SINGULAR HAMILTONIAN SYSTEMS AND SYMPLECTIC CAPACITIES

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Abstract. The purpose of this paper is to develop the basics of a theory of Hamiltonian systems with non-differentiable Hamilton functions which have become important in symplectic topology. A characteristic differential inclusion is introduced and its equivalence to Hamiltonian inclusions for certain convex Hamiltonians is established. We give two counterexamples showing that basic properties of smooth systems are violated for non-smooth quasiconvex submersions, e.g. even the energy conservation which nevertheless holds for convex submersions. This also implies that the convexity assumption determines, although not symplectically invariant, a limit case for symplectic geometry. Some applications of this theory are reviewed: symplectic capacities for general convex sets, the symplectic product and a product formula for symplectic capacities.

1. Introduction. We consider the linear space $\mathbb{R}^{2n}$ with the standard closed non-degenerate 2-form ω as symplectic manifold. One can describe ω by an almost complex structure $J : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$, $J_p^2 = -id_p\ \forall p \in \mathbb{R}^{2n}$, and the standard scalar product. Expressed in coordinate functions to be arranged in analogy to complex ones, $x = (x_1, \ldots, x_n)$ with $x_i = (p_i, q_i)$, J_p is given by a constant matrix (denoted again by J)

$$J = \begin{pmatrix} \square & & \\ & \ddots & \\ & & \square \end{pmatrix}, \quad \square = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cong i.$$

The standard scalar product and the symplectic form are then represented by

$$x.y = \sum_{i=1}^{2n} x_i y_i, \quad \omega(x,y) = Jx.y.$$

A differentiable map is called symplectic if $\varphi^*\omega = \omega$, or in the above coordinates if $d\varphi(x)^T J d\varphi(x) = J$. We denote the set of symplectic embeddings of open sets in $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$ by $\mathcal{E}_\omega(\mathbb{R}^{2n})$ and the symplectic diffeomorphisms of $\mathbb{R}^{2n}$ by $\mathcal{D}_\omega(\mathbb{R}^{2n})$.

Let $B(r) = B^{2n}(r) = \{x \in \mathbb{R}^{2n} \mid |x| \leq r\}$ be the ball and $Z(r) = B^2(r) \times \mathbb{R}^{2n-2} = \{x \in \mathbb{R}^{2n} \mid q_1^2 + p_1^2 \leq r^2\}$ be a cylinder with a symplectic base disc.

Now we can state two theorems about symplectic embeddings and diffeomorphisms as motivation for our considerations, quoted in $\mathbb{R}^{2n}$ only although they hold for general symplectic manifolds.

THEOREM A [G 85]. *If the ball $B(r)$ can be symplectically embedded into the cylinder $Z(R)$, then $r \leq R$.*

In other words, $B(r)$ cannot be squeezed symplectically into an infinitely long cylinder $Z(R)$ if $r > R$, whereas this can be done for $r \leq R$ by the identity map, expressing a rigidity of symplectic embeddings. Gromov showed that a symplectic invariant defined with J-holomorphic curves is the obstruction for such embeddings. It is different from the well known Liouville volume $\text{vol}(D) = \int_D \omega^n$ which cannot be the obstruction because $\text{vol}(B(r)) < \infty$ and $\text{vol}(Z(R)) = \infty$, independently of r and R.

Such an invariant, nowadays called *symplectic capacity*, or more precisely *symplectic capacity for $\mathcal{F}$ and $\mathcal{D}_\omega$* is a map c of a family $\mathcal{F}$ of subsets of $(\mathbb{R}^{2n}, \omega)$ to $\mathbb{R}_+$ satisfying (assuming $D, D', B(r), Z(r) \in \mathcal{F}$)

(a) $D \subset D' \Rightarrow c(D) \leq c(D')$,
(b) $\varphi \in \mathcal{D}_\omega \Rightarrow c(\varphi(D)) = c(D)$,
(c) $c(B(r)) = \pi r^2 = c(Z(r))$.

The axioms are designed in the way that the existence of a symplectic capacity implies readily theorem A. The next theorem is another consequence of the existence of a symplectic capacity.

THEOREM B [G85], [El87]. *The set of symplectic diffeomorphisms of $\mathbb{R}^{2n}$ is closed in the set of diffeomorphisms of $\mathbb{R}^{2n}$ with respect to the C^0-compact-open topology.*

In other words, a topological rigidity is stated. This raised the question whether one can define notions of C^0-symplectic manifolds and maps as a framework for symplectic topology.

Towards the notion of C^0-symplectic manifolds, a step has been done in [K90], namely the case of convex sets with non-smooth boundaries, which we report in the present article.

In order to extend the notion of symplectic maps to non-differentiable continuous maps, one can, for a given capacity c, consider a subgroup G_c of homeomorphisms of $\mathbb{R}^{2n}$ conserving the capacities of a certain class of subsets of $\mathbb{R}^{2n}$. Consider for example for $\mathcal{F}$ the set of sub-level sets $S_q = \{x \mid q(x) < 1\}$ where q varies over the set of all quadratic forms. Let B be the open ball $\{x \mid |x|^2 < 1\}$, then it is possible to formulate the *local symplectic rigidity theorem*.

THEOREM C [EH 89]. *Let φ_k in $C^0(B, \mathbb{R}^{2n})$ conserve the capacities $c(\varphi_k(S_q)) = c(S_q)$ for all quadratic forms q and converge to φ in the sup-norm. Then, for every point x*

where $d\varphi(x)$ exists, $d\varphi(x)$ is either symplectic or antisymplectic:

$$d\varphi(x)^T J d\varphi(x) = \pm J.$$

Images of smooth hypersurfaces by such maps are non-smooth in general. It is possible to explain, by the theory presented in this article, what Hamiltonian dynamics on such singular hypersurfaces means.

Hamiltonian dynamics is in fact related to symplectic capacities: Ekeland and Hofer showed in their papers [EH89] and [EH90] that both the embedding and topological rigidy can also be understood by means of periodic solutions of Hamiltonian systems instead of J-holomorphic curves. We give a very short review of their construction.

For any bounded domain $D \subset \mathbb{R}^{2n}$ and any function H from the following restricted set

$$\mathcal{H}(D) = \{H \in C^3(\mathbb{R}^{2n}, \mathbb{R}_+) \mid \exists \text{ open } U \supset \bar{D} \text{ such that } H|_U = 0,$$

$$H(x) = a|x|^2 \text{ for } x \text{ large}, \ a > \pi, \ a \notin \mathbb{N}\pi\},$$

Ekeland and Hofer construct a minimax critical value $c(H)$ of the Hamiltonian action functional $\phi_H(x) = \frac{1}{2}\int_0^1 Jx(t).\dot{x}(t)\,dt - \int_0^1 H(x(t))\,dt$ on the space of loops $E := H^{1/2}(S^1, \mathbb{R}^{2n})$. This value corresponds to a 1-periodic solution of the Hamiltonian equation $\dot{x} = JH'(x)$ running somewhere in D^c. One shows the monotonicity $H_1 \geq H_2 \Rightarrow c(H_1) \leq c(H_2)$. The real number

$$c_{EH}(D) := \inf_{H \in \mathcal{H}(D)} c(H)$$

being independent of H, is a good candidate for a symplectic "size" of the set D: In fact c_{EH} is shown to satisfy (a), (b) and (c), the axioms of symplectic capacities.

One would like to be able to define the capacity with the ideal limit Hamiltonian I_D, the characteristic function of D with value 0 on D and ∞ elsewhere, which satisfies $I_D \geq H \ \forall H \in \mathcal{H}_D$; then no infimum on Hamiltonian functions had to be taken: a simplification which also needs non-smooth Hamiltonian systems. This application is presented in sections 3.1 and 9 and was one of the basic ideas for [K90].

It is not evident that the Hamiltonian equation should pass to the above infimum. But it does so in the case where the hypersurface ∂D is C^1 and *of restricted contact type*, which means that it is regularly submersed in a family of hypersurfaces $\{H(x) = E\}$ generated by a transversal ω-contraction, $L_X\omega = \omega$: The solutions of different levels are equivalent through the contraction, i.e. existence of a solution on a nearby level means existence on ∂D.

In this situation (containing the special case of convex hypersurfaces) Ekeland and Hofer get a useful representation result for the capacity: $c_{EH}(D)$ is a multiple of the symplectic action $A(x) = \frac{1}{2}\int \dot{x}(t).Jx(t)dt$ of a T-periodic solution x (with unknown T) of $\dot{x} = JH'(x)$ on the level ∂D, i.e. $c_{EH}(D) = nA(x)$ for some unknown $n \in \mathbb{N}_+$, representing the action of an n-fold iteration of x.

The factor n could not be controlled in [EH 89] and [EH 90], even not in the case of a smooth convex domain D where one conjectured $n = 1$. This has been done by Sikorav in August 1990 for the Ekeland-Hofer capacity c_{EH}, at least for smooth convex sets. Before and independently, a converse statement has been shown in [K90], namely that the

least characteristic action c_0 on convex hypersurfaces is a symplectic capacity for possibly *non-smooth* convex domains in $\mathbb{R}^{2n}$. To this aim we studied Hamiltonian type differential inclusions as presented in sections 2 and 3 and used the dual Hamiltonian functional. In another, different approach, Hofer and Zehnder [HZ90] introduced a capacity c_{EZ}, whose restriction to smooth convex sets is also equal to c_0, i.e. $c_{EH} = c_{EZ} = c_0$ on smooth convex sets.

A further motivation is the idea that symplectic capacities are possibly calculable by lower and upper sums in analogy to measure theory. This would be an important tool, because capacities are quite difficult to calculate on examples. The idea is to determine the capacity of special cubes, which has been done in [K90], and then give a lower bound for the capacity of a general set D by calculating the capacity of a disjoint union of cubes contained in D. But also for this construction, one juxtaposes - most economically - domains with corners and therefore one needs an examination of Hamiltonian dynamics on non-smooth hypersurfaces.

Moreover, one hopes that it would be easier to determine the Hamiltonian dynamics on piecewise linear (PL) hypersurfaces in order to increase the number of known examples.

In fact, the capacity c_0 can be calculated for symplectic products, see [K90]. To obtain this, we introduced a characterization of closed characteristics on products, which implies a symplectic characterization of polydiscs, see [K95a]. Products having evidently non-smooth boundaries present an other motivation for the present article.

As a summary, we showed in [K90] that a capacity on convex sets can be obtained without an infimum on a set of Hamilton functions and without approximations of ∂K and solutions by a transversal flow. Moreover, no iterated solutions ($n \neq 1$) interfere. For the monotonicity (b), we used the equivalence of Hamiltonian and characteristic differential inclusion in the convex case (see §3.1) to get an analytical definition by the minimum of the dual Hamiltonian functional, an approach which is simpler than the one in [EH89] but is restricted to convex sets.

In sections 2 to 7, the main results of this paper are presented. The last three sections consist of a survey of applications to symplectic capacities and symplectic products. More details on these applications are given in [K95] and [K95a].

In [K91], the capacity c_0 on convex sets is extended to all subsets in $\mathbb{R}^{2n}$. We got, using [EH90] together with [Si90], or alternatively [EZ90], two *distinct* symplectic capacities u and ℓ which are shown to present an upper and lower bound for *all* capacities extending the least action on convex sets. For instance the Ekeland-Hofer and Hofer-Zehnder capacities are estimated above and below by u and ℓ and can be calculated in certain cases by means of these. The observation that u and ℓ are distinct has two interesting consequences: lower and upper sums do not converge to the same value in general, and one conjectures that c_{EH} is different from c_{HZ}.

As a simple corollary, two inequalities by Ekeland and Croke-Weinstein respectively are improved.

Another result is for example that all sets D with $B(r) \subset D$ having a periodic orbit (in the general sense of §§2 and 8) on ∂D with action strictly less than πr^2 *cannot* be symplectomorphic to a convex set, see [K91].

A rule for the capacities of some unions and differences of sets is determined, showing that a capacity does not behave like a measure theory. Moreover isotropic tori and sets with codimension 1 are calculated. Until [K90], no examples other than the ball, the cylinder $Z(r)$ and the ellipsoid which are trivial as they are given by or follow from properties (a), (b) and (c), were known.

2. Characteristic differential inclusions. In order to explain what we mean by Hamiltonian type systems with non-differentiable Hamilton functions, we introduce the characteristic differential inclusion of a Lip-submersed hypersurface S. To do this we recall first the differentiable case.

If $H \in C^1$ and $S := \{H(x) = 1\}$ is regular, $H'(x) \neq 0 \;\; \forall x \in S$, one can show that

$$(H) \qquad\qquad \dot{x}(t) = JH'(x(t)), \qquad H(x(0)) = 1,$$

has the same trajectories as the so called "characteristic equation" of the hypersurface S with outward normal $n(x)$, $x \in S$:

$$(C) \qquad\qquad \dot{x}(t) = Jn(x(t)) \qquad \text{with } x(t) \in S \; \forall t.$$

The equation (C) is characteristic for S in the sense that the set of its solutions (also called "characteristic curves" or "characteristics") is determined by the hypersurface only, more precisely by the restriction of the symplectic structure to S, and does not depend on the Hamilton function H generating S as regular hypersurface. Its solutions are parametrizations by arc length of the integral leaves of the field $\ker \omega_{|S}$. Two solutions x_1 and x_2 of (H) for different Hamilton functions H_1 and H_2 are called *geometrically equivalent* if $\operatorname{Im} x_1 = \operatorname{Im} x_2$; (C) together with the choice of an initial point fixes a representative of every equivalence class.

To show the equivalence $(H) \Leftrightarrow (C)$, we just used some basic properties of smooth Hamiltonian systems, which however *are violated*—see §§5 to 7—for non-smooth systems. In spite of these problems, a similar equivalence is proved for *convex* non-smooth systems in §3.1.

Our aim is to introduce a natural characteristic equation in the case where S is submersed by a Lipschitzian function.

First, the hypersurface may have edges and corners, where the outward normal is not unique (i.e. a set valued function). Such a generalized normal can be defined if S is given by a Lipschitz function, in particular if S is the boundary of a convex set.

For the last case, the idea is to take, at a given point x, all normals of the hyperplanes passing through x whose negative half spaces *contain* the convex set.

DEFINITION. The (*outward*) *normal cone* and the *normalized* (*outward*) *normal set* of a convex set K at a point $x \in \mathbb{R}^{2n}$ are given by

$$N_K(x) = \{p \in \mathbb{R}^{2n} \mid p.(x-y) \geq 0 \;\; \forall y \in K\},$$
$$n_K(x) = \{p \in N_K(x) \mid |p| = 1\}.$$

It is easy to see that $N_K(x)$ is a *cone* for all x, i.e. $p \in N_K(x) \Rightarrow \lambda p \in N_K(x) \quad \forall \lambda \in \mathbb{R}_+$, and that $N_K(x) = \{0\}$, $n_K(x) = \emptyset$ for all $x \in \mathring{K}$. Furthermore, for a smooth point x of ∂K, $N_K(x) = \mathbb{R}_+ n$, $n_K(x) = \{n\}$, where n is the usual normalized outward normal

vector at x. One sees readily that N_K is a set valued vector field in K with support on ∂K, thus suitable for our aim to construct a system which is characteristic for the hypersurface ∂K.

Secondly, it is clear that for instance curves passing through a corner of ∂K cannot be differentiable. The idea is to ask that γ should be differentiable only almost everywhere. One would therefore like to study

$$\text{(i)} \quad \dot\gamma(t) \in Jn_K(\gamma(t)) \quad \text{a.e.,}$$

where a solution γ is asked to be parametrized by arc length and Lipschitzian with constant 1: $|\gamma(t) - \gamma(t')| \leq \int_{t'}^{t} |\dot\gamma(t)| dt = |t - t'|$; Lipschitzian functions are almost everywhere differentiable (by Rademacher's theorem), so that the system is consistent. Here it is already plausible that the non-uniqueness problem is present, because $n_K(x)$ is set valued and also $Jn_K(x)$ may intersect the generalized tangent space of ∂K in a (non-trivial) family of directions.

But is not clear yet whether a submersed family of convex hypersurfaces would define a field with energy conserving solutions. The answer is given in section 5: It is negative.

Therefore, a natural idea to generalize (C) is to consider (i) together with the additional constraint (ii) which is automatically satisfied in the smooth case, see §3. It is called *characteristic differential inclusion for ∂K*:

$$(CI) \qquad \begin{array}{ll} \text{(i)} & \dot\gamma(t) \in Jn_K(\gamma(t)) \quad \text{a.e.,} \\ \text{(ii)} & \gamma(t) \in \partial K \quad \forall t \in [0, T_\gamma]. \end{array}$$

3. Hamiltonian inclusions. Some new facts on convex Hamiltonian systems are established in this section.

DEFINITION. The *subdifferential* of a locally Lipschitz function H is defined by

$$\partial H(x) = \left\{ p \in \mathbb{R}^{2n} \,\middle|\, \forall v \in \mathbb{R}^{2n} \ p.v \leq D_+ H(x)(v) := \lim_{h \to 0^+} \frac{H(x + hv) - H(x)}{h} \right\}.$$

Its elements are called subgradients. x is called a critical point of H if $0 \in \partial H(x)$.

For convex functions, $\partial H(x)$ is non-empty and bounded for all $x \in \mathbb{R}^{2n}$ (see [A84]). At a point where H is differentiable, $\partial H(x)$ reduces to $\{H'(x)\}$: $p.v \leq H'(x).v \ \forall v \in \mathbb{R}^{2n} \Rightarrow p = H'(x)$.

The subdifferential is thus a generalization of the gradient in the same way as the normal cone generalizes the outward normal. In fact, we will show now that, for some convex "gauge functions" H of a bounded convex domain K, $\partial H(x)$ and $n_K(x)$ differ only by the length of their elements.

3.1. *Equivalence.* For convex sets K containing 0 in their interior we now choose $H(x) = (j_K(x))^\alpha$ as Hamilton function, where $j_K(x) = \inf\{\lambda \mid \frac{x}{\lambda} \in K\}$ is the so called *gauge function* of K, and for α we assume $\alpha \geq 1$. H is convex, finite and α-homogeneous. Moreover $\mathring{K} = \{x \mid H(x) < 1\}$.

With this choice of H the following fixed energy problem is equivalent to (CI):

$$(HI) \qquad \begin{cases} -J\dot x(t) \in \partial H(x(t)) \quad \text{a.e.} \\ H(x(t)) = 1 \quad \forall t \in [0, T] \end{cases}$$

where $x(t)$ is assumed Lipschitz.

THEOREM 1 (Equivalence of (CI) and (HI)). *For H as above*:

(a) $n \in N_K(x) \Leftrightarrow \frac{\alpha}{n.x} n \in \partial H(x)$ *for* $x \in \partial K$,

(b) (CI) *and* (HI) *have the same solutions up to monotone absolutely continuous reparametrizations*.

P r o o f. (a) Observe beforehand that there is a small ball $B_\varepsilon(0) \subset K$, therefore there is $r > 0$ such that $rx \in K$ $\forall x \in \partial K$, and conclude $n.x \geq n.rn > 0$ for all non-zero $n \in N_K(x)$ and for all $x \in \partial K$. Division by $n.x$ is then well-defined; in the case $n = 0$, the quotient $\frac{n.y}{n.x}$ is defined to be 0 by continuity.

As a first step, we prove the equivalence (for $x \in \partial K$):

$$n \in N_K(x) \ \Leftrightarrow \ j_K(y) \geq \frac{n.y}{n.x} \ \forall y \in \mathbb{R}^{2n},$$

namely

$$n \in N_K(x) \Leftrightarrow n.(x - y) \geq 0 \ \forall y \in K$$
$$\Rightarrow n.(x - y) \geq 0 \ \forall y \in \partial K$$
$$\Leftrightarrow n.\left(x - \frac{y}{j_K(y)}\right) \geq 0 \ \forall y \in \mathbb{R}^{2n}$$
$$\Leftrightarrow j_K(y) \geq \frac{n.y}{n.x} \ \forall y \in \mathbb{R}^{2n}.$$

Conversely, using $1 \geq j_K(y) \ \forall y \in K$, we get

$$j_K(y) \geq \frac{n.y}{n.x} \ \forall y \in \mathbb{R}^{2n} \Rightarrow 1 \geq \frac{n.y}{n.x} \ \forall y \in K$$
$$\Leftrightarrow n \in N_K(x).$$

Therefore the above implications are equivalences.

As a second step, we apply the equivalence (true because H is convex)

$$p.x - H(x) = \min_{y \in \mathbb{R}^{2n}} \left(p.y - H(y)\right) \Leftrightarrow p \in \partial H(x)$$

(Legendre duality) for the choice $p = \frac{1}{n.x} n$ with given $n \in N_K(x)$:

$$n \in N_K(x) \Leftrightarrow 0 \geq \frac{n.y}{n.x} - j_K(y) \ \forall y \in \mathbb{R}^{2n}$$
$$\Leftrightarrow \frac{1}{n.x} n.x - H(x) = 0 \geq \frac{1}{n.x} n.y - j_K(y) \ \forall y \in \mathbb{R}^{2n}$$
$$\Leftrightarrow \frac{1}{n.x} n \in \partial j_K(x).$$

Finally, the right derivative of $H(x) = \left(j_K(x)\right)^\alpha$ for $\alpha \geq 1$ is given by

$$D_+ H(x)(v) = \alpha \left(j_K(x)\right)^{\alpha-1} D_+ j_K(x)(v) \quad \forall v \in \mathbb{R}^{2n},$$

its subdifferential is therefore $\partial H(x) = \alpha \partial j_K(x)$, from where we conclude

$$n \in N_K(x) \Longleftrightarrow \frac{\alpha}{n.x} n \in \partial H(x).$$

It is remarkable that $N_K(x)$ is a cone whose rays correspond to exactly one element of $\partial H(x)$, namely $p = \alpha \frac{n}{n.x}$. By normalization we get

$$n \in n_K(x) \Leftrightarrow \frac{\alpha}{n.x} n \in \partial H(x) \text{ and } |n| = 1.$$

(b) Problems (HI) and (CI) are restricted to the same energy hypersurface ∂K, so that (a) applies:

$$-J\dot{x}(t) \in \partial H(x(t)) \Rightarrow \frac{-J\dot{x}(t)}{|-J\dot{x}(t)|} \in n_K(x(t)),$$

$$-J\dot{\gamma}(s) \in n_K(\gamma(s)) \Rightarrow \frac{\alpha}{-J\dot{\gamma}(s).\gamma(s)} \cdot \big(-J\dot{\gamma}(s)\big) \in \partial H(\gamma(s)) \ ,$$

where vanishing numerators are excluded. The reparametrizations are thus given by the monotone functions τ and τ^{-1}:

$$\tau(t) = \int\limits_0^t \frac{\alpha}{-J\dot{\gamma}(s).\gamma(s)}\, ds, \qquad x(t) := \gamma(\tau(t)),$$

$$\tau^{-1}(s) = \int\limits_0^s \frac{1}{|\dot{x}(t)|}\, dt, \qquad \gamma(s) := x\big(\tau^{-1}(s)\big).$$

If we fix the domain of τ to be $[0, T_\gamma]$, then the minimality of the period T_x follows from the minimality of T_γ, and conversely. It is obvious that τ and τ^{-1} are absolutely continuous. ∎

R e m a r k. Observe that also the characteristic function of a set

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ \infty & \text{otherwise,} \end{cases}$$

and the normal cone are related: $\partial I_K(x) = N_K(x)$. This is (morally) used for symplectic capacities as explained in §9.

3.2. *Energy conservation.* After we showed that there exist non-differentiable quasi-convex Hamilton functions having solutions which do not conserve the energy (cf. section 5), I. Ekeland asked R. T. Rockafellar whether this would happen with convex functions also. The answer was the following theorem, the proof of which is given and reformulated for convenience. An earlier proof can be found in [Cl80], as pointed out by F. Clarke.

THEOREM 2 [R89]. *Let H be a finite convex function from $\mathbb{R}^{2n}$ to $\mathbb{R}$ and $x \in W^{1,1}([0,T], \mathbb{R}^{2n})$ a solution of*

$$(H_0) \qquad\qquad \dot{x}(t) \in J\partial H(x(t)) \quad \text{a.e. in } [0,T].$$

Then $H(x(t)) = H(x(0)) \ \forall t \in [0,T]$.

P r o o f. A solution x is Lipschitzian with a Lipschitz constant bounded by $\max\{|p| \mid p \in \partial H(x(t)), t \in [0,T]\} < \infty$ and H is locally Lipschitzian by [A84, p. 19–21]. Therefore $\theta(t) = H(x(t))$ is locally Lipschitzian and thus almost everywhere differentiable (Rademacher). With this regularity, one can establish $\dot{\theta}(t) = 0$ almost everywhere; this is done in an analogous way as in the C^1-case: Let us denote by $\mathcal{T}$ the set of full measure

where $\theta(t)$ *and* $x(t)$ are differentiable and $\dot{x}(t) \in J\partial H(x(t))$. For all $t \in \mathcal{T}$, the right and left derivatives coincide, in particular for $v = \dot{x}(t)$:

$$\dot{\theta}(t) = D_+ H(x(t))(\dot{x}(t)) = -D_+ H(x(t))(-\dot{x}(t)).$$

Because of $-J\dot{x}(t) \in \partial H(x(t))$, we get

$$-J\dot{x}(t).v \leq D_+ H(x(t))(\dot{x}(t))$$

for both $v = \pm\dot{x}(t)$. The antisymmetry of J yields

$$0 = -J\dot{x}(t).\dot{x}(t) \leq D_+ H(x(t))(\dot{x}(t)),$$
$$0 = -J\dot{x}(t).\dot{x}(t) \geq D_+ H(x(t))(\dot{x}(t)).$$

Therefore $D_+ H(x(t))(\dot{x}(t)) = 0 = \dot{\theta}(t) \ \forall t \in \mathcal{T}$ and $\theta(t) = H\big(x(t)\big) = \text{const.}$ ∎

4. Quasiconvex submersions. Now consider a family of sets $K(s) \subset \mathbb{R}^N, s \in \mathbb{R}$, continuous with respect to the Hausdorff metric on sets.

DEFINITION. A family $K(s)$ is called *regular* if $\partial K(s) \cap \partial K(s') = \emptyset, \forall s \neq s'$.

This generalizes the notion of a C^1-submersed family of hypersurfaces to the non-differentiable case. In fact, it implies that $\partial K(s)$ can be given as levels of a continuous function $f : \bigcup_s K(s) \to \mathbb{R}$ defined by

$$f(x) = s \Leftrightarrow x \in K(s).$$

If f is C^1, then $f'(x) \neq 0 \ \ \forall x \in K(s)$, which is the classical definition of the regularity of the hypersurface $K(s)$.

Now the relation between f and $K(s)$ is examined if $K(s)$ consists of convex sets.

DEFINITION. A function f and the family $\{x \mid f(x) = s\}_{s\in\mathbb{R}}$ are called *quasiconvex* if the sublevels $\{x \mid f(x) \leq s\}$ are convex for all $s \in \text{Im } f$. The family $K(s)$ or $\partial K(s)$ is called *convexifiable* in the interval I if there is a convex function $H : \mathbb{R}^N \to \mathbb{R}$ and a reparametrization $h : I \to \mathbb{R}$ such that

$$\partial K(s) = \{x = \mathbb{R}^N \mid H(x) = h(s)\} \quad \forall s \in I.$$

It is easy to give quasiconvex families which are convexifiable, e.g. $f(r) = \sqrt{r}$ or $f(r) = \sqrt{\sup_i |x_i|}$. In section 5, we present a quasiconvex example which is not convexifiable, together with a proof of this property using the energy conservation of Hamiltonian inclusions as established in theorem 2.

5. A quasiconvex example (CE1). Let $c(s) = \begin{pmatrix} -\sin s \\ -1+\cos s \end{pmatrix}$ be the parametrization by arc length of the unit circle with center $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $n_s := \begin{pmatrix} -\sin s \\ +\cos s \end{pmatrix}$ the outward unit normal. Then consider the family

$$K(s) := \{x \in \mathbb{R}^2 \mid |x| \leq |c(s)| \text{ and } n_s.x \leq 0\}, \quad s \in [0, \pi],$$

of convex sets given by intersection of halfplanes and discs of radii $|c(s)| = \sin\frac{s}{2}$.
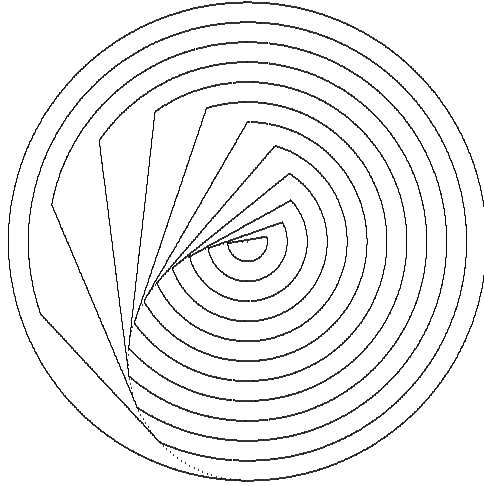
Fig. 1. A quasiconvex family in $\mathbb{R}^2$ violating energy conservation (CE1)

It is easy to show that the distance between the boundaries is positive:

$$\mathrm{dist}(\partial K(s'), \partial K(s)) = 2\big(\sin\frac{s'-s}{2}\big)^2 > 0 \qquad \forall s, s' \in [0, \pi], s' > s,$$

which means that $\partial K(s)$ are mutually disjoint and $K(s)$ defines a *regular* family of convex sets: For all $x \in B(2) = K(\pi)$ there is a unique value $s =: f(x)$ such that $x \in \partial K(s)$. One gets the representation of $K(s)$ by

$$\{f(x) \le s\} = K(s).$$

f is a quasiconvex function which is at least of class C^0 (by the continuity of n_s and $|c(s)|$ in s) but not C^1 because there are two curves of corners, the first one being $c(s)$.

The characteristic differential inclusion adapted to the family $K(s)$ is

(1) $\dot{\gamma}(t) \in J n_{K(E(t))}(\gamma(t))$ a.e.,

which has to be solved for $(E, \gamma) \in C^0([0, T], [0, \pi]) \times \mathrm{Lip}([0, T], B(2))$ including the a priori unknown energy function.

We examine two special solutions:

(i) The parametrization by arc length γ_s of $\partial K(s)$, for s fixed, defines a solution (γ_s, s) with constant energy $E(t) = s$. The inclusion (1) is in fact solved for all but two exceptional points; one observes moreover that the right and left derivatives satisfy it even everywhere.

(ii) The corner curve $c(s)$ is another solution for $t \in\,]0, \pi]$. It satisfies (1) at every time t although it consists of corners only:

$$\frac{d}{dt}c(t) = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix} = J n_t \in J n_{K(E(t))}(c(t)) \qquad \forall t \in\,]0, \pi]$$

with $E(t) = t$. But the solution c **does not conserve the "energy"**!

As a consequence of (i) and (ii), one concludes that the initial value problem (IVP) at $x_0 = c(s)$ possesses at least two solutions, $\gamma_s(t + s_0)$ and $c(t + s)$. In fact, there are infinitely many geometrically different solutions (γ, E) for

(IVP) $$\dot{\gamma}(t) \in Jn_{K(E(t))}(\gamma(t)), \qquad \gamma(0) = x_0,$$

for *every* initial point $x_0 \in \mathcal{A} := \mathrm{int}(B(2) \setminus \{0\})$.

Moreover, the boundary value problem for $x_0, x_1 \in \mathcal{A}$

(BVP) $$\dot{\gamma}(t) \in Jn_{K(E(t))}(\gamma(t)), \quad \gamma(0) = x_0, \quad \gamma(T) = x_1,$$

has infinitely many solutions (γ, E, T) if and only if $f(x_0) \leq f(x_1)$.

One finds that the function $E(t)$ is increasing for any solution (γ, E). This is due to the special property of our example that $Jn \in Jn_K$ points to the exterior of K. Consequently, every solution for the boundary values x_0, x_1 on the same level $\partial K(s)$ *conserves* the energy f: $f(x_0) = f(x_1) \Leftrightarrow E(t) = \mathrm{const} = f(x_0)$. Whence all *periodic* solutions of (1) conserve the energy in this special example.

Now we apply the theorem of Rockafellar: If there were a *convex* function k parametrizing $K(s)$, then all solutions of (HI) would conserve the energy and (HI) would be equivalent to (1) by theorem 1, which is a contradiction. This implies that CE1 is **not convexifiable in** $[0, \pi]$. We remark that this result is even more general than what can be obtained by classical methods of convex analysis, see the proof of Crouzeix [C89] as reported in [K90] that CE1 is not convexifiable in $[0, \frac{\pi}{2}]$.

The study of singular Hamiltonian systems therefore provides a new tool to decide whether a given family of convex sets in even dimensions is convexifiable or not.

6. A convex example (CE2). So far we have constructed a *quasiconvex* system whose solutions of IVP and BVP may not be unique for certain points and at the same time do not always conserve the energy; now we show that even in convex systems, where the energy is always conserved, the IVP and BVP may have an infinity of solutions.

PROPOSITION 1. *All isotropic edges of dimension* ≥ 2 *of the standard cube in* $\mathbb{R}^{2n}$ *have infinitely many solutions of the IVP at all its interior points. The BVP of a pair of points has infinitely many solutions if the second point lies in a characteristic cone of attainable points.*

P r o o f. We study the fixed energy initial value problem

$$\dot{\gamma}(t) \in J\,n_{K(s)}(\gamma(t)) \quad \text{a.e.,}$$
$$\gamma(t) \in \partial K(s), \qquad s \text{ fixed,}$$
$$\gamma(0) = x_0.$$

It is enough to study a 4-dimensional cube, which can be considered as a (restricted symplectic) edge of a higher dimensional one.

Let $K(s) = \{(p_1, q_1, p_2, q_2) \in \mathbb{R}^4 \mid p_i, q_i \in [0, s], s \in \mathbb{R}_+\}$ be the cubes with side lengths s. Its boundary $\partial K(s)$ is composed of edges (strata) of dimension 0 (corners), 1, 2 and 3 (faces). At an interior point of such an edge, the set of normalized normals is 3, 2, 1 and 0-dimensional respectively.
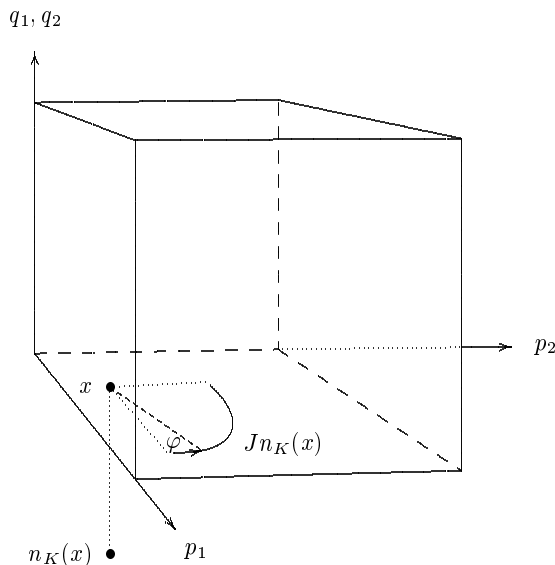
Fig. 2. Non-uniqueness in isotropic edges (CE2)

To give a counterexample against the uniqueness on any energy level, consider for example the 2-dimensional isotropic (Lagrangian) edge

$$A^2 = \{(p_1, 0, p_2, 0) \mid p_i \in [0,2]\}$$

of $K := K(2)$, see fig. 2. The set of normalized normals for an interior point $x \in \operatorname{int} A^2$ is

$$n_K(x) = \{(0, -\cos\varphi, 0, -\sin\varphi) \mid \varphi \in [0, \pi/2]\}.$$

After application of J we get

$$Jn_K(x) = \{(\cos\varphi, 0, \sin\varphi, 0) \mid \varphi \in [0, \pi/2]\}.$$

The set of solutions with initial point x is parametrized by an infinite dimensional function space $\{\phi \in C^0([0,T],[0,\frac{\pi}{2}]) \mid |\dot{\phi}(t)| = 1 \forall t\}$:

$$\gamma_\phi(t) := x + (\cos\phi(t), 0, \sin\phi(t), 0),$$

and the boundary value problem for two points $x, y \in A^2$ possesses still infinitely many solutions of this form if y lies in the "attainable cone" of points y satisfying $x - y \in Jn_K(x) = Jn_K(y)$. ∎

We remark that K is a symplectic product of two-dimensional squares and moreover a (non-smooth) completely integrable system; a similar behaviour is observed in general symplectic products (see [K94c]) even if the factors have smooth boundary.

COROLLARY 1. CE2 *does not define a flow. Nevertheless, if at every point $x \in A^2$ a preferred direction $\in JN_K(x)$ is assigned depending smoothly on x, we get a flow in A^2 which preserves the symplectic form. One observes therefore a non-uniqueness of the Hamiltonian flow.*

P r o o f. This is immediate, because every vector field defines a flow, and the deformation of curves along isotropic edges preserves the symplectic area although the Riemannian area is not conserved. ∎

COROLLARY 2 (Regularity). *The solutions of (CI) and (HI) may not be differentiable from the right everywhere, whereas a convex function defining S is*: *The regularity of the solutions is strictly smaller than that of the defining function.*

P r o o f. A curve with $\gamma(0) = x$ consisting of a sequence of connected vertical and horizontal arcs accumulating at x describes a solution of the form γ_ϕ which is not differentiable from the left nor from the right, as the limit

$$\dot{\gamma}^+(t) := \lim_{h \to 0^+} \frac{\gamma(t+h) - \gamma(t)}{h}$$

does not exist at $t = 0$ because it would be composed of two vectors, a horizontal and a vertical one. The *maximal regularity* one can hope for is strictly less than right differentiability, whereas convex functions are right differentiable. ∎

7. Conclusions. We showed that solutions of C^0-systems in general

(1) don't conserve the energy (CE1);
(2) don't have unique solutions of the initial value problem (CE1,2);
(3) don't have the same regularity as H (CE1,2);
(4) depend on the Hamilton function representing the hypersurface as energy level (CE1).

These are four violations of very basic properties of $C^{1+\varepsilon}$-systems, $\varepsilon > 0$, namely of properties which were used to show the equivalence of (H) and (C). The example for (1) could be made quasiconvex but not convexifiable, because of theorem 2. This reveals that the convex Hamiltonians form a *distinguished* limit case in symplectic topology, at least in C^0-category, although convexity is not symplectically invariant. This also means that (C) and (H) don't have equivalent generalizations to the non-smooth case, unless further assumptions on H are considered, namely H e.g. of the form $(j_K)^\alpha$ and the explicit additional energy constraint, see §3.1. We therefore adopt the point of view that the Hamiltonian description is secondary and that one should consider the *characteristic equation* or *inclusion* as primary and more geometric.

8. Closed characteristics. To give a geometric definition of a symplectic capacity, we are interested in periodic solutions of the characteristic differential inclusion (CI), called *closed characteristics*. More precisely, we consider absolutely continuous *prime periodic* solutions of

$$(*) \qquad \begin{cases} \text{(i)} & \dot{\gamma}(t) \in Jn_K(\gamma(t)) \quad \text{a.e.} \\ \text{(ii)} & \gamma(t) \in \partial K \quad \forall t \in [0, T_\gamma] \\ \text{(iii)} & \gamma(t + T_\gamma) = \gamma(t) \quad \forall t \in [0, T_\gamma] \\ & \text{and } T_\gamma > 0 \text{ is the minimal period of } \gamma \end{cases}$$

namely the set

$$\Gamma(K) := \{\gamma \in \mathrm{Lip}\,([0, T_\gamma], \mathbb{R}^{2n}) \mid (*)\}.$$

Solutions of $(*)$ have the ambiguity of the choice of an initial value, so for every solution we get an S^1-orbit in $\Gamma(K)$ with respect to the natural S^1-action which leaves $(*)$ invariant. We pass to the quotient

$$\Gamma^*(K) := \Gamma(K)/S^1$$

to get the set of geometrically different solutions of $(*)$. $\Gamma^*(K)$ can be regarded as the set of representatives of the moduli space of prime closed characteristic curves.

A symplectic size of loops is given by their symplectic action

$$A(\gamma) = \frac{1}{2} \int\limits_0^{T_\gamma} \dot{\gamma}(t).J\gamma(t)\,dt.$$

We will call loops *small* or *big* according to the absolute value of $A(\gamma)$, and symplectic actions of elements of $\Gamma(K)$ (prime closed characteristics) are called *characteristic actions* of ∂K. Observe that the parameter transformations preserving orientation and mapping degree conserve the value of $A(\gamma)$, therefore $A(\gamma)$ passes to the quotient by such reparametrizations.

A is not invariant with respect to the action of $\mathbb{N}$ given by the iteration $\gamma \mapsto \gamma^{(k)}$, as $A(\gamma^{(k)}) = kA(\gamma)$. For convenience we define

$$\Gamma^{\mathbb{N}_+}(K) = \{\gamma^{(k)} \mid \gamma \in \Gamma(K), k \in \mathbb{N}_+\}$$

as the set of iterated characteristic loops, which by definition don't solve $(*)$. Their minimal period is $\ell(\gamma^{(k)})/k$ instead of $\ell(\gamma^{(k)})$.

9. Symplectic capacities

DEFINITION. Let $\mathcal{K} = \{K \subset \mathbb{R}^{2n} \mid 0 \in \mathring{K}, \text{convex}\}$ be the set of convex bodies, where $\mathring{K}$ denotes the interior of K. Define c to be the map

$$c : \mathcal{K} \to \mathbb{R}_+, \qquad K \mapsto c(K) = \inf\{A(\gamma) \mid \gamma \in \Gamma(K)\},$$

assigning to K the *minimal characteristic action* of ∂K. We will call c *symplectic capacity* * as soon as $c(K)$ is shown to be well defined, attained by a $\gamma^* \in \Gamma(K)$ for bounded K, positive and satisfying the defining properties (a), (b) and (c). By convention, $c(K) = \infty$ if $\Gamma(K) = \emptyset$, which does not occur if K is bounded.

To show that $c(K)$ is positive, we used the gauge function $j_K(x)$ (§3) and the well known dual Hamiltonian action functional of $H = (j_K)^\alpha$, with $\alpha > 0$, see [CE80] and [E90]. We proved that $c(K)$ is a monotone function of the minimum of this functional [K90]. The well known fact that the dual functional is bounded from below as opposed to the direct one, is a remarkable advantage of the dual formulation: a minimum can be calculated by a computer whereas saddle points can't in general.

We call this the *geometric definition* as it defines c independently of Hamiltonian functions by a purely geometric term, the *symplectic area* of a surface $\Sigma \subset K$ contoured

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*It is the one we denoted in the introduction by c_0.

by the characteristic loop γ^*:

$$A(\gamma^*) = \frac{1}{2} \int_{\operatorname{Im}\gamma^*} \gamma^*.(-J)d\gamma^* = \int_\Sigma \omega.$$

It is independent of the choice of Σ with $\partial\Sigma = \operatorname{Im}\gamma^*$, because K is contractible. For arbitrary closed curves γ, $A(\gamma)$ is a relative integral invariant in terms of Poincaré [P99, p. 5]. Moreover, for closed *characteristics* $\gamma \in \Gamma^*(K)$, $A(\gamma)$ is a fundamental integral for both canonical and geometric quantization and relates classical to quantum mechanics.

Now we collect some properties of c.

THEOREM 3 [K90]. *The least characteristic action c is a symplectic capacity for convex sets $\mathcal{K}$ and embeddings $\mathcal{E}_\omega$*:

(a) $K \subset K' \Rightarrow c(K) \le c(K')$,
(b) $\phi \in \mathcal{E}_\omega \Rightarrow c(\phi(K)) = c(K)$,
(c) $c(B(r)) = \pi r^2 = (Z(r))$,

for all $K, K' \in \mathcal{K}$. Moreover, *the following properties are satisfied*:

(d) *c is Hausdorff continuous on $\mathcal{K}$,*
(e) *$c(K \times K') = \min\{c(K), c(K')\}$ for symplectic products $K \times K'$,*
(f) *After continuous extension, $c(K) = 0$ for bounded convex sets $K \subset \mathbb{R}^{2n-1} \subset \mathbb{R}^{2n}$.*

As mentioned in the introduction, Sikorav [Si90] proved that the capacity by Ekeland and Hofer, which was defined to be the infimum of a certain critical value over a set of Hamiltonian functions, equals c on smooth convex bodies and is acheived by a non-iterated solution. The characterization by a closed characteristic on non-smooth convex sets was of course not possible there, because the notion of a characteristic system as developed in the present article was not made precise yet.

10. The symplectic product. In [K90], we gave a representation of all characteristic loops on $\partial(K_1 \times K_2)$ by characteristic loops on the factors ∂K_i and conversely. This suggests a product formula for *all* capacities which are given by actions of characteristic loops. Here we will only indicate what is needed to show formula (e) for the *least* characteristic action c.

Consider two linear symplectic spaces $(\mathbb{R}^{2n_i}, \omega_i, g_i)$ with ω_i and g_i as in the introduction. In the canonical metric $g = g_1 \oplus g_2$, denoted by $x.y$ again, the splitting of $E = \mathbb{R}^{2n} = \mathbb{R}^{2n_1} \oplus \mathbb{R}^{2n_2} = E_1 \oplus E_2$ is orthogonal. Consider the orthogonal projections on these factors, expressed in coordinates by $P_1 x = \binom{x_1}{0}$ which we identify with $P_1 x = x_1$ for $x = \binom{x_1}{x_2}$.

The natural symplectic form is $\omega = \omega_1 \oplus \omega_2$, given by $J = J_1 \oplus J_2$.

Now look at the *symplectic product* $K := K_1 \times K_2 \subset \mathbb{R}^{2n_1} \times \mathbb{R}^{2n_2}$. We suppose $K_i \in \mathcal{K}(\mathbb{R}^{2n_i})$ to simplify the proof and to fit the product formula in our convex framework:

$$K \in \mathcal{K}(\mathbb{R}^{2n}) \Leftrightarrow K_i \in \mathcal{K}(\mathbb{R}^{2n_i}) \text{ for } i = 1 \text{ and } 2.$$

It is easy to show that $N_K(x_1, x_2) = N_{K_1}(x_1) \oplus N_{K_2}(x_2)$ $\forall x = (x_1, x_2) \in K_1 \times K_2$ and that the operators J, P_i and d/dt commute.

THEOREM 4. *For all* $\gamma \in \Gamma(K_1 \times K_2)$, $P_i\gamma$ *belongs to* $\Gamma^{\mathbb{N}_+}(K_i) \cup K_i$ *up to reparametrization of degree* 1, *where* K_i *denotes the set of constant functions with values in* K_i.

R e m a r k. The lack of C^1-regularity of the solutions of $(*)$ was handled by a careful control of the differential inclusion almost everywhere, more precisely, we considered a representative of the Lebesgue class of the derivative and controlled it pointwise on a set of full measure.

As an alternative approach, we smoothened $K_1 \times K_2$ by Yoshida's approximation and followed the perturbed C^1-solutions as the perturbation parameter changes. Just as in the situation of Viterbo's proof for contact manifolds [V87], these perturbed solutions approach the hypersurface from the outside. But the notion of the characteristic differential inclusion still has to be made precise and examined for the limit, i.e. with the smoothening approach one does not gain anything.

R e m a r k. A converse statement to theorem 4 can also be proved (see [K90]): Given $\Gamma(K_1)$ and $\Gamma(K_2)$, one recovers $\Gamma(K_1 \times K_2)$ by an adapted family of parameter transformations. Again, the interesting part is $\partial K_1 \times \partial K_2$, where a non-resonance condition has to be checked. This condition is similar to the usual non-resonance condition on Lagrangian tori; just remark that in fact $\partial K_1 \times \ldots \times \partial K_n$ is a Lagrangian torus in $\mathbb{R}^{2n}$ if K_i are 2-dimensional simply connected subsets of $\mathbb{R}^2$.

R e m a r k. The symplectic product is physically non-trivial as it represents a composed system with *non-zero* interaction. In terms of gauge functions, consider

$$h_i(x_i) = (j_{K_i}(x_i))^2,$$
$$H(x_1, x_2) = (j_{K_1 \times K_2}(x_1, x_2))^2 = \max\{h_i(x_i) \mid i = 1, 2\}$$
$$= h_1(x_1) + h_2(x_2) + V_{1,2}(x_1, x_2).$$

Use

$$j_{K_1 \times K_2}(x_1, x_2) = \inf\{\lambda \in \mathbb{R}_+ \mid x/\lambda \in K\} = \sup\{\lambda \mid \lambda(x_1, x_2) \in K_1 \times K_2\}$$
$$\leq \sup\{\lambda \in \mathbb{R}_+ \mid \lambda x_i \in K_i\} = j_{K_i}(x_i).$$

to show that the interaction term is $V_{1,2}(x_1, x_2) = -\min\{h_i(x_i) \mid i = 1, 2\}$ and therefore non-zero if $x \neq 0$.

R e m a r k. The symplectic product is also non-trivial for the theory of symplectic capacities in the sense that in general there is no estimate $B(r) \subset D \subset Z(r)$ for $D = K_1 \times K_2$. For all sets D with such an estimate, capacity is calculated trivially from the axioms:

$$c(B(r)) = \pi r^2 \leq c(D) \leq c(Z(r)) = \pi r^2,$$

yielding $c(D) = \pi r^2$. All examples (ellipsoid, ball and cylinder) whose capacity could be calculated before [K90] were of this trivial kind.

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