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TRANSVERSALITIES FOR LAGRANGE SINGULARITIES OF ISOTROPIC MAPPINGS OF CORANK ONE

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0. Introduction. The Lagrange singularity theory connects the Lagrange classification of Lagrange immersions with the classification of families of functions, that is, generating families. (The theory of Hörmander, Arnol'd, Zakalyukin, ..., [AGV]). Though there exist detailed studies on singular Lagrange varieties (e.g. [A1], [A2], [DP], [G1], [G2], [J], [Z]), the nice connection between Lagrange singularities and generating families seems to break, in particular, when we try to classify generic Lagrange non-immersions or generic isotropic mappings. Our attempt is then to study singularities of isotropic mappings in the framework of singularity theory of differential mappings (Thom-Mather theory: e.g. [T], [M2]), and to classify their generic singularities under the Lagrange equivalence.

In the singularity theory, the transversality theorem is a powerful tool to grasp generic conditions for differentiable mappings, with respect to partial derivatives (see, for instance, [M1], [M2]). In general, if we fix a space of mappings, a generic condition on a mapping should be described by the transversality of the jet section of the mapping to a stratification naturally defined in a jet space of sufficiently higher order. Then, "the transversality theorem" claims the density of the subspace of mappings satisfying the transversality condition, relatively to an appropriate topology on the space of mappings. As a rule, the validity of "the theorem", however, depends on the space of mappings.

Then, the purpose of this paper is to formulate explicitly and to prove the transversality theorem for the space of isotropic mappings of corank at most one, endowed with

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the Whitney C^∞ topology. The corank condition is necessary at least to assure the corresponding jet spaces are non-singular.

In what follows, all manifolds, mappings and differential forms are assumed to be of class C^∞.

Isotropic mappings arise naturally in symplectic geometry: Let X be a manifold of dimension n, and M a symplectic manifold of dimension $2n$ with the symplectic form ω. We call a mapping $\phi : X \to M$ *isotropic* if the pull-back $\phi^*\omega$ is equal to zero. An isotropic immersion is called a Lagrange immersion (cf. [W]), and therefore an isotropic mapping is regarded as a Lagrange immersion with singularities: A point $x \in X$ is *a singular point* of ϕ if the the rank of tangential mapping $(\phi_x)_* : T_xX \to T_{\phi(x)}M$ is less than n. We call the dimension of the kernel of $(\phi_x)_*$, the corank of the germ ϕ_x. By the corank of ϕ, we mean the supremum of the corank of ϕ_x, $x \in X$.

Let $J_I^k(X, M)$ stand for the set of k-jets $j^k\phi(x)$ of isotropic map-germs $\phi : X, x \to M, \phi(x)$, $x \in X$. Then, in general, $J_I^k(X, M) \subset J^k(X, M)$ is not a submanifold. We denote by $R^k(X, M)$ the set of k-jets of isotropic map-germs of corank at most one. Then we see that $R^k(X, M)$ is a submanifold of $J^k(X, M)$, and moreover the natural projection $\pi_{k,\ell} : R^k(X, M) \to R^\ell(X, M)$ is a submersion for $\ell \leq k$ (Corollary 3.3).

We denote by $C_I^\infty(X, M)^1$ the set of isotropic mappings $\phi : X \to M$ with corank $\phi \leq 1$, endowed with the Whitney C^∞ topology. For each $\phi \in C_I^\infty(X, M)^1$, we consider the jet extension $j^k\phi : X \to R^k(X, M)$.

This paper is devoted to proving the following transversality theorem:

THEOREM 0.1. *Let X be an n-manifold, M a symplectic $2n$-manifold, k a non-negative integer and U a locally finite family of submanifolds of $R^k(X, M)$. Then the subspace T_U of $C_I^\infty(X, M)^1$, consisting of mappings ϕ such that the k-jet extension $j^k\phi$ is transverse to all elements of U, is dense in $C_I^\infty(X, M)^1$.*

To apply Theorem 0.1 to the singularity theory of isotropic mappings, we need to recall several fundamental notions.

Two isotropic map-germs $\phi : X, x \to (M, \omega)$ and $\phi' : X', x' \to (M', \omega')$ are called *symplectically equivalent* if there exist a diffeomorphism-germ $\sigma : X, x \to X', x'$ and a symplectic diffeomorphism-germ $\tau : M, \phi(x) \to M', \phi'(x')$, $\tau^*\omega' = \omega$, such that $\tau \circ \phi = \phi' \circ \sigma$. Remark that any Lagrange immersion-germs are symplectically equivalent to each other.

In Lagrange singularity theory, it is also natural to consider a symplectic manifold with a Lagrange fibration. The typical example is the cotangent bundle T^*Q over an n-manifold Q with the symplectic form $\omega = d\theta$, where θ is the canonical (Liouville) form on T^*Q, and with the Lagrange fibration $\pi : T^*Q \to Q$.

For an isotropic mapping $\phi : X \to T^*Q$, a *Lagrange singular point* $x \in X$ of ϕ is a point x with $\operatorname{corank}(\pi \circ \phi)_x < n$. Here we call corank $(\pi \circ \phi)_x$, L-corank of ϕ_x. Then it is clear that $\operatorname{corank} \phi_x \leq \text{L-corank}\, \phi_x$, and therefore, a singular point of ϕ is necessarily a Lagrange singular point.

Our concern is to give a fundamental tool for classifying Lagrange singularities under Lagrange equivalence: Two isotropic map-germs $\phi : X, x \to T^*Q$ and $\phi' : X', x' \to T^*Q'$ are called *Lagrange equivalent* if there exist a diffeomorphism-germ $\sigma : X, x \to$

X', x' and a symplectic diffeomorphism-germ $\tau : T^*Q, \phi(x) \to T^*Q', \phi'(x')$, covering a diffeomorphism-germ $Q \to Q'$ such that $\tau \circ \phi = \phi' \circ \sigma$ (see [AGV]).

By Darboux's theorem, any isotropic map-germ $X, x \to M$ (resp. $X, x \to T^*Q$) is symplectically (resp. Lagrange) equivalent to an isotropic map-germ $\mathbf{R}^n, 0 \to T^*\mathbf{R}^n, 0$, where the symplectic form on $T^*\mathbf{R}^n$ is given by $\omega = d(\sum_{i=1}^n p_i dq_i)$.

Denote by $I(n)$ the set of isotropic map-germs $\mathbf{R}^n, 0 \to T^*\mathbf{R}^n, 0$. Further denote by $S^k(n)$ (resp. $L^k(n)$) the group of k-jets of symplectic equivalence $\{(\sigma, \tau)\}$ (resp. Lagrange equivalence) on $I(n)$. We set $R^k(n) = \{j^k\phi(0) \mid \phi \in I(n), \operatorname{corank}\phi \leq 1\}$. Then $R^k(n)$ is a $S^k(n)$-invariant submanifold of the usual jet space $J^k(n, 2n)$, (see Proposition 3.2).

Though the following modified form of the transversality theorem is in fact a special case of Theorem 0.1, it is more useful in the classification problem of generic singularities of isotropic mappings under the symplectic or the Lagrange equivalence.

COROLLARY 0.2. *We fix non-negative integers n and k. Let U be a stratification of $R^k(n)$. Assume U is invariant under $S^k(n)$ (resp. $L^k(n)$). We denote by $\tilde{U}$ the naturally obtained stratification of $R^k(X, M)$ (resp. $R^k(X, T^*Q)$), for an n-manifold X and a symplectic $2n$-manifold M (resp. an n-manifold Q). Then, the subspace $T_U \subset C_I^\infty(X, M)^1$ (resp. $C_I^\infty(X, T^*Q)^1$) of isotropic mappings $\phi : X \to M$ (resp. $\phi : X \to T^*Q$) of corank at most one such that the k-jet extension $j^k\phi : X \to R^k(X, M)$ (resp. $R^k(X, T^*Q)$) is transverse to $\tilde{U}$, is dense, with respect to the Whitney C^∞ topology.*

Theorem 0.1 (resp. Corollary 0.2) implies the transversality theorem for Lagrange immersions, which is an implicit base of the generic classification of Lagrange singularities of Lagrange immersions [AGV].

As another application of Theorem 0.1 or Corollary 0.2, we see that, if $n < 4$, then generic isotropic map-germs $\phi : \mathbf{R}^n, 0 \to T^*\mathbf{R}^n, 0$ of corank one are of L-corank (= $\operatorname{corank} \pi \circ \phi$) one (Corollary 3.6).

Moreover we see that, in fact, there exists a generic isotropic map-germ $\phi : \mathbf{R}^4, 0 \to T^*\mathbf{R}^4, 0$ of corank one and of L-corank 2 (Proposition 3.9). This type of singularities is regarded as a degeneration of the Lagrange singularities of type $D_4^{\pm}$ of Lagrange immersions. However, we need other techniques to obtain concrete normal forms of the singularities of this type.

In §1, we study the structure of isotropic mappings under both symplectic and Lagrange equivalences. The results obtained are used to the proof of Theorem 0.1, as well as they are interesting in theirselves.

The notion of jet is essential for the usual singularity theory. We give in §2 some simple descriptions of jet-spaces for the counterpart of the singularity theory of isotropic mappings.

We prove Theorems 0.1 in §3, using the results of §§1 and 2. We also give a simple consequence from Theorem 0.1 and Corollary 0.2.

For brevity, we denote by E_n the $\mathbf{R}$-algebra of map-germs $\mathbf{R}^n, 0 \to \mathbf{R}$, and by m_n the unique maximal ideal consisting of map-germs $\mathbf{R}^n, 0 \to \mathbf{R}, 0$.

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1. Isotropic mappings. Let $\phi_0 \in I(n)$ and ϕ_s $(s \in \mathbf{R}^\ell, 0)$ be an *isotropic deformation* of ϕ_0. By definition, we are assuming that $\Phi = (\phi_s, s) : \mathbf{R}^n \times \mathbf{R}^\ell, 0 \to T^*\mathbf{R}^n \times \mathbf{R}^\ell, 0$ is a C^∞ map-germ, and that $\phi_s^*\omega = 0$, for all $s \in \mathbf{R}^\ell, 0$, and, of course, when $s = 0$, ϕ_s is equal to the given ϕ_0.

Let $x = (x_1, \ldots, x_n)$ denote the coordinate system of $\mathbf{R}^n$, $q = (q_1, \ldots, q_n)$ the coordinate system of the base space of $T^*\mathbf{R}^n$, and $(p; q) = (p_1, \ldots, p_n; q)$ the associated canonical coordinate with the Liouville form $\theta = \sum_{i=1}^n p_i dq_i$. Then $\omega = d\theta$.

Since $\phi_s^*\omega = d_x \phi_s^*(\sum_{i=1}^n p_i dq_i) = 0$, there exists a family of (generating) functions e_s uniquely up to the addition of a function of s such that $d_x e_s = \phi_s^*(\sum_{i=1}^n p_i dq_i)$, where d_x means the exterior derivative with respect to x. Then we have

$$de_s = \sum_{i=1}^n p_i \circ \phi_s \; d(q_i \circ \phi_s) + \sum_{j=1}^\ell (r_j)_s ds_j,$$

for some function-germs $(r_j)_s \in E_n$. If we set

$$\tilde{\Phi} = (\phi_s; r_s, s) : \mathbf{R}^n \times \mathbf{R}^\ell, 0 \to T^*\mathbf{R}^{n+\ell},$$

then $\tilde{\Phi}$ is isotropic and it is a lift of Φ with respect to the projection $\pi : T^*\mathbf{R}^{n+\ell} \to T^*\mathbf{R}^n \times \mathbf{R}^\ell$ defined by $\pi(p, q; r, s) = (p, q, s)$. As easily verified, any isotropic lifts of Φ are Lagrange equivalent to $\tilde{\Phi}$. We call $\tilde{\Phi}$ an *isotropic unfolding* of ϕ_0. Then we have the following fundamental fact:

PROPOSITION 1.1. *Let $\phi : X^n, x_0 \to M^{2n}, \phi(x_0)$ be an isotropic map-germ of corank k. Then ϕ is symplectically equivalent to an isotropic unfolding of a $\phi_0 \in I(k)$ of corank k.*

Proof. From symplectic linear algebra, it is easy to see that there exist symplectic coordinates $(p_1, \ldots, p_n; q_1, \ldots, q_n)$ of $M, \phi(x_0)$ such that $(q_{k+1}, \ldots, q_n) \circ \phi$ is a submersion. Then $(x_{k+1}, \ldots, x_n) = (q_{k+1}, \ldots, q_n) \circ \phi$ is a part of a coordinate system $x = (x_1, \ldots, x_n)$ of $\mathbf{R}^n, 0$. If we set $s = (x_{k+1}, \ldots, x_n)$, and $\phi_0 = (p_1, \ldots, p_k; q_1, \ldots, q_k) \circ \phi|_{s=0}$, then ϕ is an isotropic unfolding of ϕ_0.

By the same proof as Proposition 1.1, we have

PROPOSITION 1.2. *Let $\phi : X^n, x_0 \to T^*(Q^n), \phi(x_0)$ be an isotropic map-germ. Assume the L-corank of ϕ $(= \operatorname{corank} \pi \circ \phi)$ is equal to k. Then ϕ is Lagrange equivalent to an isotropic unfolding of a $\phi_0 \in I(k)$ of L-corank k.*

Similarly we have the following, which is needed in §3.

PROPOSITION 1.3. *Let $\phi_\lambda : X^n, x_\lambda \to M^{2n}, \phi_\lambda(x_\lambda)$ (resp. $T^*(Q^n), \phi_\lambda(x_\lambda)$) be an ℓ-parameter C^∞ family of isotropic map-germs. Assume the corank of ϕ_0 (res. the L-corank of ϕ_0, i.e. the corank of $\pi \circ \phi_0$) is equal to k. Then ϕ_λ is symplectically (resp. Lagrange) equivalent by an ℓ-parameter family of symplectic (resp. Lagrange) equivalences to an*

ℓ-parameter family of isotropic unfoldings of an ℓ-parameter family ϕ'_λ of elements in $I(k)$ with corank $\phi'_0 = k$ *(resp. L-*corank $\phi'_0 = k$*).*

By Proposition 1.1 (resp. 1.2), in particular, an isotropic map-germ of corank one (resp. of L-corank one) is symplectically (resp. Lagrange) equivalent to an isotropic unfolding of a map-germ $\phi : \mathbf{R}, 0 \to T^*\mathbf{R}, 0$. Remark that ϕ is automatically isotropic, since any 2-forms vanish on $\mathbf{R}, 0$, and, by the same reason, any deformation (ϕ_s, s) : $\mathbf{R} \times \mathbf{R}^{n-1}, 0 \to T^*\mathbf{R} \times \mathbf{R}^{n-1}, 0$ is also isotropic. Simply write $\phi_s(t) = (P(t,s), Q(t,s))$, $t \in \mathbf{R}, 0$. Then the corresponding isotropic unfolding is given by $\tilde{\Phi} = (\phi_s; r, s)$, where

$$r_j = \frac{\partial}{\partial s_j}\left(\int_0^t P\frac{\partial Q}{\partial t}dt\right) - P\frac{\partial Q}{\partial s_j} = \int_0^t \left(\frac{\partial P}{\partial s_j}\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial t}\frac{\partial Q}{\partial s_j}\right)dt, \qquad 1 \le j \le n-1.$$

Thus the description of isotropic map-germs of corank one (resp. of L-corank one) is easily handled: In fact the local symplectic classification of generic isotropic mappings of corank one is given in [I1], [Z]. (See also [G2].) Further the local Lagrange classification of generic isotropic mappings of L-corank one will be given in [I3].

To describe the structure of isotropic map-germs of corank one under Lagrange equivalence, we introduce the following notation:

Let I, J be a decomposition of $\{1, \dots, n-1\}$: $I \cup J = \{1, \dots, n-1\}, \#I + \#J = n-1$. We fix a system of coordinates (p_I, q_J, t) of $\mathbf{R}^n, 0$, where where $p_I = (p_i)_{i \in I}$ and so on. Then, for a pair of functions $u, v \in E_n$, we define a map-germ $\phi_{I,J}[u,v] : \mathbf{R}^n, 0 \to T^*\mathbf{R}^n, 0$, by

$$\begin{aligned}
&p_I \circ \phi_{I,J}[u,v] = p_I,\\
&p_J \circ \phi_{I,J}[u,v] = \int_0^t \left(\frac{\partial u}{\partial q_J}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial q_J}\right)dt, \qquad p_n \circ \phi_{I,J}[u,v] = u.\\
&q_I \circ \phi_{I,J}[u,v] = \int_0^t \left(\frac{\partial u}{\partial t}\frac{\partial v}{\partial p_I} - \frac{\partial u}{\partial p_I}\frac{\partial v}{\partial t}\right)dt,\\
&q_J \circ \phi_{I,J}[u,v] = q_J, \qquad q_n \circ \phi_{I,J}[u,v] = v.
\end{aligned}$$

Then $\phi_{I,J}[u,v]$ is isotropic, of corank one and of L-corank $\#I$.

We need in §3 the following, which generalize Propositions 1.1, 1.2 and 1.3.

PROPOSITION 1.4. *Let ϕ_λ $(\lambda \in \mathbf{R}^\ell, 0)$ be an ℓ-parameter family of elements in $I(n)$ with* corank $\phi_0 \le 1$. *Then the family ϕ_λ is Lagrange equivalent to $\phi_{I,J}[u_\lambda, v_\lambda]$, for some decomposition I, J of $\{1, \dots, n-1\}$ and for some u_λ, v_λ with $u_\lambda(0) = v_\lambda(0) = 0$ (by a one-parameter family of Lagrange equivalences $(\sigma_\lambda, \tau_\lambda)$).*

Moreover, if $\phi_0 = \tau \circ \phi_{I,J}[u,v] \circ \sigma^{-1}$, then we can choose $(u_\lambda, v_\lambda, \sigma_\lambda, \tau_\lambda)$ so that $(u_0, v_0, \sigma_0, \tau_0) = (u, v, \sigma, \tau)$.

Proof. Notice that, for the case of Lagrange immersions, a similar result is well-known (cf. [AGV]).

First we see ϕ_λ is Lagrange equivalent to ϕ'_λ of the following form:

$$\phi'_\lambda : (p_I, q_J, t) \mapsto (p_I, p_J, p_n; q_I, q_J, q_n),$$

for some fixed decomposition I, J of $\{1, \dots, n-1\}$, where p_J, p_n, q_I, q_n are function-germs of $(p_I, q_J, t; \lambda)$ with $p_J|_{t=0} = p_n|_{t=0} = q_I|_{t=0} = p_n|_{t=0} = 0$.

Then there exists function-germ e' of $(p_I, q_J, t; \lambda)$ with $e'|_{t=0} = 0$ and

$$d_{(p_I,q_J,t)}e' = p_I dq_I + p_J dq_J + p_n dq_n = d(p_I q_I) - q_I dp_I + p_J dq_J + p_n dq_n.$$

Set $e = e' - p_I q_I$. Then $d_{(p_I,q_J,t)}e = -q_I dp_I + p_J dq_J + p_n dq_n$. Thus we have

$$\frac{\partial e}{\partial p_I} = -q_I + p_n \frac{\partial q_n}{\partial p_I}, \quad \frac{\partial e}{\partial q_J} = p_J + p_n \frac{\partial q_n}{\partial q_J}, \quad \frac{\partial e}{\partial t} = p_n \frac{\partial q_n}{\partial t},$$

and we see $e = \int_0^t p_n(\partial q_n/\partial t)dt$. Hence

$$q_I = -\frac{\partial e}{\partial p_I} + p_n \frac{\partial q_n}{\partial p_I} = \int\limits_0^t \left(\frac{\partial p_n}{\partial t}\frac{\partial q_n}{\partial p_I} - \frac{\partial p_n}{\partial p_I}\frac{\partial q_n}{\partial t} \right) dt,$$

$$p_J = \frac{\partial e}{\partial q_J} - p_n \frac{\partial q_n}{\partial q_J} = \int\limits_0^t \left(\frac{\partial p_n}{\partial q_J}\frac{\partial q_n}{\partial t} - \frac{\partial p_n}{\partial t}\frac{\partial q_n}{\partial q_J} \right) dt.$$

If we set $u_\lambda = p_n(\cdot, \lambda)$ and $v_\lambda = q_n(\cdot, \lambda)$, then we have the required result.

2. Isotropic jets. Recall that $I(n)$ denotes the set of isotropic map-germs $\mathbf{R}^n, 0 \to T^*\mathbf{R}^n, 0$. Set $J^k(n) = \{j^k\phi(0) \mid \phi : \mathbf{R}^n, 0 \to T^*\mathbf{R}^n, 0\} = J^k(n, 2n)$. A k-jet $z \in J^k(n)$ is called *isotropic* if $z = j^k\phi(0)$ for some $\phi \in I(n)$. We denote by $J^k_I(n)$ the set of isotropic k-jets in $J^k(n)$, for $k = 1, 2, \dots, \infty$.

Our problem in this section is to study the structure of the set $J^k_I(n)$ of isotropic k-jets.

We denote by $S^k(n)$ (resp. $L^k(n)$) the group of pairs $(j^k\sigma(0), j^k\tau(0))$ of k-jets of a diffeomorphism-germ $\sigma : \mathbf{R}^n, 0 \to \mathbf{R}^n, 0$ and a symplectic (resp. Lagrange, i.e. fiber-preserving symplectic) diffeomorphism-germ $\tau : T^*\mathbf{R}^n, 0 \to T^*\mathbf{R}^n, 0$. Naturally $S^k(n)$, therefore $L^k(n)$ acts on $J^k(n)$. Then $J^k_I(n)$ is a $S^k(n)$-invariant, therefore $L^k(n)$-invariant subset of $J^k(n)$.

We introduce an auxiliary notion:

Definition 2.1. A map-germ $\phi : \mathbf{R}^n, 0 \to T^*\mathbf{R}^n, 0$ is called *ℓ-isotropic* ($\ell = 1, 2, \dots$ $\dots, \infty$) if $\phi^*\omega \in m_n^\ell \Omega$, that is, $j^{\ell-1}(\phi^*\omega)(0) = 0$, where Ω denotes the E_n-module of germs of differential 2-form on $\mathbf{R}^n, 0$. A jet $z \in J^k(n)$ is called ℓ-isotropic if $z = j^k\phi(0)$ for some ℓ-isotropic ϕ.

Now set $J^k_{\ell-I}(n) = \{z \in J^r(n) \mid z \text{ is } \ell\text{-isotropic}\}$. Then we have a sequence of sets:

$$J^k(n) \supset J^k_{1-I}(n) \supset \dots \supset J^k_{k-I}(n) \supset J^k_{k+1-I}(n) \supset \dots \supset J^k_{\infty-I}(n) \supset J^r_I(n).$$

Then it is easy to see the following lemmata:

Lemma 2.2. *$J^k_{\ell-I}(n)$ is algebraic (resp. semi-algebraic) if $\ell \le k$ (resp. $k < \ell < \infty$).*

Lemma 2.3. *$J^1_{1-I}(n) = J^1_I(n)$, which is identified with the set of linear isotropic mappings $\mathbf{R}^n \to T^*\mathbf{R}^n$. Moreover $J^1_I(n) \subset Hom_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^{2n}) \cong \mathbf{R}^{2n^2}$ is a quadratic hypersurface with $Sing J^1_I(n) = \{$linear isotropic mappings of corank $\ge 2\}$.* ([G1], [A2]. See also Example 2.4 below.)

Set $\Sigma^i(n) = \{j^1\phi(0) \in J^1(n) \mid \text{corank } \phi = i\}$ and $\Sigma_i(n) = \{j^1\phi(0) \in J^1(n) \mid \text{L-corank } \phi = i\}$.

The natural projection $\pi_{k,\ell} : J^k(n) \to J^\ell(n)$ $(k \geq \ell)$ induces $\pi_{k,\ell} : J^k_I(n) \to J^\ell_I(n)$. Denote the inverse image $\pi^{-1}_{k,1}(\Sigma^i(n))$ (resp. $\pi^{-1}_{k,1}(\Sigma_i(n))) \subset J^k(n)$ also by $\Sigma^i(n)$ (resp. $\Sigma_i(n)$).

Now, to study (first order) singularities or Lagrange singularities of isotropic mappings, set $\Sigma^i_I(n) = \Sigma^i(n) \cap J^1_I(n)$ (resp. $\Sigma_{j,I}(n) = \Sigma_j(n) \cap J^1_I(n)$). Further set $\Sigma^i_{j,I}(n) = \Sigma^i_I(n) \cap \Sigma_{j,I}(n)$ $(i \leq j)$. We freely regard $\Sigma^i_I(n)$, $\Sigma_{j,I}(n)$ and $\Sigma^i_{j,I}(n)$ as subsets in $J^k_I(n)$ for arbitrary k.

Remark that $\Sigma^0_I(n)$ is the set of jets of Lagrangian immersions $\mathbf{R}^n, 0 \to T^*\mathbf{R}^n, 0$.

Example 2.4. To describe $J^1_I(n)$, we set $V = \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{C}^n) \cong M_n(\mathbf{C})$, the space of complex square matrices of size n. We denote by $\langle \cdot, \cdot \rangle$ the standard Hermitian structure on $\mathbf{C}^n$, and by $[\cdot, \cdot]$ the standard symplectic structure on $\mathbf{C}^n$; $[u, v] = \mathrm{Im}\langle u, v\rangle, u, v \in \mathbf{C}^n$. Then $J^1_I(n)$ is identified with the set $W \subset V$ of linear isotropic mappings. We define a polynomial map $\rho : V \to \mathrm{Alt}(n)$ by $\rho(\ell)(u, v) = [\ell u, \ell v]$, $u, v \in \mathbf{R}^n$. where $\mathrm{Alt}(n)$ is the space of skewsymmetric bilinear forms $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$. Then we see $W = \rho^{-1}(O)$ and therefore W is a real algebraic subset of V. Further we see that ρ is submersive along $\Sigma^0_I(n) \cup \Sigma^1_I(n)$. Hence $\mathrm{Sing}(W)$ is contained in $\bigcup_{j=2}^n \Sigma^j_I(n)$. Furthermore each $\Sigma^i_I(n)$ is a submanifold of V of dimension $(1/2)\{n(3n+1) - i(3i+1)\}$, $0 \leq i \leq n$. (See [I2]).

Now we pick up a wider domain than $\Sigma^0_I(n)$ in $J^k_I(n)$:

$$R^k(n) = \{j^k\phi(0) \in J^k_I(n) \mid \operatorname{corank}\phi \leq 1\} = \Sigma^0_I(n) \cup \Sigma^1_I(n),$$

which is open in $J^k_I(n)$.

Example 2.5. Let $n=2$. The ambient space $J^1(2) = J^1(2, 4)$ is identified with $\mathbf{R}^8$, by setting $q_1 = ax + by$, $q_2 = cx + dy$; $p_1 = \alpha x + \beta y$, $p_2 = \gamma x + \delta y$, where (x, y) is the coordinate of $\mathbf{R}^2$. Then $J^1_I(2)$ is given by $\alpha b - \beta a + \gamma d - \delta c = 0$. Thus $\dim J^1_I(2) = 7$ and $\mathrm{Sing}\, J^1_I(2) = \{0\}$. Set $\bar{\Sigma}^1_I(2) = \Sigma^1_I(2) \cup \Sigma^2_I(2)$. This is defined by $\alpha b - \beta a + \gamma d - \delta c = ad - bc = \alpha\delta - \beta\gamma = 0$, which is of dimension 5 and $\mathrm{Sing}\bar{\Sigma}^1_I(2) = \{0\}$. Further set $\bar{\Sigma}_{1,I}(2) = \Sigma_{1,I}(2) \cup \Sigma_{2,I}(2)$. Then $\bar{\Sigma}_{1,I}(2)$ is defined by $\alpha b - \beta a + \gamma d - \delta c = ad - bc = 0$, which is of dimension 6 and $\mathrm{Sing}\bar{\Sigma}_{1,I}(2) = \{a = b = c = d = 0\}$. Now set $X_0 = J^1_I(2), X_1 = J^1(2) \cap \{ad - bc = 0\} = \bar{\Sigma}_{1,I}(2)$, $X_2 = X_1 \cap \{\alpha\delta - \beta\gamma = 0\} = \bar{\Sigma}^1_I(2)$, $X_3 = \{a = b = c = d = 0\} = \Sigma_{2,I}(2)$, $X_4 = X_2 \cap X_3 = \Sigma_{2,I}(2) \cap \bar{\Sigma}^1_I(2) = \Sigma^1_{2,I}(2) \cup \{O\}$, and $X_7 = \{O\} = \Sigma^2_I(2)$. Further set $U_0 = X_0 - X_1, U_1 = X_1 - (X_2 \cup X_3), U_2 = X_2 - X_4, U_3 = X_3 - X_4, U_4 = X_4 - X_7 = \Sigma^1_{2,I}(2)$. Then $\{U_0, U_1, U_2, U_3, U_4\}$ is the stratification of $R^1(2)$ by the orbits of $L^1(2)$, and we see $\operatorname{codim} U_j = j$.

The following is the key to study the structure of $R^k(n)$:

Proposition 2.6. $J^k_{k-I}(n) \cap (\Sigma^0(n) \cup \Sigma^1(n)) = R^k(n)$.

Proof. Consider the natural action of $S^k(n)$ on $J^k(n)$ preserving $J^k_I(n)$ and $J^k_{\ell-I}(n)$. Assume z is a k-isotropic k-jet and $\operatorname{corank} z \leq 1$. Then, by Proposition 1.1, there exist polynomials $P_1, \ldots, P_n$ and Q_n of degree $\leq k$ such that z is (symplectically) equivalent to $j^k(P; x', Q_n)(0)$, where $P = (P_1, \ldots, P_n)$ and $x' = (x_1, \ldots, x_{n-1})$. The map-germ

$\phi = (P; x', Q_n)$ is also k-isotropic, therefore, $\phi^*\omega = \sum_{i=1}^{n-1} dP_i dx_i + dP_n dQ_n \in m_n^k \Omega$. Comparing coefficients of $dx_i \wedge dx_n$, we have

$$\frac{\partial P_i}{\partial x_n} = \frac{\partial(P_n, Q_n)}{\partial(x_i, x_n)} + \rho_i, \quad \rho_i \in m_n^k \quad (1 \le i \le n-1).$$

Set $\tilde{P}_i = P_i - \int_0^{x_n} \rho_i dx_n$, $1 \le i \le n-1$, $\tilde{P} = (\tilde{P}_1, \ldots, \tilde{P}_{n-1}, P_n)$ and $\phi' = (\tilde{P}; x', Q_n)$. Then $j^k\phi'(0) = z$ and ϕ' is isotropic. Hence z is isotropic; $z \in J_I^k(n)$. So $z \in J_I^k(n) \cap (\Sigma^0(n) \cup \Sigma^1(n)) = R^k(n)$.

The converse inclusion is clear.

Modifying the proof of Lemma 2.6, we have

LEMMA 2.7. *Let $a : \mathbf{R}^\ell, 0 \to J^k(n)$ be a C^∞-map-germ with $a(\mathbf{R}^\ell, 0) \subset R^k(n)$. Let $a(0) = j^k\phi(0)$ for some $\phi \in I(n)$. Then there exists a C^∞ family $\phi_\lambda \in I(n)$, $\lambda \in \mathbf{R}^\ell, 0$, such that $j^k\phi_\lambda(0) = a(\lambda)$ and $\phi_0 = \phi$.*

In particular, for a C^∞-curve z_λ in $R^k(n)$ through $z_0 = j^k\phi(0), \phi \in I(n)$ $(\lambda \in \mathbf{R}, 0)$, there exists a C^∞ family ϕ_λ of $I(n)$ with $j^k\phi_\lambda(0) = z_\lambda$ and $\phi_0 = \phi$.

Proof. By Proposition 1.1, using a symplectic equivalence, we may assume ϕ is of type $(P; x', Q_n)$, where $P = (P_1, \ldots, P_n)$ and $P_1, \ldots, P_n, Q_n \in m_n$. Then there exist a (not necessarily isotropic) deformation $(P_\lambda, x', Q_{n\lambda})$ of ϕ such that $j^k(P_\lambda, x', Q_{n\lambda})(0)$ is symplectically equivalent to $a(\lambda)$ by a family of symplectic equivalences $(\sigma_\lambda, \tau_\lambda)$ with (σ_0, τ_0). By the same procedure as in the proof of Proposition 2.6, we find an isotropic deformation $\phi'_\lambda = (\tilde{P}_\lambda, x', \tilde{Q}_{n\lambda})$ of ϕ such that $j^k\phi'_\lambda(0) = j^k(P_\lambda, x', Q_{n\lambda})(0)$. Applying the inverse of $(\sigma_\lambda, \tau_\lambda)$ to ϕ'_λ, we get the required family ϕ_λ.

Remark 2.8. In general it seems natural to pose the following conjecture: For any $k < \infty$, there exists $\ell = \ell(k) < \infty$, such that $J_{\ell-I}^k(n) = J_I^k(n)$. Further, $J_{\infty-I}^\infty(n) = J_I^\infty(n)$.

3. Transversality Theorem. We fix $n \ge 1$. In this section, we denote by V^k the affine space $J^k(n) = J^k(n, 2n)$ of k-jets of C^∞ map-germs from $\mathbf{R}^n, 0$ to $T^*\mathbf{R}^n, 0$. Besides, we denote by Λ^k the vector space of $(k-1)$-jets of *closed* 2-forms in $\mathbf{R}^n, 0$. Remark that, by Poincaré lemma, Λ^k is a vector subspace of the vector space of $(k-1)$-jets of 2-forms in $\mathbf{R}^n, 0$ and it is equal to the space of $(k-1)$-jets of exact 2-forms in $\mathbf{R}^n, 0$. Furthermore we see

$$\dim \Lambda^k = {}_{n+k}C_k\, n - {}_{n+k+1}C_{k+1} + 1.$$

We endow $T^*\mathbf{R}^n = \mathbf{R}^{2n}$ with the standard symplectic form $\omega = \sum_{i=1}^n dp_i \wedge dq_i$. Then define the polynomial mapping $\rho : V^k \to \Lambda^k$ by $\rho(j^k\phi(0)) = j^{k-1}(\phi^*\omega)(0)$.

First we show

LEMMA 3.1. *For a $z \in V^k$, consider the tangent mapping $\rho_* : T_zV^k(\cong V^k) \to T_{\rho(z)}\Lambda^k(\cong \Lambda^k)$. If* corank $(z) \le 1$, *then ρ_* is surjective, that is, ρ is a submersion at z.*

Proof. Set $z = j^k\phi(0)$ and $\phi = (P, Q) = (P_1, \ldots, P_n; Q_1, \ldots, Q_n)$. For a one-parameter deformation $\phi_t = (P + t\tilde{P}, Q + t\tilde{Q})$, we have

$$d(\phi^*\omega)/dt|_{t=0} = \sum_i d\tilde{P}_i dQ_i - \sum_i d\tilde{Q}_i dP_i = d\Big(\sum_i (\tilde{P}_i dQ_i - \tilde{Q}_i dP_i)\Big).$$

Therefore $\rho_*(\tilde{P}, \tilde{Q}) = j^{k-1}\{d(\sum_i(\tilde{P}_i dQ_i - \tilde{Q}_i dP_i))\}(0)$. Then, to show ρ_* is surjective, it is sufficient to show that, for any one-form germ E on $\mathbf{R}^n, 0$, there exist function-germs $\tilde{P}_i, \tilde{Q}_i, 1 \le i \le n$, and e such that

$$\sum_i (\tilde{P}_i dQ_i - \tilde{Q}_i dP_i) = E + de.$$

Since the property that ρ_* is surjective is $S^k(n)$-invariant, we may assume

$$Q = (x_1, \ldots, x_{n-1}, u(x', x_n)),$$

where $x' = (x_1, \ldots, x_{n-1})$. Set $E = \sum_{i=1}^n E_i dx_i$. If we set, in particular, $\tilde{Q} = 0$ and $\tilde{P}_n = 0$, then the equation to solve turns out to

$$\sum_{i=1}^{n-1} (\tilde{P}_i - E_i - \partial e/\partial x_i) dx_i - (E_n + \partial e/\partial x_n) dx_n = 0.$$

First set $e = -\int_0^{x_n} E_n dx_n$. Then next set $\tilde{P}_i = E_i + \partial e/\partial x_i, i = 1, \ldots, n-1$. Thus ρ_* is surjective if $\operatorname{corank}(z) \le 1$.

PROPOSITION 3.2. *$R^k(n)$ is a $S^k(n)$-invariant submanifold of $V^k = J^k(n, 2n)$. The codimension of $R^k(n)$ in V^k is equal to $\dim \Lambda^k$. Moreover the natural projection $\pi_{k,k-1} : R^k(n) \to R^{k-1}(n)$ is a surjective submersion ($k \ge 1$).*

COROLLARY 3.3. *For an n-manifold X and a symplectic $2n$-manifold M, the set $R^k(X, M)$ of k-jets $j^k\phi(x)$ of isotropic map-germs $\phi : X, x \to M$ of corank ≤ 1 is a submanifold of $J^k(X, M)$ of codimension $\dim \Lambda^k$. Moreover, the natural projection $\pi_{k,k-1} : R^k(X, M) \to R^{k-1}(X, M)$ is a surjective submersion ($k \ge 1$).*

Remark 3.4. The system $\{R^k\}$ is regarded as the *prolongation* of the non-linear first order partial differential equation: $\omega = 0$ ([Go]).

Proof of Proposition 3.2. The first part is clear from Lemma 3.1 and Proposition 2.6.

To see the second part, take $z \in R^k(n)$ and a C^∞ curve in $R^{k-1}(n)$ through $\pi_{k,k-1} z \in R^{k-1}(n)$. Let $j^k\phi(0) = z$ for a $\phi \in I(n)$. Then, by Lemma 2.7, this curve lifts to a C^∞ deformation ϕ_λ of ϕ in $I(n)$. Taking k-jets of ϕ_λ, we have a C^∞ lifting of the curve through z.

Proof of Theorem 0.1. We follow the standard argument to prove the transversality theorem, for instance, as in §3 of [M1].

First of all we remark the following: The space $C_I^\infty(X, M)$ of isotropic mappings is a Baire space with respect to the Whitney C^∞ topology. Furthermore $C_I^\infty(X, M)^1 \subset C_I^\infty(X, M)$ is also a Baire space.

The proof of this fact is similar to that of Proposition 3.1 of [M1]: The point is that the isotropic condition for a mapping is a local and closed condition about its one-jet section. The second half is clear, because $C_I^\infty(X, M)^1$ is open in $C_I^\infty(X, M)$.

Then to show Theorem 0.1, it is sufficient to prove that, for each $\phi \in C_I^\infty(X, M)^1$ and for each $x_0 \in X$, there exist a manifold E, $e_0 \in E$, and a continuous mapping $\varphi : (E, e_0) \to (C_I^\infty(X, M)^1, \phi)$ such that the induced mapping $\Phi : E \times X \to R^k(X, M)$

defined by $\Phi(e,x) = j^k(\varphi(e))(x)$ is submersive at (e_0, x_0). (See Lemma 3.2 and the argument in Proposition 3.3 of [M1].)

To complete this, we use the results in §§1 and 2 as follows: Denote by $L(n)$ the group of Lagrange equivalences on $I(n)$. For each decomposition I, J of indices $\{1,\ldots,n-1\}$, we define a map $\Psi : m_n \times m_n \times L(n) \to I(n)$ by $\Psi(u,v;\sigma,\tau) = \tau \circ \phi_{I,J}[u,v] \circ \sigma^{-1}$. The mapping Ψ induces a C^∞ mapping $\Psi^k : J^k(n,2) \times L^k(n) \to R^k(n)$ by $\Psi^k(j^k u(0), j^k v(0); j^k\sigma(0), j^k\tau(0)) = j^k(\Psi(u,v;\sigma,\tau))(0)$.

Then Ψ^k is a submersion: In fact, let $(u,v,\sigma,\tau) \in m_n \times m_n \times L(n)$, $\Psi(u,v,\sigma,\tau) = \phi \in I(n)$ and $z = j^k\phi(0) \in R^k(n)$. Let z_λ be a smooth curve in $R^k(n)$ through z. Then, by Lemma 2.7, there exists a C^∞-deformation ϕ_λ in $I(n)$ of ϕ with $j^k\phi_\lambda(0) = z_\lambda$. Then by Proposition 1.4, there exist $u_\lambda, v_\lambda, \sigma_\lambda, \tau_\lambda$ such that $\Psi(u_\lambda, v_\lambda; \sigma_\lambda, \tau_\lambda)(= \tau_\lambda \circ \phi_{I,J}[u_\lambda, v_\lambda] \circ \sigma_\lambda^{-1}) = \phi_\lambda$ and that $(u_0, v_0, \sigma_0, \tau_0) = (u, v, \sigma, \tau)$. So we have

$$\Psi^k(j^k u_\lambda(0), j^k v_\lambda(0); j^k\sigma_\lambda(0), j^k\tau_\lambda(0)) = z_\lambda.$$

This means that any C^∞ curve on $R^k(n)$ through z has a C^∞ lifting with respect to Ψ^k, and therefore Ψ^k is a submersion.

We identify $J^k(n,2)$ with the space pf pairs of polynomials of n variables with degree $\le k$ without constant terms. We set then $E = J^k(n,2) \times L^k(n) \times M$.

Let $\phi \in C_I^\infty(X,M)^1$ and $x_0 \in X$. Then we have

$$\phi_{x_0} = \tau_0 \circ \phi_{I,J}[u_0, v_0] \circ \sigma_0^{-1} : X, x_0 \to M, \phi(x_0),$$

for a diffeomorphism-germ $\sigma_0 : \mathbf{R}^n, 0 \to X, x_0$, for a symplectic diffeomorphism-germ $\tau_0 : T^*\mathbf{R}^n, 0 \to M, \phi(x_0)$, for a decomposition I, J of $\{1, 2, \ldots, n-1\}$, and for a pair of elements $u_0, v_0 \in E_n$. (See Proposition 1.4.) Set $e_0 = (u_0, v_0, 1, \phi(x_0))$, where $1 \in L^k(n)$ is the unit, and construct $\varphi : (E, e_0) \to (C_I^\infty(X,M)^1, \phi)$ by

$$\varphi(u,v;\sigma,\tau;m) = \tau_0 \circ \{b \cdot T_m \circ \tau \circ \phi_{I,J}[u,v] \circ \sigma^{-1} + (1-b)\tau_0^{-1} \circ \phi\sigma_0\} \circ \sigma_0^{-1},$$

where b is a bump function on $J^k(n,2) \times L^k(n) \times \mathbf{R}^{2n} \times \mathbf{R}^n$, which is equal to 1 near the origin and is equal to 0 off a neighbourhood of the origin, and $T_m : T^*\mathbf{R}^n \to T^*\mathbf{R}^n$ is the translation by $\tau_0^{-1}(m) - \tau_0^{-1}(\phi(x_0))$.

Then ϕ is clearly continuous and we have, near x_0,

$$\varphi(u,v;\sigma,\tau;m) = \tau_0 \circ T_m \circ \tau \circ \phi_{I,J}[u,v] \circ \sigma^{-1} \circ \sigma_0^{-1}.$$

Since, as verified above, Ψ^k is a submersion, Φ constructed from φ is a submersion at (e_0, x_0). ∎

Remark 3.5. By the same method as in the proof of Theorem 0.1, we can show also the multi-transversality theorem for isotropic mappings of corank at most one (cf. [M1]).

Lastly, we give a simple consequence of Theorem:

COROLLARY 3.6. *Let $n < 4$. Then generic isotropic map-germs $\phi : \mathbf{R}^n, 0 \to T^*\mathbf{R}^n, 0$ of corank one are of L-corank one.*

For the proof of Corollary 3.6, we repeat the constructions in §1, in the first jet level.

Denote by $J^1(n-k, R^1(k))$ the space of 1-jets of map-germs $\mathbf{R}^{n-k}, 0 \to R^1(k) \subset J^1(k)$. Then define a mapping $\delta : J^1(n-k, R^1(k)) \to R^1(n)$ as follows: Let $j^1\alpha(0) \in J^1(n -$

$k, R^1(k))$ with $\alpha : \mathbf{R}^{n-k}, 0 \to R^1(k)$. Denote by $A : \mathbf{R}^{n-k} \times \mathbf{R}^k, 0 \to T^*\mathbf{R}^k, 0$ the induced deformation. Then there exists $e : \mathbf{R}^{n-k} \times \mathbf{R}^k, 0 \to \mathbf{R}$ such that $de_s = A_s^*\theta, s \in \mathbf{R}^{n-k}, 0$. If we assume $e(0, s) = 0$, then e is uniquely determined. From

$$de = \sum_i p_i \circ A_s d(q_i \circ A_s) + \sum_i (\partial e/\partial s_i) ds_i,$$

we define an isotropic map-germ $B : \mathbf{R}^{n-k} \times \mathbf{R}^k, 0 \to T^*(\mathbf{R}^{n-k} \times \mathbf{R}^k), 0$, by

$$p_i \circ B = p_i \circ A, \quad q_i \circ B = q_i \circ A, \quad r_i \circ B = \partial e/\partial s_i, \quad s_i \circ B = s_i.$$

Clearly $j^1B(0)$ depends only on the one-jet $j^1\alpha(0)$. Then set $\delta(j^1\alpha(0)) = j^1B(0)$.

From δ, we define

$$\Delta : J^1(n-k, R^1(k)) \times L^1(n) \to R^1(n) \times L^1(n) \to R^1(n),$$

where the second mapping is the $L^1(n)$-action on $R^1(n)$. Then we easily verify the following

Lemma 3.7. (1) *Δ is a submersion onto $R^1(n) - \bar{\Sigma}_{k+1,I}(n)$.*
(2) $\Delta^{-1}(\Sigma_{j,I}(n)) = Pr^{-1}(\Sigma_{j,I}(k)) \times L^1(n)$, $0 \le j \le k$.
(3) $\Delta^{-1}(\Sigma^1_I(n)) = Pr^{-1}(\Sigma^1_I(k)) \times L^1(n)$.
(4) $\operatorname{codim} \Sigma_{j,I}(n) = \operatorname{codim} \Sigma_{j,I}(k)$ *and* $\operatorname{codim} \Sigma^1_{j,I}(n) = \operatorname{codim} \Sigma^1_{j,I}(k)$, $0 \le j \le k$.
Here $Pr : J^1(n-k, R^1(k)) \to R^1(k)$ is the natural projection.

From Lemma 3.7 and Example 2.5, we have

Lemma 3.8. $\operatorname{codim} \Sigma^1_{2,I}(n) = \operatorname{codim} \Sigma^1_{2,I}(2) = 4$.

Proof of Corollary 3.6. By Theorem 0.1 (or Corollary 0.2) and Lemma 3.8, a generic isotropic mapping of corank at most one has no singularity of type $\Sigma^1_{2,I}(n)$ if $n < 4$. Therefore a generic isotropic map-germ of corank one is necessarily of L-corank one, if $n < 4$.

We conclude this paper by showing the following:

Proposition 3.9. *There exists a generic isotropic map-germ $\phi : \mathbf{R}^4, 0 \to T^*\mathbf{R}^4$ such that* $\operatorname{corank} \phi = 1$ *and* $\text{L-corank}\, \phi = 2$.

Proof. It is sufficient to show that there exists an isotropic map-germ $\phi \in I(4)$ of corank one and of L-corank two such that the one-jet extension $j^1\phi : \mathbf{R}^4, 0 \to R^1(4)$ is transverse to $\Sigma^1_{2,I}(4)$ at 0: By any isotropic perturbations of ϕ, the Lagrange singular point of type $\Sigma^1_{2,I}$ does not vanish.

On the other hand, by Example 2.5 and Lemma 3.7, we easily have the following criterion for the transversality: Assume that $\phi \in I(4)$ is constructed from a two parameter isotropic deformation (P_1, P_2, Q_1, Q_2) of an element of $I(2)$ as in §1, where P_1, P_2, Q_1, Q_2 are function-germs of x, y, z, w, and z, w are regarded as parameters. If $\partial P_1/\partial x(0) \neq 0$, then $j^1\phi$ transversely intersects $\Sigma^1_{2,I}(4)$ at 0, if and only if the map-germ

$$f = \left(\frac{\partial P_1}{\partial x}\frac{\partial P_2}{\partial y} - \frac{\partial P_2}{\partial x}\frac{\partial P_1}{\partial y}, \frac{\partial Q_1}{\partial x}, \frac{\partial Q_2}{\partial x}, \frac{\partial Q_2}{\partial y} \right) : \mathbf{R}^4, 0 \to \mathbf{R}^4, 0$$

is of maximal rank.

Then we define $\phi \in I(4)$, for instance, by

$$p_1 = x, \quad p_2 = (1/2)y^2, \quad p_3 = -(1/2)xy^2, \quad p_4 = -(1/3)y^3,$$
$$q_1 = (1/2)(x^2 + y^2 z), \quad q_2 = xz + yw + y^2, \quad q_3 = z, \quad q_4 = w.$$

Remark that $e = (1/3)x^3 + (1/4)y^4 + (1/2)xy^2 z + (1/6)y^3 w$ is a generating function of ϕ. In this example, $P_1 = x, P_2 = (1/2)y^2, Q_1 = (1/2)(x^2 + y^2 z), Q_2 = xz + yw + y^2$, and therefore $f = (y, x, z, w + 2y)$. Hence ϕ satisfies the required transversality condition.

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