

**STUDY OF GLOBAL SOLUTIONS OF PARABOLIC
EQUATIONS VIA *A PRIORI* ESTIMATES III.
EQUATIONS OF p -LAPLACIAN TYPE**

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Introduction. In this work, we continue the series [9, 10] which is based on Fila's approach to boundedness of global solutions to nonlinear parabolic equations [3, 4]. Now we examine equations of the form

$$(0.1) \quad u_t = \operatorname{div} A(Du) + f(u, Du)$$

for a vector function A with $\partial A(p)/\partial p$ a positive definite matrix. Our model case will be $A(p) = |p|^{m-2}p$, which corresponds to the so-called p -Laplacian (although we are using m in place of p). Under various hypotheses connecting A and f , we shall show that global solutions of (0.1) with zero boundary values in a cylinder $\Omega \times (0, \infty)$ with smooth cross-section Ω are actually bounded in L^∞ . For this boundedness, we will assume that f depends only on u , but our major focus is really on an intermediate result, which we formulate here for a special case.

THEOREM 0.1. *Let u be a solution of (0.1) in a C^2 domain $\Omega \subset \mathbb{R}^n$ with*

$$(0.2) \quad u = 0 \quad \text{on } \partial\Omega \times (0, \infty)$$

for $A(p) = |p|^{m-2}p$, with $2 < n < m$, suppose there are positive functions C_0 and C_1 such that

$$(0.3) \quad |f(z, p)| \leq C_0(|z|) + \theta |p|^{(nm+m-n)/n} + C_1(\theta)$$

for all $(z, p) \in \mathbb{R} \times \mathbb{R}^n$ and all $\theta > 0$, and set

$$(0.4) \quad E(t) = \int_{\Omega \times \{t\}} |Du|^m dx.$$

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If there are positive constants τ, σ, B and k such that $\tau > \sigma, B > k$, and

$$(0.5) \quad E(\tau) = B, \quad E(\sigma) = k, \quad E(t) \leq B \quad \text{for } t \in [\sigma, \tau],$$

then

$$(0.6) \quad \tau - \sigma \geq \eta$$

for some positive constant $\eta(B, k, C_0, C_1, m, n, \Omega)$.

A similar result will be proved when the inequalities $2 < n < m$ fail to hold, but then condition (0.3) must be suitably modified. If $1 < m < n$ and $n > 2$ instead of $n < m$ and if

$$(0.3)' \quad |f(z, p)| \leq \theta|z|^{(nm+m-n)/(n-m)} + \theta|p|^{(nm+m-n)/n} + C_1(\theta)$$

replaces (0.3), then (0.5) implies (0.6). (One must also assume $m > 2n/(n+2)$ for the global boundedness result; see the remarks after Lemma 2.2.) For $n = m > 2$, (0.3) is replaced by

$$|f(x, z)| \leq \exp(\theta|z|^{n/(n-1)}) + \theta|p|^n + C_1(\theta).$$

For $n = 2$, all these implications hold with $\theta|p|^{(nm+m-n)/n}$ replaced by $|p|^{(3/2)m-1-\theta}$. Finally, if $1 = n < m$, then (0.5) implies (0.6) if (0.3) is replaced by

$$|f(x, z)| \leq C_0(|z|) + C_1|p|^m$$

for some constant C_1 .

Most of our effort will be to prove that (0.5) implies (0.6) for a suitable choice of E based on the hypotheses on A and f (and, in some cases, that B and k are sufficiently far apart). We also examine the related problem with boundary condition

$$(0.7) \quad A(Du) \cdot \gamma + \psi(u) = 0 \quad \text{on } \partial\Omega \times (0, T),$$

where γ denotes the unit inner normal to $\partial\Omega$.

In addition to structure modelled on $A(p) = |p|^{m-2}p$ or $A(p) = G(|p|)p$ for some function G , we also consider a structure typified by defining $A^i(p) = |p_i|^{m_i-2}p_i$ for suitable constants m_i or, more generally, $A^i(p) = G_i(|p_i|)p_i$ for appropriate functions G_i . Reasoning by analogy with other results concerning such problems, it seems natural to assume also that each G_i is a Young's function so that the theory of Orlicz spaces [7] can be used. Although much of our analysis is closely related to that theory, our slightly more general assumptions are a better setting for the problem at hand. Indeed, many of our technical lemmata are obvious when G (or each G_i) is a power function, but there seem to be no previous appearances of them in print.

We begin in Section 1 with a second derivative estimate which is needed to handle the correct growth condition on f with respect to the gradient. Section 2 discusses a suitable generalization of Theorem 0.1, and its corresponding global boundedness conclusion, in case A grows rapidly at ∞ , as in Theorem 0.1 with $m > n$. The case of slower growth ($m < n$ in Theorem 0.1) is covered in Section 3. The borderline case (which corresponds to $m = n$ in Theorem 0.1) is the topic of Section 4. Section 5 deals with boundary condition (0.7), and Section 6 discusses some extensions of the results to more general situations.

1. A second derivative estimate. When dealing with the equation $u_t = \Delta u + f(u, Du)$ in [9], we used the L^2 Schauder theory to estimate terms in $E'(t)$ arising from the gradient dependence of f . If the Laplacian is replaced by the p -Laplacian, the corresponding inequality is not part of the usual theory of such operators. In this section, we prove such a result, which is similar to [6, Theorem 3.1.1.1] of Grisvard for studying global second derivative estimates of solutions of linear elliptic equations in convex domains.

THEOREM 1.1. *Let Ω be a bounded C^2 domain in \mathbb{R}^n , and let A be a C^2 \mathbb{R}^n -valued function defined on $\Omega \times \mathbb{R}^n$. Suppose also that there are positive constants $\delta \leq 1$, Λ , M , and F and a C^1 , increasing function g such that*

$$(1.1) \quad \delta \leq \frac{\tau g'(\tau)}{g(\tau)} \leq F,$$

$$(1.2a) \quad |A(x, p)| + |p||A_p(x, p)| + |A_x(x, p)| \leq \Lambda g(|p|),$$

$$(1.2b) \quad a^{ij}(x, p)\xi^i\xi^j \geq \frac{g(|p|)}{|p|}|\zeta|^2$$

for all $(x, p, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n$ with $|p| \geq M$ and all $\tau \geq M$, where $a^{ij} = \partial A^i / \partial p_j$. If $u \in C^2(\bar{\Omega})$ solves

$$(1.3) \quad \operatorname{div} A(x, Du) = b(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

then

$$(1.4) \quad \int_{\Omega} \left(1 - \frac{M}{|Du|}\right)_+^2 g'(|Du|)^2 |D^2u|^2 dx \leq C(\Omega, n, \delta, \Lambda, F) \int_{\Omega} [g(|Du|)^2 + b^2] dx.$$

Proof. We note that our hypotheses are invariant under any C^2 change of independent variable. Therefore, we consider first the case that $\partial\Omega$ is just the set $\{x^n = 0\}$. Multiplying the differential equation by η , an arbitrary smooth function, yields

$$\int_{\{x^n=0\}} A^n \eta dx' = \int_{\Omega} [-A^i D_i \eta - b\eta] dx.$$

Now we take $\eta = D_k \zeta^k$ for ζ a smooth vector-valued function (and we note that the sum on k goes all the way to n). Integration by parts yields

$$(1.5) \quad \int_{\{x^n=0\}} [A^n D_k \zeta^k - A^i D_i \zeta^n] dx' = \int_{\Omega} (D_k(A^i) + b\delta_k^i) D_i \zeta^k dx.$$

A simple approximation argument shows that this inequality holds if $\zeta \in C^1$. In particular, if $\zeta^k = 0$ for $x^n = 0$ and $k < n$ and if ζ has compact support, then

$$\int_{\{x^n=0\}} [A^n D_k \zeta^k - A^i D_i \zeta^n] dx' = - \sum_{i < n} \int_{\{x^n=0\}} A^i D_i \zeta^n dx'.$$

To proceed, we suppose that h is a nonnegative $C^1([M, \infty))$ function with $h(M) = 0$ and

$$\delta - 1 \leq \frac{\tau h'(\tau)}{h(\tau)} \leq h_1 \quad \text{on } [M, \infty)$$

for some positive constant h_1 , and take $\zeta = h(|Du|)\eta^2 Du$ for η a C^2 function with compact support such that $0 \leq \eta \leq 1$. First we examine the boundary integral. Setting

$H(\tau) = \tau h(\tau)$ and summing on i only up to $n - 1$ yields

$$\begin{aligned} \int_{\{x^n=0\}} A^i D_i \zeta^n dx' &= \int_{\{x^n=0\}} A^i(x, 0, \dots, 0, D_n u) H'(D_n u) D_{in} u \eta^2 dx' \\ &\quad + \int_{\{x^n=0\}} A^i(x, 0, \dots, 0, D_n u) H(D_n u) \eta D_i \eta dx'. \end{aligned}$$

Now we define $h^i(x, \tau) = \int_0^\tau A^i(x, 0, \dots, 0, \sigma) H'(\sigma) d\sigma$ and integrate by parts to see that

$$\begin{aligned} \int_{\{x^n=0\}} A^i(x, 0, \dots, 0, D_n u) H'(D_n u) D_{in} u \eta^2 dx \\ = - \int_{\{x^n=0\}} [2h^i(x, D_n u) \eta D_i \eta + (\partial h^i(x, D_n u) / \partial x^i) \eta^2] dx'. \end{aligned}$$

Simple calculations show that

$$|A^i(x, 0, \dots, 0, D_n u) H(D_n u)| + |h^i(x, D_n u)| + |\partial h^i / \partial x^i| \leq Cg(|Du|)H(|Du|),$$

and hence

$$- \int_{\{x^n=0\}} A^i D_i \zeta^n dx' \leq C \int_{\{x^n=0\}} g(|Du|)H(|Du|)\eta dx'$$

with C depending also on h_1 . Integrating by parts and suppressing the argument $|Du|$ in g , h and H , we find that

$$\begin{aligned} \int_{\{x^n=0\}} gH\eta dx' &= \int_{\Omega} [(gH)' D_n(|Du|)\eta(x) + gH D_n \eta(x)] dx \\ &\leq C \int_{\Omega} \left(\frac{gh}{(1 - |Du|/M)} |D^2 u| \eta + |Du| gh |D\eta| \right) dx \\ &\leq \varepsilon \int_{\Omega} \frac{gh}{|Du|} |D^2 u|^2 \eta^2 dx + C(\varepsilon) \int_{\Omega} \frac{gh|Du|}{(1 - M/|Du|)^2} dx \end{aligned}$$

for any $\varepsilon > 0$. Hence, choosing $\varepsilon = \frac{\delta}{4}$ gives

$$- \int_{\{x^n=0\}} A^i D_i \zeta^n dx' \leq \frac{\delta}{4} \int_{\Omega} \frac{gh}{|Du|} |D^2 u|^2 \eta^2 dx + C \int_{\Omega} \frac{gh|Du|}{(1 - M/|Du|)^2} dx.$$

To estimate the interior integral in (1.5), we set $C_k^i = \partial A^i / \partial x^k + b\delta_k^i$ and $g^{km} = \delta^{km} - |Du|^{-2} D_k u D_m u$. It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} \int_{\Omega} (D_k(A^i) + b\delta_k^i) D_i \zeta^k dx \\ = \int_{\Omega} a^{ij} D_{jk} u D_{ik} u h \eta^2 dx + \int_{\Omega} |Du| a^{ij} D_i(|Du|) D_j(|Du|) h' \eta^2 dx \\ + \int_{\Omega} 2a^{ij} D_{jk} u D_k u h \eta D_i \eta dx + \int_{\Omega} h C_k^i g^{km} D_{im} u \eta^2 dx \\ + \int_{\Omega} C_k^i D_k u D_i(|Du|) h' \eta^2 dx + 2 \int_{\Omega} C_k^i D_k u h D_i \eta \eta dx \end{aligned}$$

$$\begin{aligned} &\geq \frac{\delta}{2} \int_{\Omega} a^{ij} D_{ik} u D_{jk} u h \eta^2 dx - C \int_{\Omega} \eta^2 \frac{|Du|h}{(1 - M/|Du|)g} C_k^i C_k^i dx \\ &\quad - C \int_{\Omega} gh|Du||D\eta|^2 dx \end{aligned}$$

with the constants depending also on h_1 . Combining these estimates and using (1.1) and (1.2) now yields

$$\int_{\Omega} h\eta^2 g'|D^2u|^2 dx \leq C \int_{\Omega} (g^2 + b^2) \frac{h|Du|}{(1 - M/|Du|)^2 g} dx.$$

An easy approximation argument shows that this inequality is valid even if h is only Lipschitz. In particular we can take $h(\tau) = (1 - M/\tau)_+^2 g(\tau)/\tau$ and note that $h(\tau) \geq Cg'(\tau)(1 - M/\tau)_+$ to see that

$$\int_{\Omega} \left(1 - \frac{M}{|Du|}\right)_+^2 g'(|Du|)^2 |D^2u|^2 \eta^2 dx \leq C(\delta, F, \Lambda) \int_{\{|Du|>M\}} [g(|Du|)^2 + b^2 \eta^2] dx.$$

A standard partition of unity argument and conversion to the original coordinates yields (1.4). ■

The technical details of our proof are modelled on Simon’s gradient bounds for nonuniformly elliptic equations [13], and a future work will show how to obtain maximum bounds on the gradient near the boundary without using barrier arguments. Other choices of h can be used to derive (1.4) or modifications of that condition. For example, $h(\tau) = (g(\tau)/\tau)(\tau/(1 + \tau^2)^{1/2})^{1-\delta}$ gives an estimate of an integral of the form $\int_{\Omega} \Phi(|Du|)|D^2u|^2 dx$ with Φ vanishing only at zero, but this estimate uses information about the behavior of the coefficients for small values of $|Du|$. For our applications, we only use the following gradient estimate, which is a simple consequence of our second derivative bound.

COROLLARY 1.2. *Under the hypotheses of Theorem 1.1, we have*

$$\begin{aligned} (1.6) \quad &\left(\int_{\Omega} g(|Du|)^{2\kappa} dx \right)^{1/\kappa} \\ &\leq C(\delta, F, g(1), \Lambda, \Omega) \left(\int_{\Omega} b^2 dx + g(M)^2 + \left(\int_{\Omega} g(|Du|) dx \right)^2 \right), \end{aligned}$$

where $\kappa = n/(n - 2)$ if $n > 2$ and κ is arbitrary for $n \leq 2$.

Proof. We start with the easily verified inequality

$$\int_{\{|Du|>2M\}} g^{2\kappa} dx \leq 4^\kappa \int_{\Omega} \left[\left(1 - \frac{M}{|Du|}\right)^2 g \right]^{2\kappa} dx.$$

Since

$$|D((1 - M/|Du|)^2 g(|Du|))| \leq (2 + F)(1 - M/|Du|)g/|Du|,$$

we infer from Theorem 1.1 and the Sobolev inequality that

$$\left(\int_{\{|Du|>2M\}} g(|Du|)^{2\kappa} dx \right)^{1/\kappa} \leq C \left(\int_{\Omega} b^2 dx + \int_{\Omega} g(|Du|)^2 dx \right).$$

Next we note that

$$\int_{\{|Du|\leq 2M\}} g(|Du|)^{2\kappa} dx \leq g(2M)^{2\kappa} |\Omega|$$

and infer that

$$\left(\int_{\Omega} g(|Du|)^{2\kappa} dx \right)^{1/\kappa} \leq C \left(\int_{\Omega} b^2 dx + g(M)^2 + \int_{\Omega} g(|Du|)^2 dx \right).$$

The proof is completed by applying Hölder's inequality. ■

In particular, for the p -Laplace operator, $g(\tau) = \tau^{p-1}$, so Corollary 1.2 in conjunction with Hölder's inequality gives an estimate for the integral of $|Du|^{(2p-2)\kappa}$.

2. Rapid growth. We begin with an estimate in case $A(p)$ grows rapidly as p goes to infinity. The model function is $A(p) = |p|^{m-2}p$ for some $m > n$. Also here and below, we define

$$Q = \Omega \times (t_0, t_1), \quad SQ = \partial\Omega \times (t_0, t_1),$$

where $t_0 < t_1$ are positive constants and Ω is a bounded domain in \mathbb{R}^n for some $n \geq 1$ with $\partial\Omega \in C^2$.

THEOREM 2.1. *Let u solve*

$$(2.1) \quad u_t = \operatorname{div} A(Du) + f(u, Du) \quad \text{in } Q, \quad u = 0 \quad \text{on } SQ.$$

Suppose there are scalar functions S , g , and G satisfying

$$(2.2) \quad A(p) = \frac{\partial S}{\partial p}(p),$$

$$(2.3a) \quad G(s) = sg(s),$$

$$(2.3b) \quad g \text{ is } C^1 \text{ and increasing on } (0, \infty),$$

$$(2.4) \quad S(p) \geq G(|p|),$$

and

$$(2.5) \quad \int_0^\infty \frac{G^{-1}(\sigma)}{\sigma^{(n+1)/n}} d\sigma < \infty,$$

and such that (1.1) and (1.2) hold. Suppose also that there are an increasing, continuous function C_0 and a positive decreasing function C_1 such that

$$(2.6a) \quad |f(z, p)| \leq C_0(|z|) + C_1(\theta) + \theta g(|p|)^{1+1/n} |p|^{1/n}$$

for all $\theta > 0$ if $n \geq 3$,

$$(2.6b) \quad |f(z, p)| \leq C_0(|z|) + C_1(\theta) + C_2 g(|p|)^{1+\theta} |p|^\theta$$

for some $\theta \in (0, 1/2)$ if $n = 2$, and

$$(2.6c) \quad |f(z, p)| \leq C_0(|z|) + C_2 G(|p|)$$

if $n = 1$. Finally, define E by

$$(2.7) \quad E(t) = \int_{\Omega \times \{t\}} S(Du) \, dx.$$

Then for any $B > 0$, there is a positive constant $C(B, C_0, C_1, g_0, \Omega, F, \delta, \Lambda)$ for which

$$(2.8a) \quad E(t) \leq B \quad \text{for } t \in [t_0, t_1]$$

implies

$$(2.8b) \quad E(t_1) - E(t_0) \leq C[t_1 - t_0].$$

Proof. Suppose first that $n \geq 3$ and that $u \in C^2(\Omega \times [t_0, t_1])$. By direct calculation and the divergence theorem,

$$E'(t) = \int_{\Omega \times \{t\}} A(Du) \cdot Du_t \, dx = - \int_{\Omega \times \{t\}} (\operatorname{div} A)u_t \, dx.$$

From the differential equation for u and Cauchy's inequality, we infer that

$$E'(t) \leq -\frac{1}{2} \int_{\Omega \times \{t\}} (\operatorname{div} A)^2 \, dx + \frac{1}{2} \int_{\Omega \times \{t\}} f^2 \, dx.$$

Next, we use (2.5), (2.6a), and the Sobolev-Orlicz imbedding theorem [2, Theorem 3.2b] to see that

$$E'(t) \leq -\frac{1}{2} \int_{\Omega \times \{t\}} (\operatorname{div} A)^2 \, dx + C(\theta) + \theta \int_{\Omega \times \{t\}} g(|Du|)^{2+2/n} |Du|^{2/n} \, dx$$

for any $\theta > 0$; C also depends, of course, on the same quantities as the constant in (2.8b). This last integral is estimated via Hölder's inequality and Corollary 1.2 to obtain

$$\begin{aligned} \int_{\Omega \times \{t\}} g(|Du|)^{2+2/n} |Du|^{2/n} \, dx &\leq \left(\int_{\Omega \times \{t\}} g(|Du|)^{2\kappa} \, dx \right)^{1/\kappa} \left(\int_{\Omega \times \{t\}} g(|Du|) |Du| \, dx \right)^{2/n} \\ &\leq C \left(\int_{\Omega \times \{t\}} (\operatorname{div} A)^2 \, dx + 1 \right). \end{aligned}$$

Hence

$$E'(t) \leq \left(C\theta - \frac{1}{2} \right) \int_{\Omega \times \{t\}} (\operatorname{div} A)^2 \, dx + C(\theta) \leq C$$

provided θ is taken sufficiently small. Integrating this inequality yields (2.8b) for smooth u . The general case for $n \geq 3$ is recovered via the regularization scheme of [8, pp. 342–343].

The cases $n = 1$ and $n = 2$ can be handled using the ideas in [9, Section 2 and Lemma 3.2]. We leave the details to the reader. ■

Note that when $n = 1$, we don't need to assume (1.1) and (1.2). The proof of [9, Lemma 2.3] shows that we may assume instead that

$$|f(z, p)| \leq C_1(|z|) + C_2\Gamma^{1/2}(|p|),$$

where

$$\Gamma(\tau) = \tau \int_0^\tau g(s)g'(s)s \, ds.$$

It follows immediately that if there are positive constants $B > k$ and $\tau > \sigma$ satisfying (0.5), then (0.6) holds with $\eta = (B - k)/C$, where C is the constant from (2.8b). In particular, Theorem 0.1 follows.

If we assume that f grows more slowly with respect to p than in Theorem 2.1, we can consider more general hypotheses on A .

THEOREM 2.2. *Let u solve (2.1), and suppose that there are scalar functions S, G, g_i , and G_i ($i = 1, \dots, n$) satisfying conditions (2.2), (2.5), and such that*

$$\begin{aligned}
 (2.9a) \quad & G_i(s) = sg_i(s), \\
 (2.9b) \quad & \text{each } g_i \text{ is } C^1 \text{ and increasing on } (0, \infty), \\
 (2.9c) \quad & G(s) = \left(\prod_{i=1}^n G_i(s) \right)^{1/n}, \\
 (2.10) \quad & S(p) \geq \sum_{i=1}^n G_i(|p_i|).
 \end{aligned}$$

Suppose also that there are a continuous increasing function C_0 and a positive constant C_2 such that

$$(2.11) \quad |f(z, p)| \leq C_0(|z|) + C_2S(p)^{1/2},$$

and define E by (2.7). Then for any positive B , there is a constant $C(B, C_0, C_2, \Omega)$ such that (2.8a) implies (2.8b).

Proof. Computing E' as in Theorem 2.1 leads to the estimates

$$\begin{aligned}
 E'(t) &\leq -\frac{1}{2} \int_{\Omega \times \{t\}} (\operatorname{div} A)^2 dx + C_1^2 \int_{\Omega \times \{t\}} S(Du) dx + \int_{\Omega \times \{t\}} C_0(|u|)^2 dx \\
 &\leq C_1^2 B + C(C_0, n)|\Omega|
 \end{aligned}$$

by the Sobolev-Orlicz imbedding theorem [15, Corollary 1]. The proof is now completed as before. ■

We now show how Theorem 2.2 leads to a global boundedness result when f is independent of p . Portions of this argument are still valid even if f depends on p , but a crucial role is played by the Lyapunov functional J defined in Lemma 2.3 below. Rather than discuss this point here, we refer the reader to [12, 16] for more details on the construction of general Lyapunov functionals.

In addition to f being independent of p , we assume that f is superlinear. In the present context, this assumption takes the following form: There are positive constants a_0, c_0, c_1 , and ε such that

$$\begin{aligned}
 (2.12a) \quad & p \cdot A(p) \leq a_0S(p) + a_1, \\
 (2.12b) \quad & zf(z) \geq (a_0 + \varepsilon)F(z) - c_1, F(z) \geq z^{2+\varepsilon} - c_0,
 \end{aligned}$$

where $F(z) = \int_0^z f(\sigma) d\sigma$. (For the equation $u_t = \Delta u + f(u)$, we have $A(p) = p$, and (2.6a) holds with $a_0 = 1/2$ and $a_1 = 0$. In this case, (2.12b) is equivalent to the single condition $\liminf_{|z| \rightarrow \infty} zf(z)/F(z) > 1/2$.) Note that (2.12a) is a strong growth condition

on S ; if $S(p) = h(|p|)$ for some function h , then h satisfies a Δ_2 -condition as defined in [7, Chapter I, §4]. In particular, h can grow no faster than polynomially.

LEMMA 2.3. *Let $n \geq 1$, let u solve*

$$(2.13) \quad u_t = \operatorname{div} A(Du) + f(u) \quad \text{in } \Omega \times (0, \infty), \quad u = 0 \quad \text{on } \partial\Omega \times (0, \infty),$$

and suppose that f and S satisfy (2.12). If u_t , $S(Du)$, and $\operatorname{div} A(Du)$ are in $L^2(\Omega \times \{T\})$ for any finite T , then $\int_{\Omega \times \{t\}} u^2 dx$ is uniformly bounded for $t \in (0, \infty)$, and

$$(2.14) \quad \liminf_{t \rightarrow \infty} E(t) < \infty.$$

Proof. As in [3, 10], we set

$$M(t) = \frac{1}{2} \int_{\Omega \times \{t\}} u^2 dx, \quad J(t) = \int_{\Omega \times \{t\}} [S(Du) - F(u)] dx.$$

After differentiating M with respect to t , using the differential equation for u and integrating by parts, we have

$$M'(t) = \int_{\Omega \times \{t\}} [-Du \cdot A(Du) + uf(u)] dx$$

for almost all t . Using (2.12a) and the first inequality of (2.12b), we find that

$$M'(t) \geq \varepsilon \int_{\Omega \times \{t\}} F(u) dx - a_0 J(t) - (a_1 + c_1)|\Omega|.$$

A simple calculation shows that $J'(t) \leq 0$, so J is nonincreasing. Taking into account the second inequality of (2.12b) and using Hölder's inequality yields $M'(t) \geq CM^{1+\varepsilon/2} - K$ for suitable positive constants C and K . As in [4, Lemma 1.2], it follows that $M(t) \leq (K/C)^{2/(2+\varepsilon)}$ because u exists for all t .

Similar calculations (cf. [10, Theorem 2.2]) show that $M'(t) \geq \varepsilon E(t) - C$, so $E(t) \rightarrow \infty$ as $t \rightarrow T^-$ implies that T must be finite, proving (2.14). (Note that these arguments also work in the slow growth case because (2.12a) is the only restriction on S .) ■

Note that (2.12b) with $A(p) = |p|^{m-2}p$ implies that $zf(z) \geq C_2 z^{1+\varepsilon} - C_3$ so (0.3)' can only be satisfied if $(nm + m - n)/(n - m) > 1$, which is equivalent to $m > 2n/(n + 2)$.

Combining the results of Theorem 2.2 and Lemma 2.3 with the techniques used in proving [9, Theorem 1.5] gives global boundedness. We outline the procedure because there are some important differences between the situation in [4, 9] and the present one.

THEOREM 2.4. *Let $n \geq 1$, suppose conditions (2.2), (2.5), (2.9), (2.10) and (2.12) are satisfied, and suppose that u solves (2.13). Suppose also that there is an increasing continuous function C_0 such that*

$$(2.15) \quad |f(z)| \leq C_0(|z|).$$

Then $E(t)$ is uniformly bounded, and u is uniformly bounded in L^∞ .

Proof. From Lemma 2.3, $E(t)$ cannot tend to infinity as $t \rightarrow \infty$. If $\lim_{t \rightarrow \infty} E(t)$ is finite, we are done. Hence we may assume that there are (finite) constants k and B so

that

$$\liminf_{t \rightarrow \infty} E(t) < k < B < \limsup_{t \rightarrow \infty} E(t).$$

For any B and k satisfying this inequality, we can find sequences (σ_m) and (τ_m) such that $\sigma_m < \tau_m < \sigma_{m+1}$, $\sigma_m \rightarrow \infty$, $E(\sigma_m) = k$, $E(\tau_m) = B$, and $E(t) \leq B$ for $t \in (\sigma_m, \tau_m)$. Since $E(\tau_m)$ is uniformly bounded, it follows from [2, Theorem 3.6] that the sequence $(u(\cdot, \tau_m))$ is uniformly bounded and equicontinuous. By taking a suitable subsequence of (τ_m) (still denoted by (τ_m)), we may therefore assume that this sequence converges uniformly to a limit function v . Our goal is to show three things about v : first, v is a solution of the limit problem

$$(2.16) \quad \operatorname{div} A(Dv) + f(v) = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega,$$

second, $\int_{\Omega} S(Dv) dx$ is bounded from above by a constant independent of B , and, third, this integral is bounded from below by a constant multiple of B .

To show that v solves (2.16), we set $u_m(X) = u(x, t - t_m)$ and $Q = \Omega \times (0, \eta)$. Then $\int_Q S(Du_m) dX \leq B$, so $\int |A(Du_m)| dX$ is uniformly bounded and hence (by extracting a subsequence as needed) there is a function $v \in L^1(Q)$ with $|A(Dv)| \in L^1(Q)$ and $A(Du_m) \rightarrow A(Dv)$ weakly in $L^1(Q)$. Therefore

$$\begin{aligned} \int_Q [A(Dv) \cdot D\varphi - f(v)\varphi] dX &= \lim_{m \rightarrow \infty} \int_Q [A(Du_m) \cdot D\varphi - f(u_m)\varphi] dX \\ &= - \lim_{m \rightarrow \infty} \int_Q u_{m,t} \varphi dX \end{aligned}$$

for any $\varphi \in C_0^1(\Omega)$.

Now we multiply the differential equation for u by u_t and integrate by parts to see that

$$\int_{\Omega \times (0, \tau_m)} (u_t)^2 dX = J(0) - J(\tau_m).$$

Since $J(\tau_m) \geq C(B)$ for some negative constant $C(B)$, it follows that $u_t \in L^2(\Omega \times (0, \infty))$. Now let C be the constant from Theorem 2.2 and set $\eta = (B - k)/C$, $t_m = \tau_m - \eta$ and $Q(m) = \Omega \times (t_m, \tau_m)$, noting that $t_m \geq \sigma_m$. It follows that

$$\lim_{m \rightarrow \infty} \int_{Q(m)} (u_t)^2 dX = 0,$$

and hence

$$\int_{\Omega} |u(x, \tau_m) - u(x, \tau_m - \theta)| dx \rightarrow 0$$

uniformly for $\theta \in [0, \eta]$ as $m \rightarrow \infty$. Combined with the equiboundedness of the collection of functions $\{u_m\}$, we find that v is independent of t and hence v is a weak solution of (2.16).

Now we take v as test function in the weak form of (2.16) and use the first inequality

of (2.12b) to see that

$$\int_{\Omega} A(Dv) \cdot Dv \, dx \geq (a_0 + \varepsilon) \int_{\Omega} F(v) \, dx - c_1 |\Omega|,$$

and therefore (2.12a) implies that

$$\int_{\Omega} S(Dv) \, dx \geq \left(1 + \frac{\varepsilon}{a_0}\right) \int_{\Omega} F(v) \, dx - \frac{a_1 + c_1}{a_0} |\Omega|,$$

or

$$(2.17a) \quad \int_{\Omega} F(v) \, dx \leq \delta \int_{\Omega} S(Dv) \, dx + k_1$$

for positive constants δ and k_1 with $\delta < 1$. On the other hand, Fatou's Lemma shows that

$$\int_{\Omega} S(Dv) \, dx \leq \liminf_{m \rightarrow \infty} \int_{\Omega} S(Du(x, \tau_m)) \, dx.$$

Since the sequence $(u(\cdot, \tau_m))$ converges pointwise to v and is uniformly bounded, it follows that $F(u(\cdot, \tau_m)) \rightarrow F(v)$ in $L^1(\Omega)$. Moreover, J is decreasing so

$$(2.17b) \quad \int_{\Omega} S(Dv) \, dx \leq \lim_{m \rightarrow \infty} J(\tau_m) + \int_{\Omega} F(v) \, dx \leq J(0) + \int_{\Omega} F(v) \, dx.$$

Combining (2.17a) with (2.17b) yields, after some rearrangement,

$$\int_{\Omega} S(Dv) \, dx \leq k_2$$

for $k_2 = (J(0) + k_1)/(1 - \delta)$, which is independent of B . Hence

$$\lim \int_{\Omega} F(u(\cdot, \tau_m)) \, dx = \int_{\Omega} F(v) \, dx \leq C$$

for some constant independent of B . Since also

$$\lim \int_{\Omega} S(Du(\cdot, \tau_m)) \, dx = \lim J(\tau_m) + \lim \int_{\Omega} F(u(\cdot, \tau_m)) \, dx \leq J(0) + C,$$

we conclude that $B \leq J(0) + C$, which means that $E(t)$ is uniformly bounded. The L^∞ bound follows from this one by [15, Corollary 1]. ■

The slow growth version of Theorem 2.3 is only slightly more delicate; in place of the uniform L^∞ estimates for u , we need to observe that the appropriate integrands are bounded by suitable L^1 functions.

3. Slow growth. In this section, we study problem (2.1) when A grows more slowly than in Section 2. A crucial element of our program is a Sobolev-type inequality involving Orlicz functions. The guiding principle is a straightforward modification of Donaldson and Trudinger's imbedding theorem [2, Theorem 3.2(a)] for Sobolev-Orlicz spaces, but there are some important technical differences. The most important is that we are not interested in results relating Orlicz norms of functions and their derivatives over a suitable domain in \mathbb{R}^n but rather the integral of a Young's function composed with the function defined on that domain. In fact, our results do not use the convexity of the Young's function so

we state them in terms of a slightly different structure, namely (2.3). Unlike the situation in Section 2, we always consider the structure from Theorem 2.1 which uses the single function g .

LEMMA 3.1. *Let g and G satisfy (2.3). Suppose H is a positive $C^1(0, \infty)$ function satisfying the differential equation*

$$(3.1) \quad (G^{-1} \circ H)(s) \frac{dH}{ds} = H(s)^{(n+1)/n}$$

for $s > 0$. Then

$$(3.2) \quad \frac{n+1}{n} \geq \frac{HH''}{(H')^2} \geq \frac{1}{n}.$$

Moreover

$$(3.3) \quad \frac{d}{ds}((H'(s))^2 H(s)^{-1/n}) \leq 2H''(s)g \circ G^{-1}(H(s)).$$

Proof. Differentiation of (3.1) followed by a use of (3.1) itself yields

$$(G^{-1})'(H)(H')^2 + G^{-1}(H)H'' = \frac{n+1}{n} H^{1/n} H' = \frac{n+1}{n} \frac{(H')^2}{H} G^{-1}(H).$$

Since G is increasing, so is its inverse and hence

$$G^{-1}(H)H'' \leq \frac{n+1}{n} \frac{(H')^2}{H} G^{-1}(H),$$

which implies the first inequality of (3.2). For the second, note that $G' \geq g$, so

$$(G^{-1})'(H) = 1/G'(G^{-1}(H)) \leq 1/g \circ G^{-1}(H) = G^{-1}(H)/H,$$

and hence

$$H'' \geq \left(\frac{n+1}{n} - 1 \right) \frac{(H')^2}{H}.$$

To prove (3.3), we suppress the argument s from H and its derivatives and note that

$$\frac{d}{ds}((H')^2 H^{-1/n}) = H' H^{-1/n} \left[2H'' - \frac{1}{n} \frac{(H')^2}{H} \right] \leq 2H'' H' H^{-1/n}.$$

Since (3.1) implies that $g \circ G^{-1}(H) = H' H^{-1/n}$, we infer (3.3). ■

Note that, even though G and H need not be Young's functions, much of the machinery of Orlicz spaces can still be used. In particular, we define

$$\|u\|_H = \inf \left\{ k > 0 : \int_{\Omega} H\left(\frac{|u|}{k}\right) dx \leq 1 \right\},$$

which is the Luxemburg norm because H is convex. Note that

$$\int_{\Omega} H\left(\frac{|u|}{\|u\|_H}\right) dx = 1$$

if u is bounded. We are now ready to prove our Sobolev inequality.

LEMMA 3.2. *Suppose the hypotheses of Lemma 3.1 are satisfied. Let $u \in C^{0,1}(\bar{\Omega})$ with $u = 0$ on $\partial\Omega$, and set $\varepsilon_1 = \min\{1/8, 1/\|u\|_H\}$. Then for any $\varepsilon \in (0, \varepsilon_1)$, we have*

$$(3.4) \quad \int_{\Omega} H'(\varepsilon|u|)^2 dx \leq 4\varepsilon \int_{\Omega} H''(\varepsilon|u|)G(|Du|) dx.$$

Proof. Without loss of generality, we may assume that u is nonnegative. Then we integrate (3.3) and apply the inequalities of Sobolev and Hölder. Since the constant in Sobolev’s inequality is less than 1 (see [5, (7.28)]), we see that

$$\int_{\Omega} (H'(\varepsilon u))^2 dx \leq C_1(\varepsilon) \int_{\Omega} H''(\varepsilon u)g \circ G^{-1}(H(\varepsilon u))|Du| dx,$$

for $C_1(\varepsilon) = 2\varepsilon[\int_{\Omega} H(\varepsilon u) dx]^{1/n}$. Because $\varepsilon_1 \leq 1/\|u\|_H$, it follows that $C_1 \leq 2\varepsilon$. Then Young’s inequality in the form [8, Lemma 1.1(e)] implies that

$$\int_{\Omega} (H')^2 dx \leq 2\varepsilon \left(\int_{\Omega} H''G(|Du|) dx + \int_{\Omega} H''H dx \right),$$

where we suppress the argument εu in H and all its derivatives. From (3.2), it follows that

$$\int_{\Omega} (H')^2 dx \leq 2\varepsilon \int_{\Omega} H''G(|Du|) dx + 4\varepsilon \int_{\Omega} (H')^2 dx.$$

The desired result follows from this inequality because $\varepsilon_1 \leq 1/8$. ■

Now we examine the situation that A has “slow growth” and f is independent of Du . Specifically, we assume of A that there are nonnegative constants g_0 and g_1 , and an increasing function g such that

$$(3.5a) \quad g(s) = g_1 s^{(n-2)/(n+2)} \quad \text{for } s \in (0, 1),$$

$$(3.5b) \quad p \cdot A(p) \geq G(|p|) - g_0 \quad \text{for all } p \in \mathbb{R}^n$$

(here G is related to g via (2.3a)), and that

$$(3.6) \quad \int_0^\infty \frac{G^{-1}(\tau)}{\tau^{(n+1)/n}} d\tau = \infty.$$

We pause here to make some observations about these conditions. The explicit form of g in (3.5a) is just a convenient technical device to facilitate the proof of Lemma 3.3 below. Moreover, for a given C^1 function g^* defined on $[0, \infty)$, we can find a C^1 function g satisfying (3.5a) such that $g(s) = g^*(s)$ for $s > 2$. If (3.5b) holds with g replaced by g^* and some constant g_0 , we can always adjust the constants g_0 and g_1 to guarantee (3.5b) with g . Condition (3.5b) says that G only has to control $p \cdot A$ for large p . When S is convex, (3.5b) follows from (2.2) and (2.4). If $G(s) = s^m$ for some constant m and all $s > 1$, then (3.6) holds if and only if $1 \leq m \leq n$.

The key step in our study of global boundedness is an estimate for $E'(t)$, once a suitable choice has been made for E . If we take H to be the even extension of the Sobolev conjugate of G , so

$$H^{-1}(s) = \int_0^s \frac{G^{-1}(\tau)}{\tau^{(n+1)/n}} d\tau$$

for $s \geq 0$, then E is given by

$$(3.7) \quad E(t) = \int_{\Omega \times \{t\}} H(\varepsilon u) \, dx$$

with ε sufficiently small.

LEMMA 3.3. *Let u be a solution of (2.1) and suppose E is given by (3.7) with $\varepsilon \leq \min\{1/8, 1/\|u\|_H\}$. Suppose also that there is an decreasing function C_1 such that*

$$(3.8) \quad f(z, p) \operatorname{sgn} z \leq C_1(\theta) + \theta|H'(\theta z)|$$

for all real z and all $\theta \in (0, 1)$. If (3.5) and (3.6) hold, then there is a constant C determined only by $C_1(\varepsilon)$, $|\Omega|$, g_0 , g_1 , and $h = G^{-1}(g_1^{(n+2)/n}/4)$ such that (2.8a) implies (2.8b).

Proof. As before, we may assume u to be C^2 , and then we only need to estimate E' . By direct calculation,

$$\begin{aligned} E'(t) &= \varepsilon \int_{\Omega \times \{t\}} H'(\varepsilon u) u_t \, dx = \varepsilon \int_{\Omega \times \{t\}} [H' \operatorname{div} A(Du) + H' f] \, dx \\ &= -\varepsilon^2 \int_{\Omega \times \{t\}} H'' Du \cdot A \, dx + \varepsilon \int_{\Omega \times \{t\}} H' f \, dx. \end{aligned}$$

Now we use (3.8) with $\theta = \varepsilon$ and then Lemma 3.2 to infer that

$$(3.9) \quad \begin{aligned} E'(t) &\leq \varepsilon^2 \left(- \int_{\Omega \times \{t\}} H'' Du \cdot A \, dx + \int_{\Omega \times \{t\}} (H')^2 \, dx + C_1(\varepsilon)|\Omega| \right) \\ &\leq \varepsilon^2 \left(- \frac{1}{2} \int_{\Omega \times \{t\}} H'' G(|Du|) \, dx + g_0 \int_{\Omega \times \{t\}} H'' \, dx \right) + C. \end{aligned}$$

Since $H(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$, we can use (3.2) to see that

$$g_0 \int_{\Omega \times \{t\}} H'' \, dx \leq 2 \int_{\{H > g_0\}} (H')^2 \, dx + \int_{\{H \leq g_0\}} H'' \, dx.$$

To estimate the second integral on the right hand side of this inequality, we note that $H(1) = g_1^{(n+2)/n}/4 = G(h)$. Therefore, on $\{1 < \varepsilon|u|, H \leq g_0\}$, we have

$$H'' \leq 2(H')^2/H = \frac{2H^{1+2/n}}{h^2} \leq \frac{2H^{1+2/n}(H^{-1}(g_0))}{h^2} = \frac{2g_0^{(n+2)/n}}{h^2}.$$

It follows that

$$g_0 \int_{\Omega \times \{t\}} H'' \, dx \leq 2 \int_{\Omega \times \{t\}} (H')^2 \, dx + [g_0 g_1^{(n+2)/n} + 2g_0^{(n+2)/n} h^{-2}] |\Omega|.$$

Using this inequality in (3.9) and applying Lemma 3.2 yields (2.8b). ■

Note that E plays two different roles in Section 2, which now must be played by two functions denoted by E and U . We use U to measure how large the solution u is and how fast it changes. In seeing how fast U (and hence u) changes, we introduce a comparable function E whose derivative is easily estimated. As we shall see in the proof

of Theorem 3.4, the correct quantity for U is $\|u\|_H$, while E is defined by (3.7). Note also that the variational hypothesis (2.2) is not used in this case.

THEOREM 3.4. *Let A and g satisfy (2.3), (3.5) and (3.6), suppose that f satisfies (3.8), suppose u solves (2.1), and set $U(t) = \|u(\cdot, t)\|_H$. If B and k are positive constants with $B > \max\{k, 8\}$, and if $\tau > \sigma$ are positive constants with*

$$(3.10) \quad U(\tau) = B, \quad U(\sigma) = k, \quad U(t) \leq B \quad \text{for } t \in (\sigma, \tau),$$

then there is a positive constant η determined only by $B, k, \Omega, C_1(1/B), g_0, g_1, n,$ and h such that (0.6) is satisfied.

Proof. Taking $\varepsilon = 1/B$, we see that

$$1 - \int_{\Omega \times \{\sigma\}} H(\varepsilon u) dx = \int_{\Omega \times \{\tau\}} H(\varepsilon u) dx - \int_{\Omega \times \{\sigma\}} H(\varepsilon u) dx \leq C[\tau - \sigma].$$

Because $H'' \geq 0$, we have $H(\varepsilon u) \leq (\varepsilon k)H(u/k)$ and therefore $1 - (k/B) = 1 - \varepsilon k \leq C[\tau - \sigma]$, which implies (0.6). ■

When f grows more rapidly with respect to p , we must modify Lemma 3.3 to obtain a hybrid of the conditions in Theorem 3.4 and those in Section 2.

LEMMA 3.5. *Suppose there are scalar functions $S, g,$ and G and a positive constant g_0 such that (2.2), (2.3), and (3.5) are satisfied. Let u be a solution of (2.1) and define E by*

$$(3.11) \quad E(t) = \int_{\Omega \times \{t\}} S(Du) dx + \lambda \int_{\Omega \times \{t\}} H(\varepsilon u) dx$$

for some positive constants ε and λ . Suppose also that $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ and that there is a positive decreasing function C_1 such that

$$(3.12) \quad |f(z, p)| \leq C_1(\theta) + \theta |H'(\theta z)| + \theta g(|p|)^{1+1/n} |p|^{1/n}$$

for all real z and all $\theta \in (0, 1)$. If there is a constant B such that

$$(3.13) \quad \int_{\Omega \times \{t\}} G(|Du|) dx \leq B,$$

and if $\varepsilon \leq 1/(8B^{1/n})$, then there is a constant $C(B, C_1, \varepsilon, \lambda, |\Omega|, g_0, g_1, n, h)$ such that (2.8b) holds.

Proof. Let us define Γ and Φ by $\Gamma(\tau) = G(\tau)/B$ and

$$\Phi^{-1}(\tau) = \int_0^\tau \frac{\Gamma^{-1}(s)}{s^{1+1/n}} ds.$$

From the Sobolev-Orlicz imbedding theorem [2, Theorem 3.4(b)], we have $\|u\|_\Phi \leq 1$ because the constant γ in that inequality is $n^{1/2}$. It is a simple matter to check that $\Phi(\tau) = H(\tau/B^{1/n})/B$, so the proof of Lemma 3.2 gives

$$\int_{\Omega \times \{t\}} (H')^2 dx \leq 4\varepsilon_1 \int_{\Omega \times \{t\}} H''G(|Du|) dx$$

for any $\varepsilon_1 \leq 1/8$ provided $\varepsilon = \varepsilon_1/B^{1/n}$. Hence the terms obtained from differentiating the first integral in E with respect to t are estimated as in Section 2 while those from the second integral are estimated as in Lemma 3.3. ■

With E defined by (3.11), the analog of Theorem 3.4 is easier because we don't need to take $B > 8$.

THEOREM 3.6. *Let A and g satisfy (1.1), (1.2), (2.2), (2.3), (2.4), (3.5) and (3.6), suppose that f satisfies (3.12), suppose u satisfies (2.1), and set*

$$(3.14) \quad U(t) = \int_{\Omega \times \{t\}} S(Du(x, t)) dx.$$

If B and k are positive constants with $B > k$, and if $\tau > \sigma$ are positive constants satisfying (3.10), then there is a positive constant δ determined only by $B, k, \Omega, C_1(1/B), g_0, g_1, n$, and h such that (0.6) holds.

Proof. Now we use Lemma 3.5 with $0 < \varepsilon < \varepsilon_0$ and $\lambda = \min\{1, (B - k)/2\}$. We then have

$$B - k = U(\tau) - U(\sigma) \leq E(\tau) - E(\sigma) + \lambda \int_{\Omega \times \{\tau\}} H(\varepsilon u) dx.$$

From our choice of ε and λ , it follows that $B - k \leq E(\tau) - E(\sigma) + \frac{1}{2}(B - k) \leq C[\tau - \sigma] + \frac{1}{2}(B - k)$. Rearranging this inequality gives (0.6) with $\eta = 1/(2C)$. ■

As previously indicated, the case $n = 2$ is handled by using the techniques of [9, Lemma 3.2] provided the term $\theta g(|p|)^{1+1/n}|p|^{1/n}$ is replaced by $g(|p|)^{1-\theta+1/n}|p|^{(1/n)-\theta}$. Because of (1.1), we have $G^{-1}(\sigma) \leq C\sigma^{1-\delta}$ for $\sigma \geq 1$, so (2.2) is always satisfied when $n = 1$.

4. The borderline case. The results of Section 3 rely on the specific form of the Sobolev-Orlicz embedding theorem. The Sobolev conjugate function H of G is used to bound the function f in our partial differential equation, and the precise form of this conjugate is used, via Lemmata 3.1 and 3.2, to control terms in the expression for E' . It is well-known that a better embedding theorem is valid in case $G(s) = s^n$, where $n \geq 2$ and $\Omega \subset \mathbb{R}^n$. (The behavior of G for small s is not relevant here.) In this section, we show how to take advantage of the better embedding.

THEOREM 4.1. *Let $n \geq 3$, and let u solve (2.1). Suppose there is a scalar function S satisfying (2.2) and (2.4) with $G(s) = s^n$. Suppose also that there is a positive decreasing function C_1 such that*

$$(4.1) \quad |f(z, p)| \leq C_1(\theta) \exp(\theta|z|^{n/(n-1)}) + \theta|p|^n$$

for all $\theta > 0$, and define E by (2.7). Then for any $B > 0$, there is a positive constant $C(B, C_0, C_1, g_0, \Omega, F, \delta, \Lambda)$ for which (2.8a) implies (2.8b).

Proof. We follow the proof of Theorem 2.1. The only difference is in the estimate of $\int f^2 dx$. We have

$$\int_{\Omega \times \{t\}} f^2 dx \leq C(\theta) + \theta \int_{\Omega \times \{t\}} |Du|^{2n} dx + C_1(\theta) \int_{\Omega \times \{t\}} \exp(2\theta|u|^{n/(n-1)}) dx,$$

and the first integral on the right is estimated as before. From the embedding theorem [14, Theorem 2], we know there is a constant $C(n)$ such that if k is a positive constant with

$$k \geq C(n) \left(\int_{\Omega \times \{t\}} |Du|^n dx \right)^{1/n},$$

then

$$\int_{\Omega \times \{t\}} \exp(|u/k|^{n/(n-1)}) dx \leq 1.$$

Using this implication with $k = 1/(2\theta)$ and $\theta \leq B^{1/n}/(2C(n))$ gives $\int_{\Omega \times \{t\}} t f^2 dx \leq C$ provided θ is sufficiently small, and then the proof is completed as in Theorem 2.1. ■

For $n = 2$, we must modify (4.1) by using $|p|^{2-\theta}$ in place of $\theta|p|^n$. Full details may be found in [9]. Note that the proof of Theorem 4.1 is valid as long as there is a function H satisfying the algebraic inequality

$$|f(z, p)| \leq C_1(\theta)H(\theta|z|)^{1/2} + \theta g(|p|)^{1+1/n}|p|^{1/n}$$

and the Sobolev inequality $\|u\|_H \leq C\|Du\|_G$. The author is currently unaware of situations in which these inequalities are satisfied which do not fall into the cases discussed in Theorem 4.1 or the previous sections of this paper.

5. Nonlinear boundary conditions. In this section we show how to modify our arguments to cover the problem

$$(5.1) \quad u_t = \operatorname{div} A(Du) \quad \text{in } Q, \quad A(Du) \cdot \gamma + f(u) = 0 \quad \text{on } SQ,$$

with Q and SQ as before, where γ denotes the unit inner normal to $\partial\Omega$.

The fast growth case is handled very much like its analog in [9].

LEMMA 5.1. *Let u be a solution of (5.1). Suppose conditions (2.2)–(2.5) are satisfied. Suppose also that there is a C^1 function C_1 such that $G \circ g^{-1}(|f(z)|) \leq C_1(|z|)$ and set*

$$(5.2) \quad E(t) = \int_{\Omega \times \{t\}} [S(Du) - f(u)Du \cdot \gamma + 2C_1(|u|)] dx.$$

Then for any positive constant B , there is a constant $C = C(C_1, B, \Omega, g_0)$ such that

$$(5.3) \quad \int_{\Omega \times \{t\}} [G(|Du|) + G(u)] dx \leq B$$

implies (2.8b).

Proof. Suppose first that f is C^1 and that u is C^2 . Then

$$\begin{aligned} E'(t) &= \int_{\Omega \times \{D\}} [u_t \cdot A(Du) - f(u)D_t u \cdot \gamma - u_t f'(u)Du \cdot \gamma + 2C'_1(u)u_t] dx \\ &= \int_{\Omega \times \{t\}} [Du_t \cdot (A(Du) - f\gamma) - u_t f'(u)Du \cdot \gamma + 2C'_1(u)u_t] dx \\ &= \int_{\Omega \times \{t\}} -u_t [\operatorname{div}(A(Du) - f(u)\gamma) + f'(u)Du \cdot \gamma + 2C'_1(u)] dx \end{aligned}$$

$$= \int_{\Omega \times \{t\}} (-u_t^2 + u_t[f \operatorname{div} \gamma + 2C_1'(u)]) \, dx.$$

Now we use the Sobolev-Orlicz imbedding theorem [2, Theorem 3.2(b)] and Cauchy's inequality to infer that $E' \leq C$, and an approximation argument implies (2.8b) for the general case. ■

Note that the condition relating f and C_1 is not used directly. For the purposes of Lemma 5.1, we could have assumed that $|f(z)| \leq C_2(|z|)$ for some increasing function C_2 ; however, the condition on $G \circ g^{-1}(|f(z)|)$ guarantees, via Young's inequality, that E is bounded from below. This lower bound is crucial for our estimate of how quickly the function U defined by

$$(5.4) \quad U(t) = \int_{\Omega \times \{t\}} [S(Du) + G(|Du|)] \, dx$$

can increase.

THEOREM 5.2. *Let u solve (5.1) and suppose conditions (0.8), (0.9), (2.2) and*

$$(5.5) \quad S(p) \geq 2G(|p|)$$

are satisfied. Suppose that there is an even C^1 function C_1 , which is increasing on $(0, \infty)$, such that

$$(5.6) \quad G \circ g^{-1}(|f(z)|) + G(z) \leq C_1(z),$$

and let $k > 0$. Then there are positive constants $C_2(k, C_1, \Omega)$ and $\eta(k, C_1, \Omega)$ such that (3.10) with $B > C_2$ implies (0.6).

PROOF. By the Sobolev-Orlicz imbedding theorem, there is a constant $C_0(k, \Omega)$ such that $|u(x, \sigma)| \leq C_0$ for $x \in \Omega$. Hence

$$\begin{aligned} E(\sigma) &\leq \int_{\Omega \times \{t\}} [S(Du) + G(|Du|) + G \circ g^{-1}(|f(u)|) + 2C_1(u)] \, dx \\ &\leq 3 \int_{\Omega \times \{t\}} [S(Du) + C_1(u)] \, dx \leq 3k + 3C_1(C_0)|\Omega|, \end{aligned}$$

while a similar argument gives $E(\tau) \geq \frac{B}{2}$. Choosing $C_2 = 8(k + C_1(C_0)|\Omega|)$ gives $E(\tau) - E(\sigma) \geq C$ and then (0.6) follows from Lemma 5.1. ■

The slow growth case is more delicate. The correct ideas are (mostly) present in [9], but an error crept into the proof of [9, Lemma 4.2]: in the estimate of the boundary integral, all powers of $|u|$ should be increased by p . As we shall see in Theorem 5.4, [9, Lemma 4.3] (and hence Theorem 4.4 there) is correct. To give a correct statement and proof of the result in [9, Lemma 4.2], we begin with a technical lemma. As before, the results of this lemma are obvious when $g(\tau) = \tau^m$ for some $m \in (0, n - 1)$.

LEMMA 5.3. *Let g and G satisfy conditions (2.3a,b), and let C_1 and θ be positive constants with $\theta \leq \frac{1}{2}$. Then there is a positive C_2 determined only by g , n , θ , and C_1 such that*

$$(5.9) \quad g \circ G^{-1}(\theta H(\theta z) + C_1) \leq \theta |H'(\theta z)| + C_2 \quad \text{for any real } z.$$

Proof. We define $k = H^{-1}(\theta^{-n} + C_1(\theta)/\theta)/\theta$ and $C_2 = g \circ G^{-1}(2C_1 + \theta^{1-n})$ and note that $g \circ G^{-1}(\theta H(\theta z) + C_1) \leq C_2$ if $|z| \leq k$. Suppressing the argument θz from H and its derivatives, we have

$$g \circ G^{-1}(2\theta H) \leq g \circ G^{-1}(H) = \frac{H'}{H^{1/n}}$$

if z is positive. Because $H(\theta z) \geq \theta^{-n}$ if $z > k$, we have (5.9) in this case as well. Finally, (5.9) holds for $z < -k$ because H is even. ■

For the slow growth case, our analysis is very close to that of Lemma 3.3.

LEMMA 5.4. *Let u be a solution of (5.1), suppose E is given by (3.7), suppose (3.6) holds, and suppose there is a decreasing function C_1 such that*

$$(5.10) \quad G \circ g^{-1}(|f(z)|) \leq \theta H(\theta z) + C_1(\theta)$$

for all real z and all $\theta \in (0, 1)$. If B is a constant such that

$$(5.11) \quad \|u(\cdot, t)\|_H \leq B$$

for all $t \in [t_0, t_1]$, then there is a constant $\varepsilon_0 = \varepsilon_0(\Omega, B)$ such that $\varepsilon \in (0, \varepsilon_0)$ implies that (2.8b) holds with C determined only by $\varepsilon, B, C_1, \Omega, g_1$, and g .

Proof. By direct calculation, followed by use of the differential equation for u and integration by parts, we have

$$\begin{aligned} E'(t) &= \int_{\Omega \times \{t\}} H' u_t \, dx = \int_{\Omega \times \{t\}} H' \operatorname{div} A(Du) \, dx \\ &= -\varepsilon \int_{\Omega \times \{t\}} H'' Du \cdot A(Du) \, dx - \varepsilon \int_{\partial\Omega \times \{t\}} H' A(Du) \cdot \gamma \, d\sigma, \end{aligned}$$

where, as before, we suppress the argument of H and its derivatives. Using the boundary condition and (5.10) yields

$$- \int_{\partial\Omega \times \{t\}} H' A(Du) \cdot \gamma \, d\sigma = \int_{\partial\Omega \times \{t\}} H' f(u) \, d\sigma \leq \int_{\partial\Omega \times \{t\}} |H'| g \circ G^{-1}(\varepsilon H + C_1) \, d\sigma.$$

(The argument ε is also suppressed from C_1 .) Now we continue to estimate this integral. First, the divergence theorem gives

$$\begin{aligned} \int_{\partial\Omega \times \{t\}} |H'| g \circ G^{-1}(\varepsilon H + C_1) \, d\sigma &= \int_{\Omega \times \{t\}} |H'| g \circ G^{-1}(\varepsilon H + C_1) \operatorname{div} \gamma \, dx \\ &\quad + \int_{\Omega \times \{t\}} \frac{d}{du} (|H'|) g \circ G^{-1}(\varepsilon H + C_1) Du \cdot \gamma \, dx \\ &\quad + \int_{\Omega \times \{t\}} |H'| \frac{d}{du} (g \circ G^{-1}(\varepsilon H + C_1)) Du \cdot \gamma \, dx. \end{aligned}$$

To estimate these integrals, we first use (5.9) with $\theta = \varepsilon$ to see that

$$g \circ G^{-1}(\varepsilon H + C_1) \leq \varepsilon |H'| + C_2.$$

Next, we have $|\frac{d}{du}|H'| = \varepsilon H''$, so

$$\begin{aligned} \frac{d}{du}(|H'|)g \circ G^{-1}(\varepsilon H + C_1)Du \cdot \gamma &= \varepsilon H''g \circ G^{-1}(\varepsilon H + C_1)|Du| \\ &\leq \varepsilon H''G(|Du|) + \varepsilon H''[\varepsilon H + C_1] \end{aligned}$$

by Young's inequality. Finally,

$$\begin{aligned} \frac{d}{du}(g \circ G^{-1}(\varepsilon H + C_1)) &= \varepsilon^2 g'(G^{-1}(\varepsilon H + C_1))(G^{-1})'(\varepsilon H + C_1)H' \\ &= \frac{g'(G^{-1}(\varepsilon H + C_1))}{G'(G^{-1}(\varepsilon H + C_1))} \varepsilon^2 H'. \end{aligned}$$

Since g and G are increasing and $sg'(s) \leq G'(s)$, it follows that

$$\frac{d}{du}(g \circ G^{-1}(\varepsilon H + C_1)) \leq \frac{\varepsilon^2 |H'|}{G^{-1}(\varepsilon H + C_1)}.$$

From this inequality and (3.2), we infer that

$$\frac{d}{du}(g \circ G^{-1}(\varepsilon H + C_1))|H'|Du \cdot \gamma \leq \frac{n\varepsilon^2 H''H}{G^{-1}(\varepsilon H + C_1)} |Du|.$$

Since C_1 is nonnegative, we infer from this inequality and Young's inequality that

$$\frac{d}{du}(g \circ G^{-1}(\varepsilon H + C_1))|H'|Du \cdot \gamma \leq n\varepsilon H''G(|Du|) + n\varepsilon H''[\varepsilon H + C_1].$$

Combining all these estimates and invoking (3.2) again yields

$$E'(t) \leq (C\varepsilon^2 - \varepsilon) \int_{\Omega \times \{t\}} H''G(|Du|) dx + C\varepsilon \int_{\Omega \times \{t\}} (H')^2 dx + C(\varepsilon).$$

Now an easy modification of Lemma 3.2 shows that

$$\int_{\Omega \times \{t\}} (H')^2 dx \leq C(\Omega)\varepsilon \left[\int_{\Omega \times \{t\}} H''G(|Du|) dx + \int_{\Omega \times \{t\}} H^{-1/n}(H')^2 dx \right]$$

provided $\varepsilon \leq C(\Omega, \|u\|_H)$. Since $H^{-1/n}(H')^2 = C(g_1, n)|\varepsilon u|^{2-2/n} \leq C$ for $|\varepsilon u| \leq 1$ and $H^{-1/n}(H')^2 \leq H(1)^{-1/n}(H')^2$ otherwise, we infer (2.8b) for ε sufficiently small. ■

Now the proof of Theorem 3.4 immediately gives the following growth rate estimate.

THEOREM 5.5. *Let A and g satisfy (2.3), (3.5) and (3.6), suppose that f satisfies (5.10), suppose u solves (5.1), and set $U(t) = \|u(\cdot, t)\|_H$. If B and k are positive constants with $B > \max\{k, 8\}$, and if $\tau > \sigma$ are positive constants satisfying (3.10), then there is a positive constant δ determined only by $B, k, \Omega, C_1, g_0, g_1, n,$ and G such that (0.6) holds.*

Note that when $A(p) = p$, we can use Theorem 5.5 in place of [9, Theorem 4.2] to prove [9, Lemma 4.3] and hence [9, Theorem 4.4].

For the borderline case, we note that (5.9) and Lemma 3.2, as well as the analog of this lemma which was used in Lemma 5.4, remain valid if

$$(5.12) \quad g(z) = z^{n-1}, \quad H(z) = \exp(|z|^{n/(n-1)}).$$

Moreover, the case $n = 2$ is included without any modifications. Hence the proof of Lemma 5.4 carries over to this case and we obtain the following results.

LEMMA 5.6. *Suppose E is given by (3.7) with g and H given by (5.12), and suppose there is a decreasing function C_1 such that (5.10) holds for all real z and all $\theta \in (0, 1)$. If u is a solution of (5.1) and if B is a constant such that (5.11) holds for all $t \in [t_0, t_1]$, then there is a constant $\varepsilon_0 = \varepsilon_0(\Omega, B)$ such that $\varepsilon \in (0, \varepsilon_0)$ implies (2.8b).*

THEOREM 5.7. *Let A satisfy (3.5b) with g and H given by (5.12), suppose that f satisfies (5.10), and set $U(t) = \|u(\cdot, t)\|_H$. If there are positive constants B and k with $B > \max\{k, 8\}$ and $\tau > \sigma$ satisfying (3.10), then there is a positive constant η determined only by $B, k, \Omega, C_1, g_0, g_1, n,$ and G such that (0.6) holds.*

Note that (5.10) with g and H given by (5.12) is equivalent to the condition

$$|f(z)| \leq C_3(\theta) \exp(\theta|z|^{n/(n-1)})$$

for all real z , all $\theta \in (0, 1)$, and a suitable function C_3 .

6. Extensions of the conditions. There are several ways in which our results can be extended to cover more general situations by very minor modification of the arguments just presented. In this section, we discuss some of these extensions briefly.

First, many of our results are true if A and f are allowed to depend also on x and t . For example, Theorem 2.1 holds with the additional hypothesis that $S_t(x, t, u, p) \leq C_3[S(x, t, u, p) + 1]$ for some constant C_3 . Theorem 3.4 and 3.6 are true even without this additional assumption.

The most interesting extension is to the case that A depends on u . In this case, our techniques seem to apply only when the equation can be written as

$$u_t = \operatorname{div} A(D\Phi(u)) + f(u, Du)$$

with A as before and Φ a strictly increasing $C^1(\mathbb{R} \setminus \{0\}) \cap C^0(\mathbb{R})$ function. Then Theorem 2.1 holds with

$$\Phi'(z)f(u, Du)^2 \leq C_0(\Phi(z)) + C_1S(\Phi'(z)p)$$

in place of (2.6) and E defined by

$$E(t) = \int_{\Omega \times \{t\}} S(D\Phi(u)) \, dx.$$

This rapid growth case is virtually identical to the one in Section 2, but the slow growth case is more delicate. The results are easy to state; if there are positive constants $\theta_1 \leq \theta_2$ such that

$$H'(s) \leq \theta_1 s H''(s), \quad \theta_1 \leq \frac{s\Phi'(s)}{\Phi(s)} \leq \theta_2,$$

and if f satisfies (3.9), then Theorems 3.4 and 3.6 are still true. To prove them in this case, we introduce a transformation T defined by

$$T[f](z) = \frac{1}{z} \int_0^z f(s) \, ds$$

for $f \in L^1_{\text{loc}}(\mathbb{R})$ and set $\psi = T[T[\Phi]]$ and $h(s) = H(\varepsilon\psi(s))$ for ε a positive constant at

our disposal. A straightforward calculation, similar to one giving [10, (4.8)], shows that

$$h'' \geq \theta_1^2 \varepsilon \max \left\{ H' \frac{\psi}{s^2}, \frac{\varepsilon}{(1 + \theta_1)^2} H'' \left(\frac{\psi}{s} \right)^2 \right\}.$$

Then the proof of Lemma 3.2 is easily modified to show that there are positive constants K and ε_0 determined only by θ_1 and θ_2 such that

$$\int_{\Omega} (h')^2 dx \leq K\varepsilon \int_{\Omega} h'' G(|D\Phi|) dx$$

for $\varepsilon \leq \min\{1/\|\Phi\|_h, \varepsilon_0\}$. From this inequality, the theorems follow just as before.

Finally, one can combine the techniques of Sections 2, 3, and 4 with those in Section 5 to consider problems of the form

$$u_t = \operatorname{div} A(Du) + f(u, Du) \quad \text{in } \Omega \times (0, \infty), \quad A(Du) \cdot \gamma + F(u) \quad \text{on } \partial\Omega \times (0, \infty).$$

We refer the interested reader to [1] for details on this situation when $A(p) = p$ and f , which only depends on u , and F are power functions.

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