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ON THE CAUCHY PROBLEM IN A CLASS OF ENTIRE FUNCTIONS IN SEVERAL VARIABLES

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Abstract. We study the integral representation of solutions to the Cauchy problem for a differential equation with constant coefficients. The Cauchy data and the right-hand of the equation are given by entire functions on a complex hyperplane of $\mathbb{C}^{n+1}$. The Borel transformation of power series and residue theory are used as the main methods of investigation.

1. Introduction. For holomorphic partial differential equations the local theory of Cauchy problem is well developed. In the non-characteristic case the classical Cauchy-Kovalevskaya theorem states existence and uniqueness of analytic solutions. Globally, if we have the entire Cauchy data on a hyperplane, the Cauchy-Kovalevskaya theorem can in certain cases be extended to yield entire solutions (cf. results by M.Miyake [1] and J.Persson [2]).

Recently, Sternin and Shatalov have given explicit solutions of the global Cauchy problem in the constant coefficient case with Cauchy data on an arbitrary analytic hypersurface in terms of the Radon-Laplace integral transform (see [3]).

In this paper a new integral representation for the solutions to a class of Cauchy problems is obtained. We assume that Cauchy data and the right-hand of a partial differential equation are entire functions. As an example we consider the case when the characteristic polynomial $P(\tau,\xi)$ is homogeneous polynomial in two variables $(\tau,\xi)\in\mathbb{C}^2$.

2. Notation and definitions. We will work in $\mathbb{C}^{n+1}$ using the following variables:

$$x=(x_1,\dots,x_n)\in\mathbb{C}^n,\quad \xi=(\xi_1,\dots,\xi_n)\in\mathbb{C}^n,\quad t\in\mathbb{C},\tau\in\mathbb{C}.$$

$\mathbb{C}^n_x$ is considered as a linear subspace of $\mathbb{C}^{n+1}$, i.e. $\mathbb{C}^n_x=\{(t,x)\in\mathbb{C}^{n+1}: t=0\}$. $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ and $\beta=(\beta_1,\dots,\beta_n)$ are multiindices, $|\alpha|=\alpha_1+\dots+\alpha_n$,

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$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. For differential operators we use the following notation:

$$D_t = \frac{\partial}{\partial t}, \qquad D_x^\alpha = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

We treat mth order, linear PDEs with constant coefficients of the following type:

$$D_t^m u + \sum_{k=0}^{m-1} \sum_{|\alpha| \le k} a_{k,\alpha} D_t^{m-k-1} D_x^\alpha u = \Phi(t,x) \tag{1}$$

or for short:

$$P(D)u = \Phi. \tag{$*$}$$

The corresponding symbol for P is

$$P(\tau,\xi) = \tau^m + \sum_{k=0}^{m-1} \Big(\sum_{|\alpha| \le k} a_{k,\alpha} \xi^\alpha \Big) \tau^{m-k-1} = \sum_{\mu=0}^{m} b_\mu(\,\xi\,)\,\tau^{m-\mu},$$

where $\tau \in \mathbb{C}$, $\xi \in \mathbb{C}^n$, $b_0(\xi) \equiv 1$, $b_\mu(\xi) = \sum_{|\alpha| \le \mu} a_{m-\mu,\alpha} \xi^\alpha$, $\mu = 1, \dots, m$. The equation $P(\tau,\xi) = 0$ is called *characteristic with respect to* (1).

We can now formulate the following Cauchy problem. Suppose that $\Phi(t,x)$ is an entire function in $\mathbb{C}^{n+1}$ and that we have entire functions $v_k(x)$ in $\mathbb{C}_x^n$. We seek the unique (entire) solution $u(t,x)$ satisfying

$$P(D)u = \Phi \quad \text{and} \tag{$*$}$$

$$D_t^k u = v_k(x), \quad k = 1, \dots, m-1, \quad \text{for } t = 0. \tag{$**$}$$

Note that since the equation (1) is normal (i.e. $b_0(\xi) = 1$), the Cauchy-Kovalevskaya theorem yields a unique, local solution $u(t,x)$ of $(*), (**)$. If we find an entire solution $\tilde{u}(t,x)$ of $(*), (**)$ then $u = \tilde{u}$ by the uniqueness theorem for holomorphic functions.

Definition. Let an entire function $F(x)$ be given by

$$F(x) = \sum_{|\alpha| \ge 0} \frac{f(\alpha)}{\alpha\,!}\, x^\alpha,$$

where $\alpha\,! = \alpha_1\,! \dots \alpha_n\,!$ A function $\check{F}(x)$ is called the *Borel transform* of $F(x)$ if

$$\check{F}(x) = \sum_{|\alpha| \ge 0} \frac{f(\alpha)}{\xi^{\alpha + I}}, \quad \text{where} \quad I = (1, \dots, 1).$$

We will need the following result ([4]): if $F(x)$ is an entire function of exponential type $\sigma = (\sigma_1, \dots, \sigma_n)$ then $\check{F}(x)$ is holomorphic in $\Gamma_\sigma = \{x \in \mathbb{C}^n : |x_j| > \sigma_j > 0, j = 1, \dots, n\}$. Choose $r_j > \sigma_j$; then $\check{F}$ is holomorphic in the closed domain $\bar{\Gamma}_r = \{x \in \mathbb{C}^n : |x_j| \ge r_j, j = 1, \dots, n\}$.

Let us denote by γ_ξ the set $\{x : \mathbb{C}^n : |\xi_j| = r_j,\ j = 1, \dots, n\}$. Under the above assumptions we have the integral formula

$$F(x) = \frac{1}{(2\pi i)^n} \int_{\gamma_\xi} \check{F}(\xi)\, e^{x\xi}\, d\xi, \tag{2}$$

where $x\xi = x_1\xi_1 + \dots + x_n\xi_n$, $d\xi = d\xi_1 \wedge \dots \wedge d\xi_n$.

3. We can now formulate our main result. We choose $r = (r_1, \ldots, r_n)$ so that the functions $\check{v}_k(x)$ are holomorphic in $\bar{\Gamma}_r$ for $k = 0, 1, \ldots, m-1$. Then we choose r_0 satisfying the following conditions:

(i) solutions of $P(\tau, \xi) = 0$ belong to $\{\tau \in \mathbb{C} : |\tau| < r_0\}$ for all $\xi \in \gamma_\xi$,
(ii) the function $\check{\Phi}(t, x)$ is holomorphic in $\bar{\Gamma}_{r_0,r} \subset \mathbb{C}^{n+1}_{t,x}$

Definition. Let $K(\check{v}(\xi), b(\xi), \tau)$ denote the function

$$K(\check{v}, b, \tau) = \sum_{k=0}^{m-1} \Big(\sum_{\mu+\nu=k} \check{v}_\mu(\xi)\, b_\nu(\xi) \Big) \tau^{m-k-1}. \tag{3}$$

Theorem. *A solution of the problem* $(*)$, $(**)$ *is given by the formula*

$$u(t, x) = \frac{1}{(2\pi i)^{n+1}} \int\limits_{\gamma_\tau \times \gamma_\xi} \frac{[K(\check{v}, b, \tau) + \check{\Phi}(\tau, \xi)]\, e^{t\tau + x\xi}\, d\tau \wedge d\xi}{P(\tau, \xi)}. \tag{4}$$

Proof. Substituting (4) into $(*)$ we obtain $P(D)u = I_1 + I_2$, where

$$I_1 = \frac{1}{(2\pi i)^{n+1}} \int\limits_{\gamma_\tau \times \gamma_\xi} K(\check{v}, b, \tau)\, e^{t\tau + x\xi}\, d\tau \wedge d\xi,$$
$$I_2 = \frac{1}{(2\pi i)^{n+1}} \int\limits_{\gamma_\tau \times \gamma_\xi} \check{\Phi}(\tau, \xi)\, e^{t\tau + x\xi}\, d\tau \wedge d\xi.$$

Since $K(\check{v}, b, \tau)$ is holomorphic with respect to τ for all $\xi \in \gamma_\xi$ we have $I_1 = 0$. From (2) we conclude $I_2 = \Phi(t, x)$. It follows that $P(D)u = \Phi(t, x)$. We only need to show that $D^k u = v_k(x)$ for $t = 0$, $k = 0, 1, \ldots, m-1$. Substituting (4) in $(*)$ we obtain $D_t^k u|_{t=0} = I_3 + I_4$, where

$$I_3 = \int\limits_{\gamma_\xi} \Big(\int\limits_{\gamma_r} \frac{K(\check{v}, b, \tau)\, \tau^k\, d\tau}{P(\tau, \xi)} \Big) e^{x\xi}\, d\xi,$$
$$I_4 = \int\limits_{\gamma_\xi} \Big(\int\limits_{\gamma_r} \frac{\check{\Phi}(\tau, \xi)\, \tau^k\, d\tau}{P(\tau, \xi)} \Big) e^{x\xi}\, d\xi.$$

Expanding K/P, $\check{\Phi}/P$ in powers of $1/\tau$ we conclude that $D_t^k u|_{t=0} = v_k(x)$ (note that we have used (2) again). ∎

4. Example. Let $P(\tau, \xi)$ be a homogeneous polynomial in two complex variables (τ, ξ). We will denote by λ_j the roots of $P(\lambda, 1) = 0$. For simplicity we assume that λ_j are simple roots. According to the above remark, we have

$$P(\tau, \xi) = \prod_{j=1}^{m} (\tau - \lambda_j\, \xi),$$
$$b_k(\xi) = (-1)^k \Big(\sum_{1 \le i_1 < \ldots < i_k \le m} \lambda_{i_1} \ldots \lambda_{i_k} \Big) \xi^k = \sigma_k(\lambda)\, \xi^k, \qquad k = 1, \ldots, m,$$
$$b_0(\xi) = 1.$$

We define $\mathfrak{P}_\mu(v) = \mathfrak{P}_1(\mathfrak{P}_{\mu-1}(v))$ and $\mathfrak{P}_1(v) = \int_0^\xi v(\xi)\,d\xi$. Under the above assumptions and notations we have

$$u(t,x) = \sum_{j=1}^{m} \frac{1}{(\prod_{\substack{k=1\\k\neq j}}^{m}(\lambda_k-\lambda_j))} \sum_{i=0}^{m-1} \Big(\sum_{\mu+\nu=i} \mathfrak{P}_\mu(v_\mu(x+\lambda_j t))\,\sigma_\nu(\lambda)\Big)\lambda_j^{m-i-1}. \tag{5}$$

P r o o f. We first compute (4) by the residue theorem in variable τ, next we use the following generalization of (2):

$$\int_\gamma \xi^\mu\,\check{F}(\xi)\,e^{x\,\xi}\,d\xi = \mathfrak{P}_\mu(F(x)). \ \blacksquare$$

In particular, for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\,\frac{\partial^2 u}{\partial x^2}$$

we have $m=2$, $\lambda_1 = a$, $\lambda_2 = -a$; then

$$u(t,x) = \frac{1}{2}[v_0(x+at)+v_0(x-at)] + \frac{1}{2a}[\mathfrak{P}_1(v_1(x+at)) - \mathfrak{P}_1(v_1(x-at))].$$

This is d'Alembert's well known formula.

References

[1] M. Miyake, *Global and local Goursat problems in the class of holomorphic or partially holomorphic functions*, J. Differential Equations 39 (1981), 445–463.

[2] J. Persson, *On the global and local non-characteristic Cauchy problems when the solutions are holomorphic functions or analytic functionals in space variables*, Ark. Mat. 9 (1971), 171–180.

[3] B. Y. Sternin and V. E. Shatalov, *On some integral transformations of complex-analytic functions*, Dokl. Akad. Nauk SSSR 280 (1985), 553–556.

[4] L. I. Ronkin, *Introduction to the Theory of Entire Functions in Several Variables*, Nauka, Moscow, 1971.