{t=t'} g_t.$$

The function F is an R-versal deformation of the germ $F(x,0)$, hence it is R-infinitesimally versal, hence for M there exist functions $v'_x \in \mathcal{E}_{n+2}$; $v'_{q_i} \in \mathcal{E}_{n+1}$, $i = k+1, \ldots, n+1$, such that

$$v'_x F'_x + v'_{q_{k+1}} F'_{q_{k+1}} + \ldots + v'_{q_{n+1}} F'_{q_{n+1}} = M. \tag{3}$$

After substitution of (3) in (2), keeping in mind the uniqueness of decompositions (2) and (3) for analytic functions, we get that $v = v_x \frac{\partial}{\partial x} + v_{q_{k+1}} \frac{\partial}{\partial q_{k+1}} + \ldots + v_{q_{n+1}} \frac{\partial}{\partial q_{n+1}} = v'_x v^0 + v'_{q_{k+1}} v^{k+1} + \ldots + v'_{q_{n+1}} v^{n+1}$. The vector fields $v^1, \ldots, v^k$ correspond to changes of $q_1, \ldots, q_k$. The lemma is proved.

Corollary 4. 1. *The v^i are polynomial vector fields.*

2. *The v^i_x are regular with respect to x of order $n+2-i$ for $i = k+1, \ldots, n+1$ and of order $n+2-k$ for $i = 0$.*

3. $v^i(0) = 0$ *for $i = 0, k+1, \ldots, n+1$.*

Consider a function $\Phi(q_1, \ldots, q_{n-1}, x)$, with $\Phi(0) = 0$. Suppose that

$$\frac{\partial \Phi}{\partial x}(0) = \ldots = \frac{\partial^{m-1}\Phi}{\partial x^{m-1}}(0) = 0 \neq \frac{\partial^m \Phi}{\partial x^m}(0).$$

For a generic pair (L_0, Φ) we have $0 < k$, $m < k$. If $m \geq 2+n-k$, then, since we are interested in the restriction of Φ to $\{F = 0\}$, we can divide Φ by F and get $m < 2+n-k$. For a generic Φ, changing only $q_1, \ldots, q_k$, we can bring it to the form

$$\Phi = q_1 + q_2 x + \ldots + q_m x^{m-1} + x^m + f, \qquad f \in \mathbf{m}_{n+2}^{m+1},$$

where $\mathbf{m}_{n+2}$ is the maximal ideal in C^∞_{n+2}, the ring of smooth functions' germs at $0 \in \mathbf{R}^{n+2}$.

Let us consider

$$\Phi_0 = q_1 + q_2 x + \ldots + q_m x^{m-1} + x^m.$$

Lemma 4. *There exists an admissible diffeomorphism g such that $g(0) = 0$ and, for the corresponding germs, $\Phi \circ g = \Phi_0$.*

Proof. Connect Φ and Φ_0 by $\Phi_t = \Phi_0 + tf$. We are going to prove the existence of a family $g_t : \mathbf{R}^{n+2} \to \mathbf{R}^{n+2}$ of admissible diffeomorphisms such that

$$\Phi_t \circ g_t = \Phi_0. \tag{4}$$

After differentiating (4) with respect to t we get

$$\tilde{v}_{xt}(x,q)\frac{\partial \Phi_t}{\partial x} + \tilde{v}_{q_1 t}(q)\frac{\partial \Phi_t}{\partial q_1} + \ldots + \tilde{v}_{q_{n+1} t}(q)\frac{\partial \Phi_t}{\partial q_{n+1}} = -f, \tag{5}$$

where

$$\begin{aligned}\tilde{v}_{xt}(x,q) &= f_0^0(t,x,q)v_x^0(x,q) + f_0^{k+1}(t,q)v_x^{k+1}(x,q) + \ldots \\ &\quad + f_0^{n+1}(t,q)v_x^{n+1}(x,q), \\ \tilde{v}_{q_i t}(q) &= f_i(t,q), \qquad i = 1,\ldots,k, \\ \tilde{v}_{q_i t}(q) &= f_0^{k+1}(t,q)v_{q_i}^{k+1}(q) + \ldots + f_0^{n+1}(t,q)v_{q_i}^{n+1}(q), \qquad i = k+1,\ldots,n+1,\end{aligned}$$

where f_i and f_0^j are some unknown functions. Let us put $f_{m+1} = \ldots = f_k = 0$. Writing (5) in detail we have

$$\begin{aligned}(6)\qquad &(mx^{m-1} + (m-1)q_m x^{m-2} + \ldots + q_2 + tf'_x) \\ &\times (x^{n+2-k} + x^{n-k}q_{k+1} + \ldots + q_{n+1})f_0^0(t,x,q) \\ &+ (mx^{m-1} + (m-1)q_m x^{m-2} + \ldots + q_2 + tf'_x) \\ &\times (x^{n+1-k}/(n+2-k) + \ldots)f_0^{k+1}(t,q) \\ &+ \ldots + (mx^{m-1} + (m-1)q_m x^{m-2} + \ldots + q_2 + tf'_x) \\ &\times (x/(n+2-k))f_0^{n+1}(t,q) + (x^{m-1} + tf'_{q_m})f_m(t,q) \\ &+ \ldots + (1 + tf'_{q_1})f_1(t,q) \\ &+ tf'_{q_{k+1}}(f_0^{k+1}(t,q)v_{q_{k+1}}^{k+1}(q) + \ldots + f_0^{n+1}(t,q)v_{q_{k+1}}^{n+1}(q)) + \ldots \\ &+ tf'_{q_{n+1}}(f_0^{k+1}(t,q)v_{q_{n+1}}^{k+1}(q) + \ldots + f_0^{n+1}(t,q)v_{q_{n+1}}^{n+1}(q)) = -f(x,q).\end{aligned}$$

PROPOSITION 1. *For any $t \in \mathbf{R}$ there exist neighbourhoods V of t and W of $0 \in \mathbf{R}^{n+2}$ such that* (6) *has a solution smooth in $V \times W$.*

Proof. In the ring C^∞_{n+2} of infinitely smooth functions' germs at $0 \in \mathbf{R}^{n+2}$ we consider the ideal $A = F\frac{\partial\Phi_t}{\partial x}C^\infty_{n+2}$. Let B denote the quotient module C^∞_{n+2}/A. For any $t \in \mathbf{R}$ the function $F\frac{\partial\Phi_t}{\partial x}$ is regular with respect to x of order $n+m+1-k$. From Malgrange's division theorem it follows that B is a finitely generated C^∞_{n+1}-module (C^∞_{n+1} stands for the ring of germs at $0 \in \mathbf{R}^{n+1}$ of infinitely smooth functions of the variables q). The functions $1, x, \ldots, x^{n+m-k}$ can be chosen as generators. In B we consider the submodule C generated by the images in B of the functions

$$\begin{aligned}e_1 &= 1 + tf'_{q_1},\ \ldots,\ e_m = x^{m-1} + tf'_{q_m}, \\ e_{m+1} &= (mx^{m-1} + (m-1)q_m x^{m-2} + \ldots + q_2 + tf'_x)\frac{x}{n+2-k} \\ &\quad + tf'_{q_{k+1}}v_{q_{k+1}}^{n+1}(q) + \ldots + tf'_{q_{n+1}}v_{q_{n+1}}^{n+1}(q), \ldots, \\ e_{m+n+1-k} &= (mx^{m-1} + (m-1)q_m x^{m-2} + \ldots + q_2 + tf'_x)\left(\frac{x^{n+1-k}}{n+2-k} + \ldots\right) \\ &\quad + tf'_{q_{k+1}}v_{q_{k+1}}^{k+1}(q) + \ldots + tf'_{q_{n+1}}v_{q_{n+1}}^{k+1}(q).\end{aligned}$$

We can see that

$$\begin{aligned}e_{m+n+1-k} &= a_{m+n+1-k}^{m+n+1-k}x^{m+n-k} \qquad (\operatorname{mod} \mathbf{m}_{n+1}B), \ldots, \\ e_1 &= a_1^1 + a_1^m x^m + \ldots + a_1^{m+n+1-k}x^{m+n-k} \qquad (\operatorname{mod} \mathbf{m}_{n+1}B),\end{aligned}$$

where $\mathbf{m}_{n+1}$ is the maximal ideal of C^∞_{n+1} and $a^i_j \in \mathbf{R}$, $a^i_i \neq 0$. Hence $B = C + \mathbf{m}_{n+1}B$. By the Nakayama Lemma we conclude that $B = C$, i.e. for any t there exist f^j_0, f_i smooth in some neighbourhood of $0 \in \mathbf{R}^{n+2}$ and solving (6). From the preparation theorem for differentiable functions depending on parameters it follows that for any t there exist neighbourhoods $t \in V \subset \mathbf{R}$ and $0 \in W \subset \mathbf{R}^{n+2}$ such that (6) has a solution smooth in $V \times W$. The proposition is proved.

The existence of $f^0_0(t,x,q)$, $f^i_0(t,q)$, $i = k+1, \ldots, n+1$, $f_i(t,q)$, $i = 1, \ldots, m$, smooth in $[0,1] \times W$ for some neighbourhood W of $0 \in \mathbf{R}^{n+2}$ can be proved in a standard way using a partition of unity. The family of vector fields $v = f^0_0 v^0 + f_1 v^1 + \ldots + f_m v^m + f^{k+1}_0 v^{k+1} + \ldots + f^{n+1}_0 v^{n+1}$, depending on the parameter t, smooth in $[0,1] \times W$, vanishes at 0 for any t, because $v^0(0) = v^{k+1}(0) = \ldots = v^{n+1}(0) = 0$ and $f_1(t,0) = \ldots = f_m(t,0) = 0$ (this can be easily got from (6)). Integrating this family we get a family of admissible diffeomorphisms g_t, preserving the origin $0 \in \mathbf{R}^{n+2}$ and such that $\Phi_t \circ g_t = \Phi_0$. Lemma 4 and Theorem 3 are proved.

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