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A GLOBAL EXISTENCE–GLOBAL NONEXISTENCE CONJECTURE OF FUJITA TYPE FOR A SYSTEM OF DEGENERATE SEMILINEAR PARABOLIC EQUATIONS

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Let, for $i = 1, 2$, $p_i \geq 1$ with $p_1 p_2 > 1$ and let $\alpha_i = \frac{p_i+1}{p_1 p_2 - 1}$. Let L_i be a uniformly elliptic second order operator on $\mathbb{R}^{N_i}$ ($N_i \geq 1$) viewed as a subspace of $\mathbb{R}^N$ where $N = N_1 + N_2 - \dim(\mathbb{R}^{N_1} \cap \mathbb{R}^{N_2})$. The coefficients of the L_i are assumed to be independent of time. We consider nonnegative solutions of

$$
\begin{aligned}
&u_t = -L_1 u + v^{p_1}, \\
&v_t = -L_2 v + u^{p_2}, \qquad\qquad (1.1)\\
&u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad v(\mathbf{x}, 0) = v_0(\mathbf{x}),
\end{aligned}
$$

where $(\mathbf{x}, t) \in \mathbb{R}^N \times [0, T)$, which are locally (in time) in L^∞. We discuss the following conjecture:

CONJECTURE. *If* $\max(\alpha_1 - \frac{1}{2}N_1, \alpha_2 - \frac{1}{2}N_2) < 0$ *then there are both global nontrivial and nonglobal solutions of this problem. If* $\max(\alpha_1 - \frac{1}{2}N_1, \alpha_2 - \frac{1}{2}N_2) \geq 0$ *then all nontrivial solutions are nonglobal.*

The local (in time) existence of locally L^∞ solutions of (1.1) has been established in [U1] by a simple modification of the contraction mapping argument used in [EH] when $L_1 = L_2 = -\Delta_N$. Therefore, we will not be concerned with this issue here.

Our interest in this problem arises from the following observations: If we consider

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nonnegative solutions of the initial value problem

$$(1.2)\qquad \begin{aligned} &u_t = -L_1 u + u^p, && (\mathbf{x}, t) \in \mathbb{R}^N \times (0, T),\\ &u(\mathbf{x}, 0) = u_0(\mathbf{x}), && \mathbf{x} \in \mathbb{R}^N, \end{aligned}$$

where $u_0 \in L^\infty(\mathbb{R}^N)$, then it is well known that if $L_1 = -\Delta$, we have the following results of Fujita and others:

(1) If $1<p<1+\frac{2}{N}$, all nontrivial, nonnegative solutions of (1.2) are nonglobal. (This is called the *subcritical case* and $1+\frac{2}{N}$ is called the *critical blow up exponent*.)

(2) If $p = 1+\frac{2}{N}$, all nontrivial, nonnegative solutions of (1.2) are nonglobal. (This is called the *critical case*.)

(3) If $p > 1+\frac{2}{N}$, there are both global nontrivial and nonglobal solutions of (1.2). (This is called the *supercritical case*.)

Remark 1. Fujita's result says the following: If $\frac{1}{p-1}$, which is the blow up rate for solutions of $y' = y^p$, is not smaller than the decay rate for solutions of $u_t = \Delta u$, then no nontrivial global solutions of (1.2) are possible while if the blow up rate is smaller than the decay rate, global, nontrivial solutions are possible. Our conjecture says something related to this. The system of ordinary differential equations $y' = z^{p_1}, z' = y^{p_2}$ has in fact two blow up rates, one for the first component, $y(t)$, and one for the second component, $z(t)$, given by the numbers α_i. Our conjecture says that if the blow up rate for either component of this system exceeds the decay rate for the corresponding linear equation ($u_t = -L_1 u$ corresponding to $y(t)$ for example) then the system does not possess global, nontrivial solutions, whereas if both blow up rates are smaller than the decay rates for the corresponding linear problem, there are global, nontrivial solutions.

It is a consequence of some estimates of [A1, A2, N] and the "variation of constants formula" that the above results of Fujita *et al.* also hold for second order uniformly elliptic operators of the form

$$L_1 u = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{x})\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^{N} b_i(\mathbf{x})\frac{\partial u}{\partial x_i}$$

where the coefficients are uniformly bounded on $\mathbb{R}^N$. In order to see this, one notes that we can represent the solution of (1.2) in the form

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^N} \Gamma(\mathbf{x}, t; \xi, 0) u_0(\xi)\, d\xi + \int_0^t \int_{\mathbb{R}^N} \Gamma(\mathbf{x}, t; \xi, s) u^p(\xi, s)\, d\xi ds$$

where $\Gamma(\mathbf{x}, t; \xi, s)$ is the fundamental solution of the linear parabolic equation $u_t = -L_1 u$. The estimates to which we just alluded assert the existence of positive constants c, δ_1, δ_2 such that

$$(1.3)\qquad c^{-1} S_{\delta_1,N}(t-s)(\mathbf{x}-\xi) \le \Gamma(\mathbf{x}, t; \xi, s) \le c S_{\delta_2,N}(t-s)(\mathbf{x}-\xi)$$

where for $t > 0$,

$$(1.4)\qquad S_{\delta,N}(t)(\mathbf{x}) = (4\pi\delta^2 t)^{-\frac{N}{2}} e^{-\frac{|\mathbf{x}|_N^2}{4\delta^2 t}}.$$

($|\cdot|_N$ denotes the Euclidean length in $\mathbb{R}^N$.) In [EH], the authors proved that when $L_1 = L_2 = -\Delta$, then for 1.1 we have

(1) If $\max(\alpha_1, \alpha_2) > \frac{N}{2}$, all nontrivial, nonnegative solutions of (1.1) are nonglobal. (Subcritical case.)

(2) If $\max(\alpha_1, \alpha_2) = \frac{N}{2}$, all nontrivial, nonnegative solutions of (1.1) are nonglobal. (Critical case.)

(3) If $0 < \max(\alpha_1, \alpha_2) < \frac{N}{2}$, there are nontrivial global, as well as nonglobal solutions of (1.1). (Supercritical case.)

Clearly, using the same estimates on fundamental solutions, if the L_i satisfy the same conditions as in [A], the same statement holds for (1.1).

The authors of [EH] rely for the proofs of the global nonexistence statements above on some modifications of the iteration arguments developed in [AW] for (1.1) when $L_1 = -\Delta$. Unfortunately, in the degenerate case considered here, these iteration arguments do *not* appear to carry over. Therefore, we shall take the subsolution approach used in [L1] in order to prove global nonexistence in the subcritical case. This argument, for a system, has as its motivation the argument used in [W] for the single equation $u_t = \Delta_N u + u^p$. However, in order to employ this argument we must restrict ourselves to the Lipschitz case ($p_i \geq 1$).

In order to set the stage for a "near proof", we note that comparison theorems applied to the "variation of constants" representation formula for (1.1) coupled with (1.3), (1.4) applied to each L_i with $N = N_i$ for $i = 1, 2$ allow us to replace (1.1) by the simpler problem:

$$\begin{aligned} &u_t = \Delta_M u + \Delta_{M_1} u + v^{p_1}, \\ &v_t = \Delta_M v + \Delta_{M_2} v + u^{p_2}, \\ &u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad v(\mathbf{x}, 0) = v_0(\mathbf{x}), \end{aligned} \tag{1.5}$$

where now we write $\mathbb{R}^N = \mathbb{R}^M \oplus \mathbb{R}^{M_1} \oplus \mathbb{R}^{M_2}$ with $N = M + M_1 + M_2$, $N_1 = M + M_1$, $N_2 = M + M_2$.

We shall write, for points $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{x} = (\mathbf{y}, \mathbf{z}_1, \mathbf{z}_2)$ with $\mathbf{y} \in \mathbb{R}^M$ and $\mathbf{z}_i \in \mathbb{R}^{M_i}$.

In order to attempt to establish the global nonexistence portion of our conjecture in the subcritical case (the case for which $\alpha_i > \frac{N_i}{2}$ for $i = 1$ or $i = 2$), we will investigate what happens to solutions of

$$\begin{aligned} &u_t^\varepsilon = \Delta_M u^\varepsilon + \Delta_{M_1} u^\varepsilon + \varepsilon^2 \Delta_{M_2} u^\varepsilon + (v^\varepsilon)^{p_1}, \\ &v_t^\varepsilon = \Delta_M v^\varepsilon + \varepsilon^2 \Delta_{M_1} v^\varepsilon + \Delta_{M_2} v^\varepsilon + (u^\varepsilon)^{p_2}, \\ &u^\varepsilon(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad v^\varepsilon(\mathbf{x}, 0) = v_0(\mathbf{x}), \end{aligned} \tag{1.6}$$

as $\varepsilon \to 0^+$.

We give a proposed proof here which is a consequence of a series of lemmas, the proofs of which are sketched. There is one gap in the proof which we are unable to fill at present.

Lemma 1. *If the solution of* (1.5) *is global for given nonnegative* $u_0, v_0 \in L^\infty(\mathbb{R}^N)$ *then, for every* $\varepsilon > 0$, *so is the solution of* (1.6) *with the same initial values.*

The idea of the proof is as follows: Define

$$\begin{aligned} U^\varepsilon(\mathbf{x},t,\tau) &= U^\varepsilon(\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,t,\tau) \\ &\equiv \int\limits_{\mathbb{R}^{M_1+M_2}} u(\mathbf{y},\mathbf{z}_1',\mathbf{z}_2',t) S_{\varepsilon,M_1}(\tau)(\mathbf{z}_1-\mathbf{z}_1') S_{\varepsilon,M_2}(t)(\mathbf{z}_2-\mathbf{z}_2')\, d\mathbf{z}_1' d\mathbf{z}_2', \\ V^\varepsilon(\mathbf{x},t,\tau) &= V^\varepsilon(\mathbf{y},\mathbf{z}_1,\mathbf{z}_2,t,\tau) \\ &\equiv \int\limits_{\mathbb{R}^{M_1+M_2}} v(\mathbf{y},\mathbf{z}_1',\mathbf{z}_2',t) S_{\varepsilon,M_1}(t)(\mathbf{z}_1-\mathbf{z}_1') S_{\varepsilon,M_2}(\tau)(\mathbf{z}_2-\mathbf{z}_2')\, d\mathbf{z}_1' d\mathbf{z}_2'. \end{aligned}$$

Then it is not too hard to see that this pair (when $t=\tau$) forms a global supersolution with the same initial values as when $\varepsilon = 0$.

Consider the following initial value problem on some interval $(0,T)$:

$$\text{(IVP)} \qquad \begin{aligned} y'(t) &= z^{p_1}(t), \qquad & z'(t) &= y^{p_2}(t), \\ y(0) &= y_0 > 0, \qquad & z(0) &= 0, \end{aligned}$$

where $p_i \geq 1, p_1p_2 > 1$. Let α_i be as above and let $\lambda_1 = \frac{p_1+1}{p_2+1}$, $\lambda_2 = \frac{1}{\lambda_1}$. Set, for $r_1 \geq 0, r_2 > 0$,

$$G_1(r_1,r_2) = \lambda_1^{-\frac{p_1}{p_1+1}} r_2^{-\frac{1}{\alpha_1}} \int\limits_1^{\frac{r_1}{r_2}} (\sigma^{p_2+1}-1)^{-\frac{p_1}{p_1+1}}\, d\sigma,$$

$$G_2(r_1,r_2) = r_2^{-\frac{1}{\alpha_1}} \int\limits_0^{\frac{r_1}{r_2^{\lambda_2}}} (\sigma^{p_1+1}\lambda_2+1)^{-\frac{p_2}{p_2+1}}\, d\sigma.$$

We have the following two lemmas whose proofs are quite standard:

Lemma 2. *Let $y(t), z(t)$ solve* (IVP) (*uniquely*) *on some interval $[0,T)$. Then they satisfy $t = G_1(y(t),y_0)$ and $t = G_2(z(t),y_0)$ and conversely.*

Lemma 3. *For $i=1,2$ the partial derivatives $G_{i,j}, G_{i,jk}$ satisfy the following*:

(1) $G_{i,1} > 0$, $G_{i,2} < 0$,
(2) $G_{i,11}G_{i,2}^2 - 2G_{i,12}G_{i,1}G_{i,2} + G_{i,22}G_{i,1}^2 \leq 0$.

Lemma 4. *Suppose $v_0 \equiv 0$. Let $D \times [0,T)$ be a parabolic cylinder in $\mathbb{R}^N \times [0,T)$ and suppose $w_\varepsilon(\mathbf{x},t) \geq 0$ is given on* (*and positive on the interior of*) *this cylinder such that $w_\varepsilon(\mathbf{x},0) \leq u_0(\mathbf{x})$ and such that*

$$w_{\varepsilon,t} = \min[(\Delta_M + \Delta_{M_1} + \varepsilon^2\Delta_{M_2})w_\varepsilon, (\Delta_M + \varepsilon^2\Delta_{M_1} + \Delta_{M_2})w_\varepsilon]$$

on the interior of this cylinder. Let $(\underline{u},\underline{v})$ be defined by $t = G_1(\underline{u},w_\varepsilon)$, $t = G_2(\underline{v},w_\varepsilon)$ on $D\times[0,T_1)$ for some $T_1 \leq T$. Then, on this smaller cylinder, $(\underline{u},\underline{v})$ is a subsolution of (1.6), *i.e. $u \geq \underline{u}$ and $v \geq \underline{v}$ on $D \times [0,T_1)$.*

The following corollary is an immediate consequence of this lemma and the properties of G_1.

Corollary 4. *If $(\underline{u},\underline{v})$ is a global subsolution on D and w_ε is as in the previous lemma, then there is a constant $C = C(p_1,p_2)$ such that $w_\varepsilon(\mathbf{x},t) \leq Ct^{-\alpha_1}$ on $D\times(0,T_1)$.*

LEMMA 5. *There exist initial values* $u_0, v_0 \in L^\infty(\mathbb{R}^N)$ *such that the corresponding solution of* (1.5) *is nonglobal.*

This is seen using Lemma 1 and the results of [EH, FLU]. By the same argument we obtain

LEMMA 6. *If* $\max(\alpha_1, \alpha_2) \geq \frac{1}{2}(M + M_1 + M_2)$, *every nontrivial*, *nonnegative solution of* (1.5) *is nonglobal.*

Now we wish to define a function w_ε and a region D for which we may apply Lemma 4. We consider the case $M_1 M_2 > 0$ only. Let

$$W_\varepsilon(\mathbf{x}, t) = S_{1,M}(t)(\mathbf{y}) S_{1,M_1}(t)(\mathbf{z}_1) S_{\varepsilon,M_2}(t)(\mathbf{z}_2).$$

LEMMA 7. *The function* W_ε *satisfies*

$$W_{\varepsilon,t} = \min[(\Delta_M + \Delta_{M_1} + \varepsilon^2 \Delta_{M_2}) W_\varepsilon, (\Delta_M + \varepsilon^2 \Delta_{M_1} + \Delta_{M_2}) W_\varepsilon]$$

on the following subset of $\mathbb{R}^N \times (0, \infty)$:

$$S = \{(\mathbf{x}, t) = (\mathbf{y}, \mathbf{z}_1, \mathbf{z}_2, t) \mid 2t\varepsilon^2(1 - \varepsilon^2) \leq |\mathbf{z}_2|^2_{M_2} - \varepsilon^4 |\mathbf{z}_1|^2_{M_1}\}.$$

(*This is the region on which* $\Delta_{M_1} W_\varepsilon \leq \Delta_{M_2} W_\varepsilon$.)

Assume that we have a global solution of (1.5). We may assume at the outset, by comparison, that the initial values have compact support (in $\mathbf{z}_1, \mathbf{z}_2$) and are of class C^∞. Moreover, from the variation of constants formula for (1.5), a second application of the comparison principle and the autonomy of the system in time, we may assume that $v_0 \equiv 0$ and that $u_0 > 0$ has support in $\{\mathbf{x} \in \mathbb{R}^N \mid |\mathbf{z}_1|^2_{M_1} + |\mathbf{z}_2 - \mathbf{z}_2^0|^2_{M_2} \leq R^2\}$ where $\mathbf{z}_2^0$ is such that for some $\delta > 0$, we have $|\mathbf{z}_2^0|_{M_2} - (1+\delta)R > 0$ and vanishes otherwise. (The latter may be accomplished by a translation.) For $\mathbf{x} \in D$ where $D = \{\mathbf{x} \in \mathbb{R}^N \mid |\mathbf{z}_1|^2_{M_1} + |\mathbf{z}_2|^2_{M_2} \leq \delta^2 R^2\}$, we define for $2t\varepsilon^2(1 - \varepsilon^2) < (|\mathbf{z}_2^0|_{M_2} - (1 + \delta)R)^2 \equiv K$,

$$w_\varepsilon(\mathbf{x}, t) = \int\limits_{\mathbb{R}^M} \int\limits_{\{|\mathbf{z}_2'|^2_{M_2} \geq 2t\varepsilon^2(1-\varepsilon^2)\}} \int\limits_{\{|\mathbf{z}_1'|^2_{M_1} \leq \varepsilon^{-4}(|\mathbf{z}_2'|^2_{M_2} - 2t\varepsilon^2(1-\varepsilon^2))\}} W_\varepsilon(\mathbf{x}', t) u_0(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}'.$$

LEMMA 8. *The function* w_e *satisfies the hypotheses of Lemma* 4 *on* $D \times (0, T_1(\varepsilon))$ *where*

$$T_1(\varepsilon) \leq T(\varepsilon) \equiv \frac{K}{2\varepsilon^2(1 - \varepsilon^2)}.$$

Moreover, *on* D, $w_\varepsilon(\mathbf{x}, 0) = 0$ *although for* $t > 0$, $w_\varepsilon > 0$ *on* D.

Remark 2. It is at this point that our proof of the conjecture is incomplete. We would like to be able to assert that $\liminf_{\varepsilon \to 0^+} T_1/T = c(u_0, D, M, M_i) > 0$. If we could do this, then our conjecture would be completely established.

Suppose that the statement in Remark 2 is in force so that $T_1(\varepsilon) = c\varepsilon^{-2}$ as $\varepsilon \to 0$. We may then apply Lemma 4 with $(w_\varepsilon(\mathbf{x}, 0), 0)$ as initial values for $(\underline{u}, \underline{v})$. We set $2\varepsilon^2 t = C \leq \frac{K}{2}$. We have from the corollary of Lemma 4 and the definition of $W_\varepsilon, w_\varepsilon$ that when $\mathbf{x} = \mathbf{0}$,

$$C^{-\frac{M_2}{2}} \int\limits_{\mathbb{R}^M} \int\limits_{\{|\mathbf{z}_2'|^2_{M_2} \ge C(1-\varepsilon^2)\}} \int\limits_{\{|\mathbf{z}_1'|^2_{M_1} \le \varepsilon^{-4}(|\mathbf{z}_2'|^2_{M_2} - C(1-\varepsilon^2))\}} e^{-\frac{|\mathbf{y}'|^2_M + |\mathbf{z}_1'|^2_{M_1}}{4t} - \frac{|\mathbf{z}_2'|^2_{M_2}}{2C}} u_0(\mathbf{y}', \mathbf{z}_1', \mathbf{z}_2'))\, d\mathbf{z}_1' d\mathbf{z}_2' d\mathbf{y}' \le C'(M, M_1, M_2, p_1, p_2) t^{\frac{M+M_1}{2} - \alpha_1}.$$

Thus as $t \to \infty$ and $\varepsilon \to 0$ we obtain the desired contradiction if the power of t on the right hand side of the last inequality is negative. If $M_1 M_2 = 0$, the above argument is easily modified.

Finally,

Lemma 9. *If $\alpha_i < \frac{N_i}{2}$ for $i = 1, 2$, then* (1.5) *has global, small data solutions.*

This was established in [U1] by standard comparison arguments.

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