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ON THE GLOBAL SOLVABILITY OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS IN THE SPACE OF REAL ANALYTIC FUNCTIONS

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Abstract. This article surveys results on the global surjectivity of linear partial differential operators with constant coefficients on the space of real analytic functions. Some new results are also included.

0. Introduction. In this talk, we first briefly review in §1 the old subject of general surjectivity of a linear partial differential operator $P(D)$ with constant coefficients in various spaces of (generalized) functions on a given open set. Then in §2 we introduce results concerning the case of the space of real analytic functions in precise formulation. Finally, in §3 we discuss recent developments related to micro-local analysis. This is, so to speak, an application of singularity theory. We thus believe that this talk fits the general subject of this Semester.

The author participated about 15 years ago in a meeting organized by L. Cattabriga at Trento in Italy and encountered many people by whose influences he undertook this research. The author would therefore like to record here his special memory of the late Prof. L. Cattabriga.

1. General surjectivity. When a fundamental solution $E(x)$ was constructed for a general operator $P(D)$ with constant coefficients by Hörmander, the natural problem to be attacked next was that of solvability of the equation $P(D)u = f$ for the given right-hand side f. But this contained another difficulty when f was a general function which may grow up near the boundary of the domain, because the simple formula

$$u = E * f \tag{1}$$

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will not work for such f. A duality argument was introduced by Ehrenpreis and Malgrange to overcome this. It consisted in proving the following equivalence:

$$P(D) : F(\Omega) \to F(\Omega) \text{ is surjective}$$
$$\Leftrightarrow {}^tP(D) : F'(\Omega) \to F'(\Omega) \text{ is injective and has closed range.}$$

Using this, the surjectivity was established for the space $C^\infty(\Omega)$ of infinitely differentiable functions on a convex open set, and further, a characterization called P-convexity was given for the surjectivity in general domains. This method could be applied without modification to the space of distributions of finite order on the whole domain $\mathcal{D}'_{\mathcal{F}}(\Omega)$, or to the space $\gamma^s(\Omega)$ of Gevrey functions with projective limit type topology. For the space $\mathcal{D}'(\Omega)$ of distributions or the space $\Gamma^s(\Omega)$ of inductive limit type Gevrey spaces or the space $\mathcal{A}(\Omega)$ of real analytic functions, the duality argument did not work because of the failure of the closed range theorem in these spaces. But Hörmander introduced a new idea of using approximation by a topology of an Abelian group finer than the original one as a topological linear space, and established surjectivity for the space $\mathcal{D}'(\Omega)$ on a convex domain, together with a criterion of surjectivity for general domains. This idea was later made into an abstract theory of topological inductive limit functors by Palamodov. Ehrenpreis used a different tool of analytically uniform structure, which is in short to find an equivalent expression by a family of (possibly uncountably many) seminorms for the given topology of the dual space which is originally of projective limit type, and he thereby established the surjectivity for the space $\mathcal{D}'(\mathbb{R}^n)$.

The researchers intended to treat the space of real analytic functions. But methods mentioned above which worked for the space of distributions did not apply. The first positive result in this respect was given only in 1971 by De Giorgi and Cattabriga who proved the surjectivity for any operator on $\mathbb{R}^2$. But what was more frightening was that soon after that Cattabriga and Piccinini found a counterexample to the surjectivity. It was the familiar heat equation on the whole space. Their argument suggested that some kind of further convexity condition is necessary. As a matter of fact, a real analytic convex open set Ω does not have a fundamental system of convex complex neighborhoods, and all difficulty comes from this fact. Examining their argument, Hörmander soon introduced a Mittag-Leffler type argument to establish the necessary and sufficient condition for the surjectivity in $\mathcal{A}(\Omega)$ for convex Ω. It was a kind of Phragmén-Lindelöf type principle posed on functions holomorphic on the null variety $N(P)$ of $P(\zeta)$. His result was very abstract though he gave various concrete examples to which his criterion applies. His most important discovery was that for convex Ω the surjectivity depends only on the principal part.

Just in the same time another approach was produced, namely to establish surjectivity for a limited class of operators. Andersson did it for the locally hyperbolic operators on $\mathbb{R}^n$, whereas Kawai treated it on a general bounded open set which is not necessarily convex. Both methods depended on a kind of micro-localization, but whereas Andersson made use of approximation procedure to solve each decomposed part, Kawai only used the convolution with the well examined fundamental solution, hence presented a totally new approach from the micro-local viewpoint. Since the news of counterexample reached

Japan in delay, some of us were believing at that time that Kawai's new method would in principle solve the problem of surjectivity for any operator. But it became clear that he had picked up favorable operators by chance.

The general problem of surjectivity was then studied in the space $\Gamma^s(\Omega)$. Cattabriga and De Giorgi found that the same counterexample of Cattabriga-Piccinini serves as a counterexample for the surjectivity in $\Gamma^s(\Omega)$. This seemed to show that the difficulty to the surjectivity was purely of topological nature. As a sufficient condition Cattabriga gave hybrid type operators, where the hyperbolic-like part and the elliptic-like part of the null variety are separated by a gap. Later, his results were generalized to a Phragmén-Lindelöf type principle similar to Hörmander's by Zampieri for the sufficient condition, and by Braun-Vogt-Meise to give a necessary and sufficient condition. Indeed, the latter is an application of their more general theory of inductive limit functors extending that of Palamodov. It was hoped that the result of Hörmander itself followed as an application of their general theory. But there seems to exist a small gap. It will be more interesting if there is a structural difference between these two spaces. Braun recently gave an interesting example of a 4-th order operator which is surjective in $\Gamma^s(\mathbb{R}^n)$ for $1 \leq s < 2$ or $s \geq 6$ but not for $2 \leq s < 6$. It is interesting to compare this with Zampieri's former result that surjectivity in $s = 1$ implies surjectivity in $1 \leq s < \sigma$ for some $\sigma > 1$.

There still remain some unsolved problems. It is interesting to clarify the relation between Hörmander's theory and Kawai's micro-local approach. Zampieri showed that for some special case Hörmander's condition can be paraphrased by means of propagation of analytic singularities as used by Kawai but in a form more refined to micro-local irreducible components. Also he introduced a very interesting example showing that Kawai's formulation is not so precise when the local propagation cone has a large dimension. Stimulated by this, the speaker introduced a kind of stratification and improved Kawai's theory to cover Zampieri's example. The biggest problem remaining is to find a micro-local means to show the necessity part at the same level of precision. Note that Hörmander's theory never applies to non-convex domains. Hence it is not known if the surjectivity only depends on the lower order part for the general non-convex domain. It is also very interesting to interpret from the micro-local viewpoint operators which are not locally hyperbolic but which are covered by Hörmander's criterion. Zampieri gave some result on this, too, but the study for main examples is still open.

2. Results on surjectivity in real analytic functions. We list up here results on the surjectivity of $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ which were not precisely formulated in the Introduction. Hörmander's result in its most primitive form is as follows:

THEOREM (Hörmander). *Let $\Omega \subset \mathbb{R}^n$ be a convex open set. $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is surjective if and only if for any compact subset $K \subset \Omega$ we can find another compact subset $K' \subset \Omega$ and $\delta > 0$ such that for every holomorphic function $F(\zeta)$ on $N(P)$ the following* (a), (b) *imply* (c):

(a) $|F(\zeta)| \leq C \exp\{\delta|\zeta| + H_K(\mathrm{Im}\zeta)\}$ *for* $\zeta \in N(P)$.
(b) $|F(\xi)| \leq C$ *for* $\xi \in N(P) \cap \mathbb{R}^n$.
(c) $|F(\zeta)| \leq C_\varepsilon \exp\{\varepsilon|\zeta| + H_{K'}(\mathrm{Im}\,\zeta)\}$ *for* $\zeta \in N(P)$.

This can be paraphrased via plurisubharmonic functions instead of holomorphic functions, which allows one to replace $N(P)$ by $N(P_m)$, the null variety of the principal part. That is, the surjectivity depends only on the principal symbol provided convex Ω are considered. The final form of the theorem localized to the plurisubharmonic functions on the variety $N(P)$ was given by Andreotti-Nacinovich [AN] and Zampieri [Z2]. Andreotti-Nacinovich [AN] and Miwa [Mi] independently gave the generalization to the case of general systems of equations with constant coefficients.

To state Kawai's result, we need to prepare some technical terms.

Definition. $P(D)$ is called *locally hyperbolic at* ξ_0 *in the direction* ν if there exists $\varepsilon > 0$ such that

$$|\xi - \xi_0| \le \varepsilon, 0 < t < \varepsilon \quad \text{imply} \quad P_m(\xi + t\nu) \neq 0. \tag{2}$$

$P(D)$ is said to be *locally hyperbolic* if it is locally hyperbolic at every point $\xi_0 \in S^{n-1}$ in some direction ν depending on ξ_0.

When $P(D)$ is locally hyperbolic, there exists a conically convex neighborhood Δ_{ξ_0} of ν in S^{n-1} such that (2) holds with ν replaced by each $\eta \in \Delta_{\xi_0}$ if ε is chosen anew. The dual cone $K_{\xi_0} = \Delta_{\xi_0}^\circ$ of Δ_{ξ_0} is called a *local propagation cone* of $P(D)$ at ξ_0. In this case the localization $(P_m)_{\xi_0}$ of P_m at ξ_0, namely the first non-trivial coefficient of the Taylor expansion of $P_m(\xi)$ at ξ_0:

$$P_m(\xi_0 + t\xi) = (P_m)_{\xi_0}(\xi)t^a + o(t^a)$$

becomes a hyperbolic operator with the propagation cone K_{ξ_0}. In general, there are several choices for such K_{ξ_0}. In any case, the opposite cone $-K_{\xi_0}$ also serves as such. But it should be noted that not all the propagation cones of the localization $(P_m)_{\xi_0}$ are necessarily local propagation cones of the original operator at ξ_0. By definition we can always arrange them at least locally in such a way that Δ_{ξ_0} depends lower semi-continuously (hence K_{ξ_0} upper semi-continuously) on ξ_0. Assuming that we can arrange it even globally on S^{n-1} for a given locally hyperbolic operator, Kawai constructed a pair of fundamental solutions $E^{\pm}(x)$ which satisfies the following good estimate for its micro-local singularity for the given choice of upper semi-continuous local propagation cones:

$$WF_A E^{\pm}(x) \subset \bigcup_{\xi \in S^{n-1}} \pm K_\xi \times \{\xi\}. \tag{3}$$

Employing these "good fundamental solutions" he proved the following

Theorem (Kawai). *Let $P(D)$ be a locally hyperbolic operator satisfying the above condition, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that there exists a covering of $\partial\Omega \times S^{n-1}$ by closed subsets $X^{\pm}$ such that*

$$(x, \xi) \in X^{\pm} \quad \textit{implies} \quad (\{x\} \pm K_\xi) \cap \Omega = \emptyset. \tag{4}$$

Then $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is surjective.

For a given locally hyperbolic operator, it is not obvious if there exists a global arrangement of local propagation cones in upper semi-continuous way (although we do not have any counter-example). But actually Kawai's argument implies that we only need micro-local good fundamental solutions, i.e. $E_j(x), j = 1, \dots, N$, on each element of an open covering $\{\Delta_j,\ j = 1, \dots, N\}$ of S^{n-1} such that

$$P(D)E_j - \delta \quad \text{is micro-analytic in} \quad \mathbb{R}^n \times \Delta_j$$

and that

$$(3)' \qquad WF_A E_j \subset \Big(\bigcup_{\xi \in \Delta_j} \pm K_\xi \times \{\xi\} \Big) \cup (\mathbb{R}^n \times (S^{n-1} \setminus \Delta_j)).$$

Then assuming the existence of a covering of $\partial\Omega \times S^{n-1}$ by closed subsets $X_j^{\pm}$, $j = 1, \dots, N$, such that $X_j^{\pm} \subset R^n \times \Delta_j$ and that

$$(4)' \qquad (x, \xi) \in X_j^{\pm} \quad \text{implies} \quad (\{x\} \pm K_\xi) \cap \Omega = \emptyset,$$

we can prove the same conclusion. This way of presentation is rather awkward. Kawai actually tried to give his final result in neater form, namely assuming only the condition

(5) at every point $x \in \partial\Omega$ and $\xi \in S^{n-1}$, $(\{x\} + K_\xi) \cap \Omega = \emptyset$ for some choice of the local propagation cone K_ξ.

But he found that then the possibility of locally upper semi-continuous choice of the local propagation cones satisfying $(4)'$ is not obvious. To prove this from the above pointwise condition (5), he had to add an assumption that the boundary $\partial\Omega$ is of class C^1. It was remarked however by Zampieri that a domain with C^1 boundary can satisfy the above pointwise condition if and only if all the local propagation cones are half lines. Hence the final formulation of Kawai excludes many interesting examples. An attempt was made to prove this procedure of pointwise to locally uniform condition. But finally Zampieri found an example of a locally hyperbolic operator which satisfies pointwise condition (5) but we can never find an arrangement of local propagation cones in locally upper semi-continuous way satisfying the assumption $(4)'$ of the above theorem. Our latest study began with this. It will be explained in detail in the next section.

Not so much is known about the necessity on general non-convex domains. The following is the most general result for the moment:

Theorem (Zampieri [Z1]). *The surjectivity of* $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ *implies the surjectivity of* $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$.

Together with Kawai's sufficiency result stated above and Malgrange's result on P-convexity, this implies the following beautiful result:

Corollary. *Let* $\Omega \subset \mathbb{R}^2$ *be a domain. Then* $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ *is surjective if and only if every characteristic line of* $P(D)$ *intersects* Ω *in a connected interval.*

To paraphrase Hörmander's condition in terms of the propagation of analytic singularities is an interesting work. Zampieri gave the following result:

Theorem (Zampieri [Z3]). *Let* $P(D)$ *be an operator such that the irreducible factor of* $P_m(\xi)$ *in the sense of analytic germs at every point* $\xi \in S^{n-1}$ *is locally hyperbolic. Assume that the multiplicity of every characteristic of* P *is at most* 2. *Then for a convex open set* Ω *the following are equivalent.*

a) $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ *is surjective.*

b) *At every point* (x, ξ) *of* $\partial\Omega \times S^{n-1}$, *for each irreducible factor of* $P_m(\xi)$ *at* ξ *in the above sense we can find a local propagation cone* K *such that* $(\{x\} + K) \cap \Omega = \emptyset$.

This theorem is proved by a direct paraphrase of Hörmander's condition in a localized form. Hence it does not work for non-convex domains. Since the proof depends heavily on the restriction of multiplicity, we do not know if the assumption of multiplicity is a technical one or not. It is a fascinating problem to find an idea to prove such a form of necessity condition directly by micro-local approach.

3. Improved micro-local theory. To explain further developments, it is better to review a little the proof of Kawai's theorem. For the sake of simplicity, in the sequel we restrict ourselves as in [Kaw1] to the case where global good fundamental solutions $E^{\pm}(x)$ are available. His argument resorts to the simple formula (1): Let $f \in \mathcal{A}(\Omega)$ be a given right-hand side. By the flabbiness of the sheaf of hyperfunctions, we can take an extension $\tilde{f}$ of f as a hyperfunction with support in $\overline{\Omega}$. Then by the flabbiness of the sheaf of micro-functions, we can decompose $\tilde{f}$ according to the covering in such a way that

$$\tilde{f} = f^{+} + f^{-}, \qquad WF_A f^{\pm} \subset X^{\pm}.$$

Then we set

$$u = E^{+} * f^{+} + E^{-} * f^{-}.$$

In this convolution neither of the factors has compact support even if Ω is bounded. But the analytic singular support is bounded and for this reason the convolution can be well-defined modulo real analytic functions. By this definition, it is clear from the estimate (3) and the assumption (4) that we can find a hyperfunction representative u which is real analytic inside Ω. Hence adjusting finally the real analytic modulo term $g = P(D)u - f \in \mathcal{A}(\mathbb{R}^n)$ by solving the equation $P(D)v = g$ on a sufficiently large convex compact set containing $\overline{\Omega}$, we obtain another hyperfunction u which satisfies $P(D)u = f$ exactly in Ω.

We made two contributions to the sufficiency part of the surjectivity in the space of real analytic functions. The first one is an extension of Kawai's theory to unbounded domains. For this, we introduced in [Kan1] the sheaf of Fourier hyperfunctions with generalized growth condition at infinity defined on the directional compactification D^n of $\mathbb{R}^n$ which extends the usual sheaf of hyperfunctions, and introduced the theory of analytic wavefront sets adapted to them, especially at infinity. We established in [Kan5] the flabbiness of the sheaf of Fourier micro-functions thus defined on $D^n \times S^{n-1}$. Note that the original Fourier hyperfunctions are introduced by Sato-Kawai with the infra-exponential growth condition (see [Kaw1]). However, the good fundamental solutions constructed naturally do not satisfy this condition. So a generalization of the growth condition was indispensable. The convolution between a generalized Fourier hyperfunction of increasing type and another of decreasing type is possible if the decay order of the latter dominates the growth order of the former, and the estimation formula of the analytic wavefront set works as well for such a convolution, even at infinity. Since we can extend a given f to a Fourier hyperfunction with arbitrarily preassigned decay order, we can thus extend Kawai's theory almost literally to the case of unbounded domains. The speaker got these ideas from a related question of Bratti and from Lieutenant's talk (see [L1]) at the meeting mentioned in the Introduction.

The second contribution is a refinement of Kawai's theory in such a form that it can treat Zampieri's example. We present here his interesting example: Consider

$$P(D) = D_1^2 D_2^2 - D_2^2 D_3^2 - D_3^4 - D_3^2 D_4^2.$$

This is simply characteristic except for the points

$$(\pm 1, 0, 0, 0), \qquad (0, \pm 1, 0, 0).$$

The local propagation cones at the former pair are half-lines. Hence they cause no geometrical problem although they are doubly characteristic. On the contrary, those at the latter pair are two dimensional cones

$$\pm K_\xi = \{\pm x_1 \geq |x_3|,\ x_2 = x_4 = 0\}.$$

An elementary calculation shows the remarkable fact that

$$\pm K'_\xi = \{\pm x_3 \geq |x_1|,\ x_2 = x_4 = 0\}$$

are not local propagation cones, although they are propagation cones of the localization $\zeta_1^2 - \zeta_3^2$. By this reason, we can show that for a domain

$$\Omega = \{|x_1 + x_3| < 1, |x_1 - x_3| < 1, |x_2| < 1, |x_4| < 1\},$$

the arrangement of local propagation cones fails at the face $x_1 + x_3 = 1$. The difficulty comes from the fact that the lower semi-continuity of local propagation cones fails when the order of localization changes. But the lower semi-continuity holds on any stratum on which the order of localization is constant. With this observation, for a given operator $P(D)$ we introduce the following stratification:

$$S^{n-1} = \Xi_0 \sqcup \Xi_1 \sqcup \ldots \sqcup \Xi_s,$$

where

$$\Xi_j = \{\xi \in S^{n-1}; \text{the localization } (P_m)_\xi \text{ is of order } m_j\}$$

with

$$0 = m_0 < m_1 < \ldots < m_s.$$

When ξ is restricted to one stratum, we have the lower semi-continuity of the local propagation cones $\xi \mapsto K_\xi$ in the following sense: *For any $a \in K_{\xi_0}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\xi - \xi_0| < \delta$, then K_ξ contains some point of the ε-neighborhood of a.* Thanks to this fact, we can show that

$$X_k^{\pm} = \{(a, \xi) \in \partial\Omega \times \Xi_k; \{a\} \pm K_\xi \cap \Omega = \emptyset\} \tag{6}$$

becomes relatively closed in $\partial\Omega \times \Xi_k$. Thus we can decompose $\tilde{f}$ by WF_A according to this stratification and the covering. We can solve the equation $P(D)u = \tilde{f}$ modulo micro-analytic terms step by step from the strata of higher dimension employing the same simple formula of convolution with good fundamental solutions. The outline of this argument is as follows: Assume that we have obtained a micro-local solution $u_k \in \mathcal{B}(\mathbb{R}^n)$ of $P(D)u_k = \tilde{f}$ on $\mathbb{R}^n \times (S^{n-1} \setminus \overline{\Xi_{k+1}})$ in the following sense:

1) $WF_A u_k \subset (\mathbb{R}^n \setminus \Omega) \times S^{n-1}$.
2) $WF_A(P(D)u_k - \tilde{f}) \subset \partial\Omega \times \overline{\Xi_{k+1}}$.

We construct u_{k+1} from this u_k employing the good fundamental solutions as follows. (The argument also shows how to construct the first one u_0.) Put

$$g_{k+1} = P(D)u_k - \tilde{f}.$$

By 2) we have $WF_A g_{k+1} \subset \partial\Omega \times \overline{\Xi_{k+1}}$. According to (6) for $k+1$, we decompose g_{k+1} into the sum of $g_{k+1}^{\pm}$ such that

$$WF_A g_{k+1}^{\pm} \subset X_{k+1}^{\pm} \cup (\partial\Omega \times \overline{\Xi_{k+2}}).$$

Then put

$$v_{k+1} = g_{k+1}^{+} * E^{+} + g_{k+1}^{-} * E^{-}.$$

(The meaning of convolution is as explained before. Namely, it is well defined modulo $\mathcal{A}(\mathbb{R}^n)$.) By the geometric condition the analytic wavefront set with the directional components in Ξ_{k+1} does not propagate into Ω. Hence we have

$$WF_A v_{k+1} \subset \{(\mathbb{R}^n \setminus \Omega) \times \overline{\Xi_{k+1}}\} \cup (\Omega \times \overline{\Xi_{k+2}}).$$

Now choose $w_{k+1} \in \mathcal{B}(\mathbb{R}^n)$ such that

$$\begin{aligned} WF_A(v_{k+1} - w_{k+1}) &\subset \overline{\Omega} \times \overline{\Xi_{k+2}}, \\ WF_A w_{k+1} &\subset (\mathbb{R}^n \setminus \Omega) \times \overline{\Xi_{k+1}}. \end{aligned}$$

From the construction we have

$$\begin{aligned} P(D)w_{k+1} - g_{k+1} &= (P(D)v_{k+1} - g_{k+1}) + P(D)(w_{k+1} - v_{k+1}) \\ &\equiv P(D)(w_{k+1} - v_{k+1}) \bmod \mathcal{A}(\mathbb{R}^n), \end{aligned}$$

hence

$$\begin{aligned} WF_A(P(D)w_{k+1} - g_{k+1}) &\subset \{(\mathbb{R}^n \setminus \Omega) \times \overline{\Xi_{k+1}}\} \cap (\overline{\Omega} \times \overline{\Xi_{k+2}}) \\ &\subset \partial\Omega \times \overline{\Xi_{k+2}}. \end{aligned}$$

Hence $u_{k+1} = u_k - w_{k+1}$ satisfies the induction hypothesis 1)–2) with k replaced by $k+1$. Thus after a finite number of steps we finally obtain the desired micro-local solution $u = u_s$.

We present here some remarks concerning the necessity for the surjectivity. The following results imply that the surjectivity is localizable in some sense:

Proposition. *Assume that for $f \in \mathcal{A}(\Omega)$ we can find a neighborhood U of $\partial\Omega$ and $u \in \mathcal{A}(\Omega \cap \mathcal{U})$ such that $P(D)u = f$ in $\Omega \cap U$. Then $P(D)u = f$ is actually solvable in $u \in \mathcal{A}(\Omega)$.*

Proposition. *Assume that $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is surjective. Then for every point $x \in \partial\Omega$ and for any neighborhood V of x, we can find another neighborhood W of x such that for every $f \in \Omega \cap V$ the equation $P(D)u = f$ has a solution u in $\Omega \cap W$.*

These are proved employing Malgrange's vanishing theorem for the first cohomology group with coefficients in $\mathcal{A}$. We show this for the latter: In view of this vanishing theorem and the Mayer-Vietoris theorem, we can decompose $f \in \mathcal{A}(\Omega \cap V)$ as

$$f = g + h, \qquad g \in \mathcal{A}(\Omega),\ h \in \mathcal{A}(V).$$

By the assumption, $P(D)v = g$ has a solution $v \in \mathcal{A}(\Omega)$. On the other hand, if we choose a convex neighborhood $W \subset\subset V$, we can find a solution $w \in \mathcal{A}(W)$ of $P(D)w = h$. Then $u = v + w$ restricted to $\Omega \cap W$ will be the desired solution.

The converse of the latter Proposition is not true. In fact, in view of Zampieri's result introduced in §2 (see Corollary) and the famous remark about the non-localizability of P-convexity (see Example with the unique Figure 1 in §3.7 of the book [H2]), we see that the surjectivity $P(D) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is also non-localizable. But we expect that for a real analytic boundary, the condition of surjectivity will be localizable in the above form along the boundary.

In the above proofs, we repeatedly employed the classical result on the surjectivity of $P(D)$ on the space $\mathcal{A}(K)$ of real analytic functions on a convex compact set. Note that this surjectivity is not trivial unlike the case of $C^\infty(K)$, because of the impossibility of the choice of a compact support modification. It is also interesting to study the condition of surjectivity for non-convex K. Some results in this respect are given in [Kaw1] for sufficiency and in [Kan3] for necessity.

Our final remark is the role of WF_A at the boundary for a real analytic function. In [L2], Lieutenant introduced this notion with the aim of the micro-local study of the surjectivity in $\mathcal{A}(\Omega)$ for general operators. Roughly speaking, for $f \in \mathcal{A}(\Omega)$ we say that $(x, \xi) \notin WF^b_A f$ at $x \in \partial\Omega$ if f can be continued to a complex neighborhood of Ω which has positive angle with the real axis in the direction ξ. (A similar notion was introduced by Schapira [Sch1-2] with another motivation.) Regrettably, Lieutenant's idea was not well pursued. We therefore give here a prototype of such a result.

Theorem. *Let Ω be a convex open set. Let $f \in \mathcal{A}(\Omega)$. Assume that every point $(x, \xi) \in WF^b_A f$ has a neighborhood $U \times \Delta \subset \mathbb{R}^n \times S^{n-1}$ such that $P(D)$ is locally hyperbolic in Δ and that the geometric condition for its local propagation cones is satisfied on $(\partial\Omega \cap U) \times \Delta$. Then $P(D)u = f$ has a solution in $\mathcal{A}(\Omega)$.*

In fact, by means of the micro-local good fundamental solutions we solve $P(D)u = f$ micro-locally on a neighborhood of $WF^b_A f$. Then what remains on the right-hand side is a real analytic function in Ω which can be continued to a convex complex neighborhood of Ω. Thus it can be solved by the classical duality argument in the convex complex domain.

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