GEOMETRY IN NONLINEAR CONTROL
AND DIFFERENTIAL INCLUSIONS
BANACH CENTER PUBLICATIONS, VOLUME 32
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1995

BV SOLUTIONS OF MULTIVALUED DIFFERENTIAL EQUATIONS ON CLOSED MOVING SETS IN BANACH SPACES

H. BENABDELLAH, C. CASTAING and M. A. GAMAL IBRAHIM

Département des Sciences Mathématiques, Case 051, Université Montpellier II
34095 Montpellier Cedex 05, France

Abstract. This paper is concerned with the existence of BV and right continuous solutions for some classes of multivalued differential equations on closed moving sets in Banach spaces.

1. Introduction. Let E be a Banach space, μ a positive Radon measure on $[0,T]\subset\mathbb{R}$ $(T>0)$. A multifunction $t\mapsto C(t)$ is given, defined on $[0,T]$ and with values in E such that the sets $C(t)$ are nonempty closed, and a multifunction F from the graph of C to E such that the sets $F(t,x)$ are nonempty convex and compact. The following problem arises from the sweeping (or Moreau) process [M2] and the calculus of variations, see Brandi–Cesari–Salvadori [BCS] and Moussaoui [Mo].

Given $x_0\in C(0)$, the problem is to find a BV and right continuous $X:[0,T]\to E$ such that $X(0)=x_0$, $X(t)=x_0+\int_{]0,t]}X'(s)\mu(ds)$, $\forall t\in[0,T]$, where X' belongs to $L^1_E([0,T],\mu)$, and

$$\begin{cases} X(t)\in C(t), & \forall t\in[0,T],\\ X'(t)\in F(t,X(t)) & \mu\text{-a.e. on } [0,T]. \end{cases} \tag{1.1}$$

In this paper we given several sufficient conditions for the existence of BV and right continuous (briefly BVRC) solutions of (1.1). When μ is the Lebesgue measure, the existence of absolutely continuous solutions of (1.1) will receive a particular treatment. After the preliminary paper by Nagumo [N], there were many results in the literature concerning absolutely continuous solutions of (1.1). We refer to Aubin–Cellina [AC], Bony [Bo], Brezis [Br], Deimling [De2], Larrieu [L], Martin [Ma], Methlouthi [Me]. For the complete bibliography of the subject, we refer to [AC] and [De2].

1991 *Mathematics Subject Classification*: 49A50, 34A60.
The paper is in final form and no version of it will be published elsewhere.

When μ is a Radon measure, the second author [Ca2] stated the existence of BVRC solutions of (1.1) in the case where F is upper semicontinuous for the right topology on the right closed graph of a multifunction C under a suitable tangential condition and proved that the existence of BVRC solutions for the sweeping process (see [M2])

$$(\mathcal{SW}) \qquad \begin{cases} X(0) = x_0 \in C(0), \\ X'(t) \in -N_{C(t)}(X(t)) \quad \mu\text{-a.e.,} \end{cases}$$

where $N_{C(t)}(X(t))$ is the normal cone of the moving closed convex set $C(t)$ in $\mathbb{R}^d$ at $X(t)$ with $t \mapsto C(t)$ bounded variation and right continuous for the Hausdorff distance, can be deduced from the existence of BVRC solutions of (1.1) by taking for F the subdifferential of the function $x \mapsto d(x, C(t))$. Further, the existence of BVRC solutions for (1.1) has applications in the calculus of variations. See Brandi–Cesari–Salvadori [BCS] and Moussaoui [Mo] and the references therein. So the previous considerations show the interest of finding BV solutions of (1.1). We also refer to Moreau's paper [M2] for the importance of the replacement of the Lebesgue measure in the sweeping process by a Radon measure for the treatment of an elastoplastic mechanical system.

This paper is organized as follows. In Section 3 some basic differential and integral inequalities are presented. In Section 4 some useful compactness results and lower semicontinuity of integral functionals are recalled. Section 5 is concerned with several new existence results for BVRC (resp. absolutely continuous) solutions of (1.1).

The main difficulties which arise in the investigation of BVRC solutions of (1.1) are important because $\dim(E)$ is infinite and the replacement of the Lebesgue measure by Radon positive measures possibly possessing some atoms, requires nonclassical techniques involving recent results concerning the differential measures of vector valued BV functions developed by Moreau–Valadier [MV3].

2. Notations and preliminaries. We introduce the following notation.

- E is a separable Banach space.
- $\overline{B}(x, r)$ is the closed ball of center x and radius r, $\overline{B}_E$ the closed unit ball.
- $c(E)$ (resp. $k(E)$, $ck(E)$, $ckw(E)$, $b(E)$) is the set of all nonempty closed (resp. compact, convex compact, convex weakly compact, bounded) subsets of E.
- $\mathcal{R}_k(E)$ (resp. $\mathcal{R}_{kw}(E)$) is the set of all nonempty closed convex subsets of E such that their intersections with any closed ball of E are compact (resp. weakly compact).
- $\overline{\text{co}}(A)$ is the closed convex hull of a subset A of E.
- $|K| := \sup\{\|x\| : x \in K\}$ where K is a subset of E.
- α is the Kuratowski measure of noncompactness defined on bounded subsets of E.

See e.g. [Ma], p. 16.

• A *subdivision* of $I := [0,T]$ is a finite sequence $(t_0, \ldots, t_n)$ such that $0 = t_0 < t_1 < \ldots < t_n = T$.

• The *variation* of a function $u : [0,T] \to E$ is the supremum over the set of subdivisions of I of the numbers $\sum_{i=1}^{n} \|u(t_i) - u(t_{i-1})\|$. The variation of u is denoted by $\mathrm{var}(u; 0, T)$. The function u has *bounded variation* (BV) if $\mathrm{var}(u; 0, T) < \infty$. If u is BV, its left hand limit $u^-(t)$ exists at any $t > 0$. By convention $u^-(0) = 0$. When u is BV and right continuous, then there is a vector measure denoted by Du (differential measure) such that

$$\forall a \le b, \qquad Du([a,b]) = u(b) - u^-(a)$$

with $Du(\{0\}) = 0$ and $\forall t$, $u(t) = u(0) + Du([0,t])$.

• μ is a positive Radon measure on $[0,T]$. If for some $u' \in L^1_E([0,T], \mu)$, one has $u(t) = u(0) + \int_{]0,t]} u'(s)\mu(ds)$, $\forall t \in [0,T]$, then u is BVRC with $Du/d\mu = u'$ where $Du/d\mu$ is the Radon–Nikodym derivative of the differential measure Du with respect to the scalar measure μ.

• A multifunction C from a topological space U to $ck(E)$ is *upper semicontinuous* (usc) at x_0 if for each $\varepsilon > 0$, there is a neighbourhood $V_\varepsilon(x_0)$ of x_0 such that $C(x) \subset C(x_0) + \varepsilon \overline{B}_E$ whenever $x \in V_\varepsilon(x_0)$.

• $[0,T]_d$ (resp. $[0,T]_g$) denotes $[0,T]$ equipped with the right (resp. left) topology

• $\mathcal{B}(X)$ denotes the σ-algebra of borelian subsets of a topological space X.

3. Basic inequalities. In this section we establish some basic differential and integral inequalities that occur in later sections.

The following result is due to M. D. P. Monteiro Marquès [MM].

LEMMA 3.1. *Let* $g \in L^1_{\mathbb{R}^+}(I, \mu)$ *and* $\beta \ge 0$ *be such that*, $\forall t$, $0 \le g(t)\mu(\{t\}) \le \beta < 1$. *Let* $\varphi \in L^\infty_{\mathbb{R}^+}(I, \mu)$ *satisfy*

$$\forall t, \qquad \varphi(t) \le \alpha + \int_{]0,t]} g(s)\varphi(s)\, \mu(ds)$$

where α *is a positive constant number. Then*

$$\forall t, \qquad \varphi(t) \le \alpha \exp\left(\frac{1}{1-\beta} \int_{]0,t]} g(s)\, \mu(ds)\right).$$

P r o o f. Let $\varepsilon > 0$. The function

$$t \mapsto \psi(t) := \alpha + \varepsilon + \int_{]0,t]} g(s)\varphi(s)\, \mu(ds)$$

is increasing, right continuous and we have $\psi \ge \alpha + \varepsilon > 0$. It follows that the function $t \mapsto \mathrm{Log}\, \psi(t)$ is increasing and BVRC. Since $D\psi$ is absolutely continuous with respect to μ and $\frac{d}{dx}(\operatorname{Log} x) = \frac{1}{x}$, thus by virtue of a result due to Moreau–Valadier ([MV2], Theorem 8, p. 16-18)

$$\frac{D \operatorname{Log} \psi}{d\mu}(t) \in \left\{ \frac{1}{x} \frac{D\psi}{d\mu}(t) : x \in [\psi^-(t), \psi(t)] \right\} \qquad \mu\text{-a.e.}$$

Since ψ is increasing, we obtain

$$0 \leq \frac{D \operatorname{Log} \psi}{d\mu}(t) \leq \frac{1}{\psi^-(t)} \frac{D\psi}{d\mu}(t) = \frac{g(t)\varphi(t)}{\psi(t) - g(t)\varphi(t)\mu(\{t\})} \qquad \mu\text{-a.e.}$$

By our assumption on g and φ, we then have

$$\frac{D \operatorname{Log} \psi}{d\mu}(t) \leq \frac{g(t)\varphi(t)}{\varphi(t) - \beta\varphi(t)} = \frac{g(t)}{1-\beta} \qquad \mu\text{-a.e.}$$

Consequently,

$$\forall t, \qquad \operatorname{Log} \psi(t) \leq \operatorname{Log}(\alpha + \varepsilon) + \frac{1}{1-\beta} \int_{]0,t]} g(s)\, \mu(ds).$$

Therefore

$$\forall t \in I, \qquad \varphi(t) \leq \psi(t) \leq (\alpha + \varepsilon) \exp\left(\frac{1}{1-\beta} \int_{]0,t]} g(s)\, \mu(ds) \right).$$

Since ε is arbitrary, it follows that

$$\forall t \in I, \qquad \varphi(t) \leq \alpha \exp\left(\frac{1}{1-\beta} \int_{]0,t]} g(s)\, \mu(ds) \right).$$

Remark 3.2. If μ is the Lebesgue measure λ on $[0,T]$, the previous inequality reduces to Gronwall's inequality by taking $\beta = 0$.

Now we introduce the function

$$\forall x \in \mathbb{R}, \qquad \theta(x) = \begin{cases} (e^x - 1)/x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then θ is increasing, continuous and $\theta(]-\infty, 0]) =]0,1]$. The following inequality is concerned with BVRC functions.

PROPOSITION 3.3. *Let $\varrho : I \to \mathbb{R}$ be a BVRC function such that $|D\varrho| \ll \mu$ and that the Radon-Nikodym density $D\varrho/d\mu$ of its differential measure $D\varrho$ with respect to μ belongs to $L^1_{\mathbb{R}}(I,\mu)$. Let a and b in $L^1_{\mathbb{R}}(I,\mu)$ be such that*

$$(*) \qquad \frac{D\varrho}{d\mu}(t) \leq \theta(-\mu(\{t\})a(t))a(t)\varrho(t) + b(t) \qquad \mu\text{-a.e.}$$

Then

$$\forall t, \qquad \varrho(t) \leq \varrho(0) \exp\Big(\int_{]0,t]} a(s)\, \mu(ds) \Big) + \int_{]0,t]} \exp\Big(\int_{[s,t]} a(r)\, \mu(dr) \Big) b(s)\, \mu(ds).$$

Proof. Set $\varphi(t) = \exp\big(-\int_{]0,t]} a(s)\,\mu(ds)\big)$ and $f(t) = -\int_{]0,t]} a(s)\,\mu(ds)$, $\forall t \in I$.

By virtue of a result due to Moreau–Valadier ([MV2], Theorem 3.7, p. 16-15) the function φ is BVRC and

$$\begin{aligned}\frac{D\varphi}{d\mu}(t) &= \frac{Df}{d\mu}(t)\int_0^1 \exp(rf^-(t)+(1-r)f^+(t))dr\\ &= -a(t)\int_0^1 \exp(f(t)+r\mu(\{t\})a(t))dr.\end{aligned}$$

If $\mu(\{t\})a(t)=0$, then $\frac{D\varphi}{d\mu}(t) = -a(t)\varphi^-(t)$. If $\mu(\{t\})a(t)\neq 0$, then

$$\begin{aligned}\frac{D\varphi}{d\mu} &= -a(t)\bigg[\frac{1}{\mu(\{t\})a(t)}\exp(f(t)+r\mu(\{t\})a(t))\bigg]_{r=0}^{r=1}\\ &= \frac{-a(t)}{\mu(\{t\})a(t)}[\exp f^-(t) - \exp f(t)]\\ &= -a(t)\exp(f^-(t))\frac{1-\exp(-\mu(\{t\})a(t))}{\mu(\{t\})a(t)}\\ &= -\theta(-\mu(\{t\})a(t))a(t)\varphi^-(t).\end{aligned}$$

Finally, for all t we get

$$\frac{D\varphi}{d\mu}(t) = -\theta(-\mu(\{t\})a(t))a(t)\varphi^-(t). \tag{3.3.1}$$

Since φ^- is ≥ 0, by our assumption $(*)$ we have

$$\varphi^-(t)\frac{D\varrho}{d\mu}(t) \leq \theta(-\mu(\{t\})a(t))a(t)\varrho(t)\cdot\varphi^-(t) + b(t)\varphi^-(t)$$

for μ-a.e. t. By (3.3.1) it follows that

$$\varphi^-(t)\frac{D\varrho}{d\mu}(t) + \frac{D\varphi}{d\mu}(t)\varrho(t) - \varphi^-(t)b(t) \leq 0 \qquad \mu\text{-a.e.} \tag{3.3.2}$$

Let us introduce the function

$$t \mapsto \Phi(t) := \varphi(t)\varrho(t) - \int_{]0,t]} \varphi^-(s)b(s)\,\mu(ds).$$

Since φ and ϱ are BVRC, the function $\varphi\varrho$ is BVRC too. Moreover, by a formula due to Moreau ([M1], Proposition 5.b) concerning the differential measure of the product $\varphi\varrho$, we have

$$D(\varphi\varrho) = \varphi^- D\varrho + \varrho D\varphi.$$

It follows that

$$\frac{D(\varphi\varrho)}{d\mu} = \varphi^-\frac{D\varrho}{d\mu} + \varrho\frac{D\varphi}{d\mu}. \tag{3.3.3}$$

Further the function $t \mapsto \int_{]0,t]} \varphi^-(s)b(s)\,\mu(ds)$ is BVRC and the Radon–Nikodym density of its differential measure with respect to μ is $\varphi^- b$. So by (3.3.3) we get

$$\frac{D\Phi}{d\mu} = \frac{D(\varphi\varrho)}{d\mu} - \varphi^- b = \varphi^- \frac{D\varrho}{d\mu} + \varrho\frac{D\varphi}{d\mu} - \varphi^- b$$

so that inequality (3.3.2) is equivalent to

$$\frac{D\Phi}{d\mu}(t) \le 0 \qquad \mu\text{-a.e.} \tag{3.3.4}$$

Integrating we get

$$\forall t \in I, \qquad \Phi(t) - \Phi(0) = \int\limits_{]0,t]} \frac{D\Phi}{d\mu}(s)\,\mu(ds).$$

Hence

$$\forall t \in I, \qquad \varphi(t)\varrho(t) - \int\limits_{]0,t]} \varphi^-(s)b(s)\,\mu(ds) - \varrho(0) \le 0,$$

that is,

$$\forall t \in I, \qquad \varrho(t) \le \frac{1}{\varphi(t)}\varrho(0) + \frac{1}{\varphi(t)} \int\limits_{]0,t]} \varphi^-(s)b(s)\,\mu(ds).$$

Since $1/\varphi(t) = \exp\big(\int_{]0,t]} a(s)\,\mu(ds)\big)$, we have

$$\forall t \in I, \qquad \frac{1}{\varphi(t)} \int\limits_{]0,t]} \varphi^-(s)b(s)\,\mu(ds) = \int\limits_{]0,t]} \exp\Big(\int\limits_{[s,t]} a(r)\,\mu(dr)\Big) b(s)\,\mu(ds).$$

Finally, we get

$$\forall t \in I, \qquad \varrho(t) \le \varrho(0)\exp\Big(\int\limits_{]0,t]} a(s)\,\mu(ds)\Big) + \int\limits_{]0,t]} \exp\Big(\int\limits_{[s,t]} a(r)\,\mu(dr)\Big) b(s)\,\mu(ds)$$

as desired.

The following result is an application of Proposition 3.3.

Proposition 3.4. *Let $\varrho \in L^\infty_{\mathbb{R}}(I,\mu)$, $a \in L^1_{\mathbb{R}}(I,\mu)$ and $c \in L^1_{\mathbb{R}^+}(I,\mu)$ be such that the product ac belongs to $L^1_{\mathbb{R}}(I,\mu)$ and*

$$\forall t \in I, \qquad \varrho(t) \le a(t) + \int\limits_{]0,t]} \theta(-\mu(\{s\})c(s))c(s)\varrho(s)\,\mu(ds).$$

Then

$$\forall t \in I, \qquad \varrho(t) \le a(t) + \int\limits_{]0,t]} \exp\Big(\int\limits_{[s,t]} c\,d\mu\Big)\theta(-\mu(\{s\})c(s))c(s)a(s)\,\mu(ds).$$

In particular, if $\forall t \in I$, $a(t) = a_0$, then

$$\forall t \in I, \qquad \varrho(t) \le a_0 \exp\Big(\int\limits_{]0,t]} c(s)\,\mu(ds)\Big).$$

Comment. If μ is the Lebesgue measure λ on $[0,T]$ and if a is nonnegative, then the inequality in Proposition 3.4 is reduced to the following Gronwall's inequality

$$\forall t \in I, \qquad \varrho(t) \le a(t) + \int\limits_{]0,t]} \exp\Big(\int\limits_{[s,t]} c(r)\lambda(dr)\Big)c(s)a(s)\,\lambda(ds)$$

since c is nonnegative and $0 < \theta(-\mu(\{t\})c(t)) \le 1$ for all $t \in I$.

Proof. Set

$$\forall t \in I, \qquad q(t) = \int\limits_{]0,t]} \theta(-\mu(\{s\})c(s))c(s)\varrho(s)\,ds.$$

Then

$$\frac{Dq}{d\mu}(t) = \theta(-\mu(\{t\})c(t))c(t)\varrho(t) \qquad \mu\text{-a.e.}$$

By our assumption, we have $\varrho(t) \le a(t) + q(t)$, $\forall t \in I$. Therefore, we obtain

$$\frac{Dq}{d\mu}(t) \le \theta(-\mu(\{t\})c(t))c(t)q(t) + b(t) \qquad \mu\text{-a.e.} \tag{3.4.1}$$

where $b(t) = \theta(-\mu(\{t\})c(t))c(t)a(t)$, $\forall t \in I$. Since q is BVRC, a and c belong to $L^1_{\mathbb{R}}(I,\mu)$, it follows from (3.4.1) and Proposition 3.3 that

$$\forall t \in I, \qquad q(t) \le \int\limits_{]0,t]} \exp\Big(\int\limits_{[s,t]} c(r)\,\mu(dr)\Big)b(s)\,\mu(ds)$$

because q is BVRC and $q(0) = 0$. Since for all $t \in I$, $\varrho(t) \le a(t) + q(t)$, we obtain the desired inequality

$$\forall t \in I, \qquad \varrho(t) \le a(t) + \int\limits_{]0,t]} \exp\Big(\int\limits_{[s,t]} c(r)\,\mu(dr)\Big)\theta(-\mu(\{s\})c(s))c(s)a(s)\,\mu(ds).$$

If $a(t) = a_0$, $\forall t \in I$, we get

$$\begin{aligned}\varrho(t) &\le a_0 + \int\limits_{]0,t]} \exp\Big(\int\limits_{[s,t]} c(r)\,\mu(dr)\Big)b(s)\,\mu(ds)\\ &= a_0 + a_0 \int\limits_{]0,t]} \exp\Big(\int\limits_{]0,t]} c\,d\mu - \int\limits_{]0,s]} c\,d\mu\Big)\theta(-\mu(\{s\})c(s))c(s)\,\mu(ds).\end{aligned}$$

Now set, $\forall t \in I$, $\varphi(t) = \exp\big(-\int_{]0,t]} c\,d\mu\big)$. Then by (3.3.1) of the proof of Proposition 3.3, we have

$$\forall t \in I, \qquad \frac{D\varphi}{d\mu}(t) = -\theta(-\mu(\{t\})c(t))c(t)\varphi^-(t).$$

It follows that for all $t \in I$,

$$\varrho(t) \le a_0 + a_0 \exp\Big(\int\limits_{]0,t]} c\,d\mu\Big) \int\limits_{]0,t]} \Big(-\frac{D\varphi}{d\mu}(s)\Big)\mu(ds)$$

$$= a_0 + a_0 \exp\Big(\int\limits_{]0,t]} c\,d\mu\Big)(1-\varphi(t)) = a_0 \exp \int\limits_{]0,t]} c\,d\mu,$$

which the desired inequality.

Here is a useful corollary of Proposition 3.4. Let us introduce the function

$$\overline{\theta}(x) = \begin{cases} \dfrac{\mathrm{Log}(1+x)}{x} & \text{if } x \in]-1,0[\cup]0,\infty[, \\ 1 & \text{if } x = 0. \end{cases}$$

It is obvious that $\overline{\theta}$ is a strictly decreasing and continuous function such that $\overline{\theta}(]-1,0]) = [1,\infty[$.

COROLLARY 3.5. *Let* $\varphi \in L^\infty_{\mathbb{R}}(I,\mu)$, $a \in \mathbb{R}$ *and* $g \in L^1_{\mathbb{R}+}(I,\mu)$ *be such that*

$$\forall t \in I, \qquad \mu(\{t\})g(t) < 1 \quad \textit{and} \quad \int\limits_I \overline{\theta}(-\mu(\{t\})g(t))g(t)\,\mu(dt) < \infty.$$

Suppose further that

$$\forall t \in I, \qquad \varphi(t) \le a + \int\limits_{]0,t]} g(s)\varphi(s)\,\mu(ds).$$

Then

$$\forall t \in I, \qquad \varphi(t) \le a \exp \int\limits_{]0,t]} \overline{\theta}(-\mu(\{s\})g(s))g(s)\,\mu(ds).$$

Proof. Set $c(t) = \overline{\theta}(-\mu(\{t\})g(t))g(t)$, $\forall t \in I$. Then by our assumptions $c \in L^1_{\mathbb{R}}(I,\mu)$ and by obvious properties of functions θ and $\overline{\theta}$, we have $g(t) = \theta(-\mu(\{t\})c(t))c(t)$, $\forall t \in I$, so that

$$\forall t \in I, \qquad \varphi(t) \le a + \int\limits_{]0,t]} \theta(-\mu(\{s\})c(s))c(s)\varphi(s)\,\mu(ds).$$

Then by Proposition 3.4, it is immediate that

$$\forall t \in I, \qquad \varphi(t) \le a \exp \int\limits_{]0,t]} \overline{\theta}(-\mu(\{s\})g(s))g(s)\,\mu(ds).$$

Remark. Corollary 3.5 generalizes Lemma 3.1 since the condition $0 \le g(t)\mu(\{t\}) \le \beta < 1$, for all $t \in I$, implies that the function $t \mapsto \overline{\theta}(-\mu(\{t\})g(t))g(t)$ is integrable. Indeed, $-\beta \in]-1,0]$ and since $\overline{\theta} :]-1,0] \to [1,\infty[$ is a decreasing function, we have

$$\forall t \in I, \qquad \overline{\theta}(-\mu(\{t\})g(t))g(t) \le \overline{\theta}(-\beta)g(t).$$

There is another variant of the previous result.

PROPOSITION 3.6. *Let* $c \in L^1_{\mathbb{R}+}(I,\mu)$, $p \in L^\infty_{\mathbb{R}+}(I,\mu)$ *and* α *a positive number. Assume that* $\forall t \in I$, $p(t) \le \alpha + \int_{]0,t[} p(s)c(s)\,\mu(ds)$. *Then for all* t *in* I, $p(t) \le \alpha \exp(\int_{]0,t]} c(s)\,\mu(ds))$.

P r o o f. Assume first that α is strictly positive. Define

$$\forall t \in I, \qquad f(t) = \alpha + \int_{]0,t]} c(s)p(s)\,\mu(ds),$$
$$\forall x \in]0,\infty[, \qquad \gamma(x) = -\operatorname{Log}(x).$$

Then f is BVRC and $Df/d\mu = cp$. Since γ is continuous and convex on $]0,\infty[$, by Moreau–Valadier ([MV2], Theorem 2.5, p. 16-11), $\gamma \circ f$ is BV, $D(\gamma \circ f)$ is absolutely continuous with respect to μ and for μ-a.e. t, we have

$$\frac{D(\gamma \circ f)}{d\mu}(t) \geq \sup\Big\{\Big\langle g, \frac{Df}{d\mu}(t)\Big\rangle : g \in \partial\gamma(f^-(t))\Big\} = \Big\langle -\frac{1}{f^-(t)}, \frac{Df}{d\mu}(t)\Big\rangle$$
$$= -\frac{c(t)p(t)}{\alpha + \int_{]0,t[} c(s)p(s)\,\mu(ds)}.$$

Hence

$$\frac{D(\gamma \circ f)}{d\mu}(t) \geq -c(t) \quad \text{if } p(t) > 0 \quad \text{and} \quad \frac{D(\gamma \circ f)}{d\mu}(t) \geq 0 \geq -c(t) \quad \text{if } p(t) = 0.$$

So

$$\frac{D(\gamma \circ f)}{d\mu}(f) \geq -c(t) \qquad \mu\text{-a.e.}$$

This implies that $(\gamma \circ f)(t) - (\gamma \circ f)(0) \geq -\int_{]0,t]} c(s)\,\mu(ds)$ for all $t \in I$. Hence we obtain

$$\operatorname{Log}\Big[\alpha^{-1}\Big(\alpha + \int_{]0,t]} c(s)p(s)\,\mu(ds)\Big)\Big] \leq \int_{]0,t]} c(s)\,\mu(ds).$$

Consequently,

$$\alpha^{-1}p(t) \leq \alpha^{-1}\Big(\alpha + \int_{]0,t]} c(s)p(s)\,\mu(ds)\Big) \leq \exp\Big(\int_{]0,t]} c(s)\,\mu(ds)\Big),$$

which is the stated inequality for $\alpha > 0$.

If $\alpha = 0$, we have $p(t) \leq \varepsilon + \int_{]0,t]} c(s)p(s)\,\mu(ds)$ for any $\varepsilon > 0$ and any t. Hence

$$\forall t \in I, \qquad p(t) \leq \varepsilon \exp\Big(\int_{]0,t]} c(s)\,\mu(ds)\Big).$$

Then $p(t) = 0$ for all $t = I$.

We retain the previous notations. Let $c \in L^1_{\mathbb{R}^+}(I,\mu)$ and define $\omega : I \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\omega(t,x) = \theta(-\mu(\{t\})c(t))c(t)x.$$

Further, let $r : I \to \mathbb{R}^+$ with $r(0) = 0$ and $\forall t \in]0,T]$, $r(t) = \int_{]0,T]} r'(s)\,\mu(ds)$ where $r' \in L^1_{\mathbb{R}}(I,\mu)$ satisfies $r'(t) \leq \omega(t, r(t))$ μ-a.e. Then by Proposition 3.3, we have

$$\forall t \in I, \qquad r(t) \le r(0) \exp\Big(\int\limits_{]0,t]} c(s)\, \mu(ds)\Big).$$

Hence $r \equiv 0$. This leads us to the following definition.

DEFINITION 3.7. Let $\mathrm{KAM}(I,\mu)$ be the set of all Carathéodory mappings $\omega : I \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\forall t \in I$, $\omega(t,0) = 0$ and that the only function $r : I \to \mathbb{R}^+$ with $r(0) = 0$ satisfying $r(t) = \int_{]0,t]} r'(s)\, \mu(ds)$, $\forall t \in]0,T]$ where $r' \in L^1_{\mathbb{R}}(I,\mu)$ and $r'(t) \le \omega(t, r(t))$ μ-a.e., is the function identically equal to zero. The functions $\omega \in \mathrm{KAM}(I,\mu)$ are called *Kamke functions*.

4. Lower semicontinuity and compactness results. In this section we recall some useful results on the lower semicontinuity of integral functionals and on the weak compactness theorems in L^1_E.

DEFINITIONS 4.1 ([ACV], p. 174). A subset $\mathcal{H}$ of $L^1_E(I,\mu)$ is $\mathcal{R}_{kw}(E)$-*tight* if for any $\varepsilon > 0$ there is a measurable multifunction L_ε from I to $\mathcal{R}_{kw}(E)$ such that

$$\forall u \in \mathcal{H}, \qquad \mu\{t \in I : u(t) \notin L_\varepsilon(t)\} \le \varepsilon.$$

THEOREM 4.2 ([Ca1, Ca3], [ACV], Théorème 6). *Let $\mathcal{H}$ be a bounded uniformly integrable and $\mathcal{R}_{kw}(E)$-tight subset of $L^1_E(I,\mu)$. Then $\mathcal{H}$ is relatively weakly compact and if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}$, there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ which is weakly convergent.*

THEOREM 4.3. *Let Γ be a measurable multifunction from I to $ck(E)$ such that $|\Gamma|$ is integrable where $|\Gamma|(t) = |\Gamma(t)|$ for all t in I. Let*

$$\int\limits_I \Gamma(t)\, \mu(dt) := \Big\{ \int\limits_I \sigma(t)\, \mu(dt) : \sigma \in S^1_\Gamma \Big\}$$

where S^1_Γ is the set of integrable selections of Γ. Then $\int_1 \Gamma(t)\, \mu(dt)$ is a convex compact subset of E.

Proof. By Theorem 4.2, S^1_Γ is convex weakly compact in $L^1_E(I,\mu)$. Hence $\int_I \Gamma(t)\, \mu(dt)$ is convex weakly compact in E. For any x' in E', by Strassen's formula ([CV], Theorem V.14), we have $\delta^*(x', \int_I \Gamma(t)\, \mu(dt)) = \int_I \delta^*(x', \Gamma(t))\, \mu(dt)$. By our assumption, it follows from Lebesgue's theorem that $\delta^*(\cdot, \int_I \Gamma(t)\, \mu(dt))$ is continuous on the unit ball B' of E' for the topology of compact convergence. Hence $\int_I \Gamma(t)\, \mu(dt)$ is convex compact.

Remark. Theorem 4.3 is actually valid if Γ is scalarly integrable satisfying (i) for every $g \in L^\infty_{\mathbb{R}}(I,\mu)$ and for every scalarly integrable selection f of Γ, the weak integral $\int fg\, d\mu$ belongs to E and (ii) the set $\{\delta^*(x', \Gamma) : x' \in \overline{B}_{E'}\}$ is uniformly integrable in $L^1_{\mathbb{R}}(I,\mu)$.

The following theorem occurs frequently in the proof of convergence of the approximated solutions in the last section.

THEOREM 4.4. *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded uniformly integrable sequence in $L^1_E(I,\mu)$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $]0,1]$ with $\lim_{n\to\infty}\varepsilon_n = 0$ and $\Gamma : I \to ckw(E)$. Assume that*

$$(*)\qquad \forall n,\quad u_n(t)\in\Gamma(t)+\varepsilon_n\overline{B}_E\quad \mu\text{-a.e.}$$

Then the sequence $(u_n)_{n\in\mathbb{N}}$ is relatively $\sigma(L^1,L^\infty)$-compact.

Proof. Since $\lim_{n\to\infty}\varepsilon_n = 0$, it is immediate by the Grothendieck Lemma ([G], p. 296) that the sequence $(u_n(t))_{n\in\mathbb{N}}$ is relatively weakly compact by our assumption $(*)$ for almost all t. Since $(u_n)_{n\in\mathbb{N}}$ is bounded and uniformly integrable in $L^1_E(I,\mu)$, $(u_n)_{n\in\mathbb{N}}$ is relatively $\sigma(L^1,L^\infty)$-compact by Theorem 4.2.

The following lower semicontinuity result will be used later.

THEOREM 4.5 ([V], Theorem 3, p. 3.6). *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of measurable function from I to E which converges in measure to u_∞. Let $(v_n)_{n\in\mathbb{N}}$ be an $\mathcal{R}_{kw}(E)$-tight sequence in $L^1_E(I,\mu)$ which converges $\sigma(L^1,L^\infty)$ to v_∞. If ψ is a $\mathcal{T}_\mu(I)\otimes\mathcal{B}(E)\otimes\mathcal{B}(E)$-measurable integrand on $I\times E\times E$, lower semicontinuous on $E\times(E,\sigma(E,E'))$, if $\psi(t,u_\infty(t),\cdot)$ is a.e. convex and if the sequence $(\psi(\cdot,u_n(\cdot),v_n(\cdot))^-)_{n\in\mathbb{N}}$ is uniformly integrable then*

$$\int_I \psi(t,u_\infty(t),v_\infty(t))\,\mu(dt)\le\liminf_{n\to\infty}\int_I \psi(t,u_n(t),v_n(t))\,\mu(dt).$$

Here is a useful application of Theorem 4.5.

THEOREM 4.6. *Let F be a multifunction from $I\times E$ to the set of nonempty closed convex subset of E satisfying*:

(i) *F is $\mathcal{T}_\mu(F)\otimes\mathcal{B}(E)$-measurable.*
(ii) *For any $t\in I$, $F(t,\cdot)$ is scalarly upper semicontinuous.*

If $(u_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions from I to E which converges in measure to u_∞, if $(v_n)_{n\in\mathbb{N}}$ is an $\mathcal{R}_{kw}(E)$-tight sequence in $L^1_E(I,\mu)$ which converges $\sigma(L^1,L^\infty)$ to v_∞, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of measurable sets in I such that $\lim_{n\to\infty}\mu(A_n)=\mu(I)$ and such that for each $n\in\mathbb{N}$, $v_n(t)\in F(t,u_n(t))$ for a.e. t in A_n, and if the sequence $(d(v_n(\cdot),F(\cdot,u_n(\cdot))))_{n\in\mathbb{N}}$ is uniformly integrable, then

$$v_\infty(t)\in F(t,u_\infty(t))\qquad \mu\text{-a.e.}$$

Proof. By our assumption, the integrand $\psi:(t,x,y)\mapsto d(y,F(t,x))$ is $\mathcal{T}_\mu(I)\otimes\mathcal{B}(E)\otimes\mathcal{B}(E)$-measurable and $\psi(t,\cdot,\cdot)$ is $E\times(E,\sigma(E,E'))$ lower semicontinuous and $\psi(t,x,\cdot)$ is convex. For each n, we have

$$(*)\qquad \int_I d(v_n(t),F(t,u_n(t))\,\mu(dt)=\int_{A_n} d(v_n(t),F(t,u_n(t))\,\mu(dt) + \int_{I\setminus A_n} d(v_n(t),F(t,u_n(t)))\,\mu(dt).$$

Since $(d(v_n(\cdot), F(\cdot, u_n(\cdot))))_{n\in\mathbb{N}}$ is uniformly integrable and $\lim_{n\to\infty} \mu(I \setminus A_n) = 0$, we get

$$\liminf_{n\to\infty} \int_I d(v_n(t), F(t, u_n(t))\, \mu(dt) \le 0$$

since the first integral in $(*)$ over A_n is equal to zero. Then by Theorem 4.5 we obtain

$$\int_I d(v_\infty(t), F(t, u_\infty(t))\, \mu(dt) \le \liminf_{n\to\infty} \int_I d(v_n(t), F(t, u_n(t))\, \mu(dt) \le 0.$$

5. Existence theorem

5.a. *BV solutions.* Let us introduce some notations:

$$I = [0, T], \qquad I^\bullet = [0, T[,$$
$$I^+_{t,\varepsilon} = I \cap [t, t+\varepsilon], \quad I^-_{t,\varepsilon} = I \cap [t-\varepsilon, t] \quad \text{for } t \in I \text{ and } \varepsilon > 0.$$

Let $C : I \to c(E)$ be a multifunction and G its graph. Define $G^\bullet := G \cap [0, T[\times E$. For any $\tau \in I$ and any $\varepsilon > 0$, denote by $J^-_\varepsilon([0,\tau])$ (resp. $J^+_\varepsilon([0,\tau])$ the set of all increasing right continuous (resp. left continuous) functions $\theta : [0,\tau] \to [0,\tau]$ such that $\theta(0) = 0$, $\theta(\tau) = \tau$ and for all $t \in [0,\tau]$, $\theta(t) \in [t-\varepsilon, t]$ (resp. $\theta(t) \in [t, t+\varepsilon]$). It is obvious that $J^-_\varepsilon([0,\tau])$ and $J^+_\varepsilon([0,\tau])$ are nonempty.

Let us recall the following two functions:

$$\chi(t) = \begin{cases} (e^t - 1)/t & \text{if } t \in \mathbb{R} \setminus \{0\}, \\ 1 & \text{if } t = 0, \end{cases}$$
$$\overline{\chi} = \begin{cases} (\mathrm{Log}(1+t))/t & \text{if } t \in]-1, 0[\cup]0, +\infty[, \\ 1 & \text{if } t = 0. \end{cases}$$

Then $\chi : \mathbb{R} \to \mathbb{R}^{+*}$ and $\overline{\chi} :]-1, +\infty[\to \mathbb{R}^{+*}$ are continuous, strictly increasing and strictly decreasing respectively and such that

$$\chi(]-\infty, 0]) =]0, 1] \quad \text{and} \quad \overline{\chi}(]-1, 0]) = [1, +\infty[.$$

Let $g : I \to \mathbb{R}^+$ be a measurable function. By obvious properties of χ and $\overline{\chi}$, it can be checked that the following two conditions are equivalent

(a) $\forall t \in I$, $0 \le \mu(\{t\})g(t) < 1$ and $\int_I \overline{\chi}[-\mu(\{t\})g(t)]g(t)\mu(dt) < \infty$.
(b) There is c in $L^1_{\mathbb{R}^+}(I, \mu)$ such that $\forall t \in I$, $g(t) = \chi[-\mu(\{t\})c(t)]c(t)$.

Indeed, it is enough to note that for any positive measurable function c on I, we have

$$\forall t \in I, \qquad c(t) = \overline{\chi}[-\mu(\{t\})g(t)]g(t) \Leftrightarrow g(t) = \chi[-\mu(\{t\})c(t)]c(t).$$

Our first result is the basic existence of ε-approximated BV and right continuous (BVRC) solutions to the problem (1.1).

Proposition 5.1. *Let $C : I \to c(E)$ be a multifunction with left closed graph G, that is, G is closed in $[0, T]_g \times E$. Let c in $L^1_{\mathbb{R}^+}(I, \mu)$ be such that $\forall t \in I$, $0 \le \mu(\{t\})c(t) < 1$ and $\int_I \overline{\chi}[-\mu(\{t\})c(t)]c(t)\, \mu(dt) < \infty$. Let $F : I \times E \to ck(E)$*

be such that, $\forall x \in E$, $F(\cdot, x)$ is scalarly μ-measurable. Suppose also that the following conditions are satisfied:

(C.1) $\forall (t,x) \in I \times E$, $|F(t,x)| \le c(t)(1 + \|x\|)$.

(C.2) $\forall (t,x) \in G$, $\inf_{y \in C(t)} d(y, x + \mu(\{t\})F(t,x)) = 0$.

(C.3) *For each (t,x) in $G^\bullet$, each $\varepsilon > 0$, there is $(t_\varepsilon, x_\varepsilon) \in G$ such that $0 < t_\varepsilon - t \le \varepsilon$ and that*

$$x_\varepsilon - x \in \int\limits_{]t,t_\varepsilon[} F(s,x)\,\mu(ds) + \mu(]t,t_\varepsilon[)\varepsilon \overline{B}_E.$$

Then for any $x_0 \in C(0)$, there exists a constant $m > 0$ such that for any $\varepsilon \in]0,1]$, there are $\theta \in J_\varepsilon^-([0,T])$ and a BVRC function $X : I \to E$ with the following properties:

(i) $\forall t \in I$, $X(t) = x_0 + \int_{]0,t]} X'(s)\,\mu(ds)$ *with* $X' \in L^1_E(I,\mu)$.
(ii) $\forall t \in I$, $X^-(\theta(t)) \in C(\theta(t))$.
(iii) $\|X'(t)\| \le mc(t) + 1$ *μ-a.e.*
(iv) $X'(t) \in F(t, X^-(\theta(t))) + \varepsilon \overline{B}_E$ *μ-a.e.*

Proof. Let $\varepsilon > 0$ and $\tau \in I$. Denote by $\mathcal{P}_\varepsilon([0,\tau])$ the set of all pairs (θ, X) where θ belongs to $J_\varepsilon^-([0,\tau])$ and X is a mapping from $[0,\tau]$ to E such that

$$\forall t \in [0,\tau], \quad X(t) = x_0 + \int\limits_{]0,t]} X'(s)\,\mu(ds) \quad \text{with } X' \in L^1_E(I,\mu),$$
$$\forall t \in [0,\tau], \qquad X^-(\theta(t)) \in C(\theta(t))$$
$$X'(t) \in F(t, X^-(\theta(t))) + \varepsilon \overline{B}_E \qquad \mu\text{-a.e. } t \in [0,\tau].$$

Then for establishing our proposition we need to prove the following assertions:

(A) *There is a constant $m > 1$ such that for any $\varepsilon \in]0,1]$, any $\tau \in I$ and any (θ, X) in $\mathcal{P}_\varepsilon([0,\tau])$, we have $\forall t \in [0,\tau]$, $F(t, X^-(\theta(t))) \subset mc(t)\overline{B}_E$.*

(B) *For any $\varepsilon \in]0,1]$, the set $\mathcal{P}_\varepsilon([0,T])$ is not empty.*

Let $\varepsilon \in]0,1]$, $\tau \in I$ and $(\theta, X) \in \mathcal{P}_\varepsilon([0,\tau])$. Clearly $X^-(\theta(\cdot))$ belongs to $L^\infty_E(I,\mu)$ and

$$\|X^-(\theta(t))\| + 1 \le 1 + \|x_0\| + \int\limits_{]0,\theta(t)[} \|X'(s)\|\,\mu(ds) \le 1 + \|x_0\| + \int\limits_{]0,t[} \|X'(s)\|\,\mu(ds).$$

By (C.1) and by the definition of $\mathcal{P}_\varepsilon([0,\tau])$ we have

$$\forall t \in [0,\tau], \qquad \|X^-(\theta(t))\| + 1 \le 1 + \|x_0\| + \varepsilon\mu(I) + \int\limits_{]0,t]} c(s)(1 + \|X^-(\theta(s))\|)\,\mu(ds).$$

It follows from Proposition 3.4 that

$$\forall t \in [0,\tau], \qquad \|X^-(\theta(t))\| + 1 \le (1 + \|x_0\| + \varepsilon\mu(I)) \exp\Big(\int\limits_I g(s)\,\mu(ds)\Big)$$

where $\forall t \in I$, $g(t) = \overline{\chi}(-\mu(\{t\})c(t))c(t)$. Set $m = (1 + \|x_0\| + \mu(I))\exp(\|g\|_1)$. Then according to (C.1) we get

$$\forall t \in [0,\tau], \qquad F(t, X^-(\theta(t))) \subset c(t)(1 + \|X^-(\theta(t))\|)\overline{B}_E \subset mc(t)\overline{B}_E.$$

This proves assertion (A).

Let $\varepsilon \in]0,1]$ and denote by $\mathcal{P}_\varepsilon$ the union of all sets $\mathcal{P}_\varepsilon([0,\tau])$ for $\tau \in I$. Clearly $\mathcal{P}_\varepsilon$ is not empty. We introduce a partial order in $\mathcal{P}_\varepsilon$ as follows. For any pair (θ_i, X_i) in $\mathcal{P}_\varepsilon$ with $(\theta_i, X_i) \in \mathcal{P}_\varepsilon([0,\tau_i])$ $(i = 1,2)$, let $(\theta_1, X_1) \le (\theta_2, X_2)$ if $\tau_1 \le \tau_2$, $\theta_1 = \theta_2|_{[0,\tau_1]}$ and $X_1 = X_2|_{[0,\tau_1]}$. Let $\mathcal{C} := \{(\theta_\gamma, X_\gamma) : \gamma \in D\}$ be a chain (totally ordered subset) in $\mathcal{P}_\varepsilon$. Let $\tau = \sup_{\gamma\in D} \tau_\gamma$. If there is an element $\widetilde{\gamma}$ in D such that $\tau = \tau_{\tilde{\gamma}}$, then $(\theta_{\tilde{\gamma}}, X_{\tilde{\gamma}})$ is an upper bound for $\mathcal{C}$. Suppose that, $\forall \gamma \in D$, $\tau_\gamma < \tau$. Define $\theta : [0,\tau] \to [0,\tau]$ and $X : [0,\tau[\to E$ by

$$\forall \gamma \in D, \qquad \theta|_{[0,\tau_\gamma]} = \theta_\gamma \quad \text{and} \quad \theta(\tau) = \tau,$$
$$\forall \gamma \in D, \qquad X|_{[0,\tau_\gamma]} = X_\gamma.$$

Then θ belongs to $J_\varepsilon^-([0,\tau])$. Now we prove that $\mathcal{C}$ admits an upper bound. Choose a sequence $(k_n)_{n\in\mathbb{N}}$ in D such that $\forall n \in \mathbb{N}$, $\tau_{k_n} < \tau_{k_{n+1}}$ and $\tau = \sup \tau_{k_n}$. For any positive integers $m < n$, we have $X'_{k_n} = X'_{k_m}$ μ-a.e. on $[0,\tau_{k_m}]$. Let N be a μ-negligible set such that $X'_{k_n} = X'_{k_m}$ for all integers $m < n$ and all $t \in [0,\tau_{k_m}]\backslash N$. Then define a mapping $X' : [0,\tau[\to E$ by $\forall n \in \mathbb{N}$, $\forall t \in [0,\tau_{k_n}]\backslash N$, $X'(t) = X'_{k_n}(t)$ and $\forall t \in N$, $X'(t) = 0$. Clearly X' is measurable and $\|X'(t)\| \le mc(t) + 1$ μ-a.e. thanks to (A). Then we have

$$\forall t \in [0,\tau[, \qquad X(t) = x_0 + \int_{]0,t]} X'(s)\,\mu(ds),$$
$$\forall t \in [0,\tau[, \qquad X^-(\theta(t)) \in C(\theta(t)),$$
$$X'(t) \in F(t, X'(\theta(t)) + \varepsilon\overline{B}_E \qquad \mu\text{-a.e. on } [0,\tau[.$$

Let

$$X^-(t) = x_0 + \int_{]0,t[} X'(s)\,\mu(ds), \qquad \forall t \in [0,\tau].$$

Then for (t,t') in $[0,\tau[^2$ with $t < t'$, we have

$$\|X^-(t') - X^-(t)\| \le \int_{[t,t'[} (mc(s) + 1)\,\mu(ds).$$

It follows that the left limit, u, of $t \mapsto X^-(t)$ at τ exists and we have

$$u = \lim_{\substack{t\to\tau \\ t<\tau}} \Big(x_0 + \int_{]0,t[} X'(s)\,\mu(ds)\Big) = \lim_{n\to\infty} \Big(x_0 + \int_{]0,\tau_{k_n}[} X'(s)\,\mu(ds)\Big).$$

Since $X^-(\tau_{k_n}) \in C(\tau_{k_n})$ and G is left closed, $(\tau, u) \in G$. Now we extend X' and X to $[0,\tau]$ (without changing the notations) so that the previous relations hold for $[0,\tau]$. If $\mu(\{\tau\}) = 0$, define $X'(\tau) = 0$ and $X(\tau) = u$. Then $X(\tau) = x_0 + \int_{]0,\tau]} X'(s)\,\mu(ds)$ and $X^-(\tau) = u \in C(\tau)$ and also $X'(t) \in F(t, X^-(\theta(t))) + \varepsilon\overline{B}_E$

μ-a.e. on $[0,\tau]$. Assume now that $\mu(\{\tau\}) > 0$. Thanks to condition (C.2), choose $y_\varepsilon \in C(\tau)$ such that

$$d(y_\varepsilon, u + \mu(\{\tau\})F(\tau, u)) < \mu(\{\tau\})\varepsilon.$$

Define $X'(\tau) := \frac{1}{\mu(\{\tau\})}(y_\varepsilon - u)$ and $X(\tau) = y_\varepsilon$. Then

$$\begin{aligned} X(\tau) = u + \mu(\{\tau\})X'(\tau) &= x_0 + \int\limits_{]0,\tau[} X'(s)\,\mu(ds) + \mu(\{\tau\})X'(\tau) \\ &= x_0 + \int\limits_{]0,\tau]} X'(s)\,\mu(ds), \end{aligned}$$

so that $X^-(\tau) = u \in C(\tau)$. Moreover, we have

$$\begin{aligned} d(X'(\tau), F(\tau, X^-(\tau))) &= d\biggl(\frac{y_\varepsilon - u}{\mu(\{\tau\})}, F(\tau, u)\biggr) \\ &= \frac{1}{\mu(\{\tau\})} d(y_\varepsilon, u + \mu(\{\tau\})F(\tau, u)) < \varepsilon, \end{aligned}$$

that is, $X'(\tau) \in F(\tau, X^-(\tau)) + \varepsilon\overline{B}_E$. Hence we can extend (θ, X) to $[0,\tau]$ in such a way that (θ, X) belongs to $\mathcal{P}_\varepsilon([0,\tau])$. Obviously (θ, X) is an upper bound for $\mathcal{C}$. Then by Zorn's lemma, $(\mathcal{P}_\varepsilon, \leq)$ admits a maximal element $(\theta_\varepsilon, X_\varepsilon) \in \mathcal{P}_\varepsilon([0,\tau_\varepsilon])$ with $\tau_\varepsilon \in I$.

To finish the proof we need to show that $\tau_\varepsilon = T$. Assume the contrary, that is, $\tau_\varepsilon < T$. Choose $\delta_\varepsilon > 0$ with $\delta_\varepsilon < \inf(\varepsilon, T - \tau_\varepsilon)$. According to (C.3), there are $(\widetilde{\tau}, \widetilde{x}) \in G$ and an integrable selection Y of the multifunction $F(\cdot, X_\varepsilon^-(\tau_\varepsilon))$ and $\widetilde{y} \in \varepsilon\overline{B}_E$ such that $\tau_\varepsilon < \widetilde{\tau} \leq \tau_\varepsilon + \delta_\varepsilon$ and

$$\widetilde{x} - X^-(\tau_\varepsilon) = \int\limits_{]\tau_\varepsilon,\tilde{\tau}[} Y(s)\,\mu(ds) + \mu(]\tau_\varepsilon, \widetilde{\tau}[)\widetilde{y}.$$

According to (C.2) there exists $\widetilde{z} \in C(\widetilde{\tau})$ such that $\widetilde{z} \in \widetilde{x} + \mu(\{\widetilde{\tau}\})F(\widetilde{\tau}, \widetilde{x}) + \mu(\{\widetilde{\tau}\})\varepsilon\overline{B}_E$. Take $\widetilde{w} \in F(\widetilde{\tau}, \widetilde{x}) + \varepsilon\overline{B}_E$ such that $\widetilde{z} = \widetilde{x} + \mu(\{\widetilde{\tau}\})\widetilde{w}$. Now define $\widetilde{\theta} : [0,\widetilde{\tau}] \to [0,\widetilde{\tau}]$, $\widetilde{X} : [0,\widetilde{\tau}] \to E$ and $\widetilde{X}' : [0,\widetilde{\tau}] \to E$ as follows:

$$\widetilde{\theta}(t) = \begin{cases} \theta_\varepsilon(t) & \text{for } t \in [0,\tau_\varepsilon], \\ \tau_\varepsilon & \text{for } t \in]\tau_\varepsilon, \widetilde{\tau}[, \\ \widetilde{\tau} & \text{for } t = \widetilde{\tau}, \end{cases}$$

$$\widetilde{X}(t) = \begin{cases} X_\varepsilon(t) & \text{for } t \in [0,\tau_\varepsilon], \\ X_\varepsilon^-(\tau_\varepsilon) + \int_{]\tau_\varepsilon,t]}(Y(s) + \widetilde{y})\,\mu(ds) & \text{for } t \in]\tau_\varepsilon, \widetilde{\tau}[, \\ \widetilde{z} & \text{for } t = \widetilde{\tau}, \end{cases}$$

$$\widetilde{X}'(t) = \begin{cases} X'_\varepsilon(t) & \text{for } t \in [0,\tau_\varepsilon], \\ Y(t) + \widetilde{y} & \text{for } t \in]\tau_\varepsilon, \widetilde{\tau}[, \\ \widetilde{w} & \text{for } t = \widetilde{\tau}. \end{cases}$$

Then it is easy to check that $(\widetilde{\theta}, \widetilde{X}) \in \mathcal{P}_\varepsilon([0,\widetilde{\tau}])$. This contradicts the fact that $(\theta_\varepsilon, X_\varepsilon)$ is maximal.

The main use of ε-approximated BVRC solutions to the existence of BVRC solutions for (1.1) is the convergence of ε-approximated solutions by our next proposition.

PROPOSITION 5.2. *Let $C : I \to c(E)$ be a multifunction with left closed graph G. Let c in $L^1_{\mathbb{R}+}(I,\mu)$ be such that $\forall t \in I$, $0 \le \mu(\{t\})c(t) < 1$ and $\int_I \overline{\chi}(-\mu(\{t\})c(t))c(t)\,\mu(dt) < \infty$. Let $\Gamma : I \to ck(E)$ be a scalarly measurable multifunction such that, for all t, $\Gamma(t) \subset c(t)\overline{B}_E$. Let $F : I \times E \to ck(E)$ be a scalarly $\mathcal{T}_\mu \otimes \mathcal{B}(E)$-measurable multifunction such that for each $t \in I$, $F(t,\cdot)$ is upper semicontinuous. Suppose that the following conditions are satisfied*:

(C.1) $\forall (t,x) \in I \times E,\ F(t,x) \subset (1+\|x\|)\Gamma(t)$.

(C.2) $\forall (t,x) \in G,\ \inf_{y\in C(t)} d(y, x + \mu(\{t\})F(t,x)) = 0$.

(C.3) *For each $(t,x) \in G^\bullet$ and each $\varepsilon > 0$ there is $(t_\varepsilon, x_\varepsilon) \in G$ such that $0 < t_\varepsilon - t \le \varepsilon$ and that*

$$x_\varepsilon - x \in \int\limits_{]t,t_\varepsilon[} F(s,x)\,\mu(ds) + \mu(]t,t_\varepsilon[)\varepsilon\overline{B}_E.$$

Then for any $x_0 \in C(0)$, there is a BVRC $X : I \to E$ with the following properties:

$$\forall t \in I, \qquad X(t) = x_0 + \int\limits_{]0,t]} X'(s)\,\mu(ds)$$

with $X' \in L^1_E(I,\mu)$, $\forall t \in I$, $X^-(t) \in C(t)$ and $X'(t) \in F(t, X^-(t))$ μ-a.e.

Proof. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $]0,1]$ with $\lim_{n\to\infty}\varepsilon_n = 0$. By Proposition 5.1, there are a constant $m > 1$, a sequence $(\theta_n)_{n\in\mathbb{N}}$ with $\theta_n \in J^-_{\varepsilon_n}(I)$, $\forall n$, and a sequence $(X_n)_{n\in\mathbb{N}}$ of mappings from I to E satisfying $\forall t \in I$, $X_n(t) = x_0 + \int_{]0,t]} X'_n(s)\,\mu(ds)$ with $X'_n \in L^1_E(I,\mu)$, $\forall t \in I$, $X^-_n(\theta_n(t)) \in C(\theta_n(t))$, $X'_n(t) \in F(t, X^-_n(\theta_n(t))) + \varepsilon_n\overline{B}_E$ μ-a.e. with

$$\|X'_n(t)\| \le mc(t) + 1 \qquad \mu\text{-a.e.},$$

where $m = (1 + \|x_0\| + \mu(I))\exp(\|g\|_1)$ and $g(t) = \overline{\chi}(-\mu(\{t\})c(t))c(t)$, $\forall t \in I$. Now we prove the following main fact: For any $t \in I$, the sequence $(X_n(t))_{n\in\mathbb{N}}$ is relatively compact in E and the sequence $(X'_n)_{n\in\mathbb{N}}$ is relatively weakly compact in $L^1_E(I,\mu)$.

By (C.1) we have for a.e. t,

$$\begin{aligned} X'_n(t) &\in F(t, X^-_n(\theta_n(t)) + \varepsilon_n\overline{B}_E \subset (1 + \|X^-_n(\theta_n(t))\|)\Gamma(t) + \varepsilon_n\overline{B}_E \\ &\subset m\Gamma(t) + \varepsilon_n\overline{B}_E \subset (mc(t)+1)\overline{B}_e \end{aligned}$$

by the proof of (A) in Proposition 5.1. Since $\lim_{n\to\infty}\varepsilon_n = 0$ and $m\Gamma(t)$ is compact, it follows from Theorem 4.4 that the sequence (X'_n) is relatively weakly compact. Moreover, by Theorem 4.3, for any t in I, the integral $\int_{]0,t]} \Gamma(s)\,\mu(ds)$

is convex and compact in E and

$$X_n(t) = x_0 + \int_{]0,t]} X'_n(s)\,\mu(ds) \in x_0 + m \int_{]0,t]} \Gamma(s)\,\mu(ds) + \varepsilon_n \mu(I)\overline{B}_E$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, it is immediate that the sequence $(X_n(t))_{n\in\mathbb{N}}$ is relatively compact. Without loss of generality we can suppose that $(X'_n)_n$ converges to X' for $\sigma(L^1, L^\infty)$. Define $X(t) = x_0 + \int_{]0,t]} X'(s)\,\mu(ds)$, $\forall t$. Then for any t, $(X_n(t))_{n\in\mathbb{N}}$ converges to $X(t)$ for $\sigma(E, E')$. Since the sequence $(X_n(t))_{n\in\mathbb{N}}$ is relatively compact, we have $\lim_{n\to\infty} X_n(t) = X(t)$ for the norm topology. Also we have $\lim_{n\to\infty} X^-_n(t) = X^-(t)$, $\forall t$, for the norm topology. An easy computation gives

$$\|X^-_n(t) - X^-_n(\theta_n(t))\| \le \int_{[\theta_n(t),t[} (mc(s)+1)\,\mu(ds)$$

for all $t \in I$. Hence $\lim_{n\to\infty} \|X^-_n(t) - X^-_n(\theta_n(t))\| = 0$. So

$$\begin{aligned}\lim_{n\to\infty} X^-_n(\theta_n(t)) &= \lim_{n\to\infty} X^-_n(t)\\ &= \lim_{n\to\infty} \Big(x_0 + \int_{]0,t[} X'_n(s)\,\mu(ds)\Big) = x_0 + \int_{]0,t[} X'(s)\,\mu(ds).\end{aligned}$$

Since $X^-_n(\theta_n(t)) \in C(\theta_n(t))$ and G is left closed, we have $X^-(t) \in C(t)$. It remains to show that $X'(t) \in F(t, X^-(t))$ for a.e. t. Observe that F is $\mathcal{T}_\mu \otimes \mathcal{B}(E)$-measurable, so the previous inclusion follows directly from Theorem 4.6. Indeed, set for $(t,x,y) \in I \times E \times E$, $\psi(t,x,y) = d(y, F(t,x))$. Then it is obvious that ψ satisfies the assumptions of Theorem 4.5. Since $X'_n(t) \in F(t, X^-_n(\theta_n(t)) + \varepsilon_n \overline{B}_E$ μ-a.e., this implies that

$$\psi(t, X^-_n(\theta_n(t)), X'_n(t)) \le \varepsilon_n \le 1$$

for all $n \in \mathbb{N}$ and μ-a.e t in I. Hence

$$\liminf_{n\to\infty} \int_I \psi(t, X^-_n(\theta_n(t)), X'_n(t))\,\mu(dt) \le 0.$$

Since $\lim_{n\to\infty} X'_n(\theta_n(t)) = X^-(t)$ for all $t \in I$, and $\lim_{n\to\infty} X'_n = X'$ for $\sigma(L^1, L^\infty)$, by Theorem 5.4, we obtain

$$\int_I \psi(t, X^-(t), X(t)\,\mu(dt) \le \liminf_{n\to\infty} \int_I \psi(t, X'_n(\theta_n(t)), X'_n(t))\,\mu(dt).$$

Hence $d(X'(t), F(t, X^-(t)) = 0$ μ-a.e. This ends the proof.

R e m a r k 5.3. The global measurability assumption on F can be weakened. Indeed, one can replace this assumption by the following: For any measurable mapping $X : I \to E$, the multifunction $t \mapsto F(t, X(t))$ is scalarly measurable. In fact, we only need to show that $X'(t) \in F(t, X^-(t))$ μ-a.e. Let $(e'_k)_{k\in\mathbb{N}}$ be a dense sequence in E' for the Mackey topology. For any measurable A in I, and for any $k \in \mathbb{N}$, we have

$$\begin{aligned}\int_A \langle e'_k, X'(t)\rangle\,\mu(dt) &= \lim_{n\to\infty} \int_{n\to\infty} \langle e'_k, X'_n(t)\rangle\,\mu(dt)\\ &\le \limsup_{n\to\infty} \int_A \delta^*(e'_k, X^-_n(\theta_n(t)))\,\mu(dt)\\ &\le \int_A \limsup_{n\to\infty} \delta^*(e'_k, F(t, X^-_n(\theta_n(t))))\,\mu(dt)\end{aligned}$$

by Fatou's lemma because we have the estimate

$$F(t, X^-_n(\theta_n(t))) \subset (1 + \|X^-_n(\theta_n(t))\|)\Gamma(t) \subset mc\overline{B}_E$$

for all $t \in I$ and all $n \in \mathbb{N}$. By the upper semicontinuity of $F(t,\cdot)$, we get

$$\int_A \langle e'_k, X'(t)\rangle\,\mu(dt) \le \int_A \delta^*(e'_k, F(t, X^-(t)))\,\mu(dt).$$

Equivalently, $X'(t) \in F(t, X^-(t))$ μ-a.e.

Remark 5.4. If E is $\mathbb{R}^d$, one can only suppose that F is separately measurable and separately upper semicontinuous. Indeed, by the Mazur lemma and by the upper semicontinuity of $\Phi(t,\cdot,\cdot) := (x,r) \mapsto F(t,x) + r\overline{B}_E$ from $E \times [0,1]$ to $ck(\mathbb{R}^d)$ for fixed t in I, it is immediate that $X'(t) \in F(t, X^-(t))$ μ-a.e. since we have $X'_n(t) \in \Phi(t, X^-_n(\theta_n(t)), \varepsilon_n)$ μ-a.e. with $\lim_{n\to\infty} X^-_n(\theta_n(t)) = X^-(t)$, $\forall t \in I$, $\lim_{n\to\infty} \varepsilon_n = 0$, and $X'_n \to X'$ for $\sigma(L^1, L^\infty)$.

Proposition 5.2 considers the existence of BVRC solutions for (1.1) when F is scalarly $\mathcal{T}_\mu \otimes \mathcal{B}(E)$-measurable with $F(t,\cdot)$ upper semicontinuous on E and the graph of the constraint C is left closed. Now we present an analogous result for the case where the graph G of the constraint C is right closed and F is globally upper semicontinuous.

PROPOSITION 5.5. *Let $C : I \to c(E)$ be a multifunction such that the graph G of C is right closed* (*that is*, *G is closed in $[0,T]_d \times E$*). *Let c be a positive number*, *$F : G \to ck(E)$ an upper semicontinuous multifunction and ω a Kamke function. Assume that the following two conditions are satisfied*:

(C.1) *For each $\varepsilon > 0$, each (t,x) in $G^\bullet$, and each $t' \in I^+_{t,\varepsilon}$, there is $x' \in C(t')$ such that*

$$x' - x \in \mu(]t,t'])[(F(t',x') \cap c\overline{B}_E) + \varepsilon\overline{B}_E].$$

(C.2) *For μ-a.e. t in I and for all bounded subsets B of E, one has*

$$\inf_{\delta>0} \alpha[F(G \cap (I^+_{t,\delta} \times B)) \cap c\overline{B}_E] \le \omega(t, \alpha(B)).$$

Then for any $x_0 \in C(0)$ there is a BVRC mapping $X : I \to E$ with the following properties:

(i) $\forall t \in I,\ X(t) = x_0 + \int_{]0,t]} X'(s)\,\mu(ds)$ *with* $X' \in L^1_E(I,\mu)$.

(ii) $\forall t \in I,\ X(t) \in C(t)$.

(iii) $X'(t) \in F(t, X(t))$ μ*-a.e.*

Proof. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $]0,1]$ such that $\lim_{n\to\infty}\varepsilon_n = 0$. For each $n \in \mathbb{N}$, consider a subdivision

$$0 = t_0^n < t_1^n < \ldots < t_{\nu_n}^n = T$$

such that $t_i^n - t_{i-1}^n \le \varepsilon_n$ for $i = 1, \ldots, \nu_n$. According to (C.1) there is a sequence $(x_i^n)_{0\le i\le \nu_n}$ such that

$$x_0^n = x_0, \quad (t_i^n, x_i^n) \in G \quad \text{for } i = 0, \ldots, \nu_n, \tag{5.5.1}$$

$$x_i^n - x_{i-1}^n \in \mu(]t_{i-1}^n, t_i^n])[(F(t_i^n, x_i^n) \cap c\overline{B}_E) + \varepsilon_n \overline{B}_E] \quad \text{for } 1 \le i \le \nu_n. \tag{5.5.2}$$

For each $1 \le i \le \nu_n$, there is y_i^n such that $y_i^n \in (F(t_i^n, x_i^n) \cap \overline{B}_E) + \varepsilon_n \overline{B}_E$ and that $x_i^n - x_{i-1}^n = \mu(]t_{i-1}^n, t_i^n])y_i^n$. Define the functions $\theta_n : I \to I$, $X'_n : I \to E$ and $X_n : I \to E$ by

$$\theta_n(0) = 0, \quad \theta_n(t) = t_i^n \quad \text{if} \quad t \in]t_{i-1}^n, t_i^n],\ 1 \le i \le \nu_n,$$
$$X'_n(0) = y_1^n, \quad X'_n(t) = y_i^n \quad \text{if} \quad t \in]t_{i-1}^n, t_i^n],\ 1 \le i \le \nu_n,$$
$$X_n(0) = x_0, \quad X_n(t) = x_{i-1}^n + \mu(]t_{i-1}^n, t])y_i^n \quad \text{if} \quad t \in]t_{i-1}^n, t_i^n],\ 1 \le i \le \nu_n.$$

Then θ_n is increasing with $0 \le \theta_n(t) - t < \varepsilon_n$ and X_n is BVRC which satisfies

$$X_n(t) = x_0 + \int_{]0,t]} X'_n(s)\,\mu(ds), \qquad \forall t \in I, \tag{5.5.3}$$

$$(\theta_n(t), X_n(\theta_n(t))) \in G, \qquad \forall t \in I, \tag{5.5.4}$$

$$X'_n(t) \in F(\theta_n(t), X_n(\theta_n(t)) \cap c\overline{B}_E \qquad \mu\text{-a.e.} \tag{5.5.5}$$

Now to complete the proof we need to show the convergence of the sequence $(X_n)_{n\in\mathbb{N}}$ of approximate BVRC solutions by our basic result in the following proposition.

PROPOSITION 5.6. *Let the assumptions of Proposition* 5.5 *be fulfilled. Let* $x_0 \in C(0)$. *Let* $(X_n)_{n\in\mathbb{N}}$ *and* $(X'_n)_{n\in\mathbb{N}}$ *be two sequences of mappings from* I *to* E *satisfying the following three conditions*:

(a) $\forall t \in I,\ X_n(t) = x_0 + \int_{]0,t]} X'_n(s)\,\mu(ds)$ *with* $X'_n \in L^1_E(I,\mu)$.

(b) $\forall n \in \mathbb{N}, \forall t \in I,\ (\theta_n(t), X_n(\theta_n))) \in G$.

(c) $X'_n(t) \in [F(\theta_n(t), X_n(\theta_n(t))) \cap c\overline{B}_E] + \varepsilon_n \overline{B}_E$ μ*-a.e.*

Then there are a mapping $X : I \to E$ *and a mapping* $X' \in L^1_E(I,\mu)$, *a subsequence* (X_{n_k}) *of* (X_n), *and a subsequence* (X'_{n_k}) *of* (X'_n) *such that* (X_{n_k}) *converges pointwise to* X *and* (X'_{n_k}) *weakly converges to* X' *in* $L^1_E(I,\mu)$. *Further* X *is a BVRC solution for* (1.1).

Proof. By assumption (a) and (c), we have

$$\|X_n(t) - X_n(t')\| \leq \int_{]t,t']} \|X_n'(s)\|\, \mu(ds) \leq (c+1)\mu(]t,t']) \tag{5.6.1}$$

for all $n \in \mathbb{N}$ and all $(t,t') \in I \times I$ with $t \leq t'$. For each $t \in I$, define

$$A_0(t) := \{X_n(t) : n \in \mathbb{N}\} \quad \text{and} \quad \varrho(t) = \alpha(A_0(t)).$$

Then $A_0(t)$ is relatively compact for all t in I if and only if the function ϱ is identically equal to zero. For $(t,t') \in I \times I$ with $t \leq t'$, we have

$$A_0(t') \subset A_0(t) + \{X_n(t') - X_n(t) : n \in \mathbb{N}\}.$$

Then by (5.6.1) it follows that

$$A_0(t') \subset A_0(t) + (c+1)\mu(]t,t'])\overline{B}_E.$$

Hence we get

$$\alpha(A_0(t')) \leq \alpha(A_0(t)) + 2(c+1)\mu(]t,t']).$$

That is, $\varrho(t') \leq \varrho(t) + 2(c+1)\mu(]t,t'])$ and analogously

$$\varrho(t) \leq \varrho(t') + 2(c+1)\mu(]t,t']),$$

so that $|\varrho(t) - \varrho(t')| \leq 2(c+1)\mu(]t,t'])$ for all (t,t') in $I \times I$ with $t \leq t'$. Therefore ϱ is a BVRC function on I and its differential measure $D\varrho$ satisfies $D\varrho(]t,t']) \leq 2(c+1)\mu(]t,t'])$ and $D\varrho(\{0\}) = \varrho^+(0) - \varrho(0) = 0$. Then it follows from Moreau–Valadier ([MV2], lemme 2.3) that

$$|D\varrho| \leq 2(c+1)\mu.$$

By Radon–Nikodym's theorem, $D\varrho$ admits a density $\dot{\varrho} := D\varrho/d\mu \in L^1_{\mathbb{R}}(I,\mu)$ with respect to μ. Moreover, by virtue of a result due to Jeffery ([J], Theorem 5, p. 655 and Theorem 9, p. 662), see also ([EJ], Theorem 3.2, p. 228) there is a μ-negligible set N such that

$$\dot{\varrho}(t) = \lim_{\varepsilon \to 0^+} \frac{D\varrho[t,t+\varepsilon]}{\mu[t,t+\varepsilon]}, \qquad \forall t \in I \setminus N. \tag{5.6.2}$$

By (C.2) we can suppose that

$$\inf_{\delta>0} \alpha[F(G \cap (I^+_{t,\delta} \times B)) \cap c\overline{B}_E] \leq \omega(t, \alpha(B))$$

whenever $t \in I \setminus N$ and $B \in b(E)$. Now we prove the following main fact:

$$\dot{\varrho}(t) \leq \omega(t, \varrho(t)) \qquad \mu\text{-a.e. on } I. \tag{5.6.3}$$

If $t = 0$ and if $\mu(\{0\}) > 0$, then $\dot{\varrho}(0) = (\varrho(0) - \varrho(0))/\mu(\{0\}) = 0$. Let $t \in]0,T[\, \setminus N$. Let $\varepsilon > 0$ and $h > 0$ with $[t,t+h] \subset I$. Define

$$B_{t,h} = A_0(I^+_{t,h}) = \bigcup_{s \in I^+_{t,h}} A_0(s).$$

Then $B_{t,h}$ is obviously bounded in E. By (C.2) and (5.6.2) we can choose $\delta_\varepsilon \in]0, h/2[$ such that

$$\alpha[F(G \cap (I^+_{t.2\delta} \times B_{t,h})) \cap c\overline{B}_E)] \leq \omega(t, B_{t,h}) + \varepsilon, \tag{5.6.4}$$

$$\dot{\varrho}(t) \leq \frac{\varrho(t+\delta) - \varrho^-(t)}{\mu([t, t+\delta])} + \varepsilon \tag{5.6.5}$$

whenever $\delta \in]0, \delta_\varepsilon]$.

Now fix $\delta \in]0, \delta_\varepsilon]$ and set

$$\mathcal{X}_n(t) = \left\{ \frac{X_k(t+\delta) - X_k^-(\delta)}{\mu([t, t+\delta])} : k \geq n \right\}.$$

Take $p \in \mathbb{N}$ such that $n \geq p$ implies $\varepsilon_n \leq \delta$. We have by easy properties of the measure of noncompactness

$$\begin{aligned} \varrho(t+\delta) \leq \alpha\{X_n(t+\delta) - X_n^-(t) : n \in \mathbb{N}\} + \alpha\{X_n^-(t) - X_n(t') : n \in \mathbb{N}\} \\ + \alpha\{X_n(t') : n \in \mathbb{N}\}. \end{aligned}$$

Hence we have the estimate

$$\varrho(t+\delta) \leq \mu([t, t+\delta])\alpha(\mathcal{X}_p(t)) + \varrho(t') + \alpha[\{X_n^-(t) - X_n(t') : n : \mathbb{N}\}].$$

Since $\|X_n^-(t) - X_n(t')\| \leq (c+1)\mu(]t', t[)$ for all $n \in \mathbb{N}$ and for $0 \leq t' < t$, we get

$$\varrho(t+\delta) \leq \varrho(t') + \mu(]t, t+\delta])\alpha(\mathcal{X}_p(t)) + 2(c+1)\mu(]t', t[) \tag{5.6.6}$$

for $t' \leq t$. Then by taking the limit of the second member of (5.6.6) as $t' \to t^-$ we obtain

$$\varrho(t+\delta) \leq \varrho^-(t) + \mu(]t, t+\delta])\alpha(\mathcal{X}_p(t)). \tag{5.6.7}$$

Therefore it follows from (5.6.5) that

$$\dot{\varrho}(t) \leq \alpha(\mathcal{X}_p(t)) + \varepsilon. \tag{5.6.8}$$

Now we estimate $\alpha(\mathcal{X}_p(t))$ by the mean value theorem. We have

$$\begin{aligned} \mathcal{X}_p(t) &\subset \bigcup_{n \geq p} \overline{\text{co}} X_n'(t)([t, t+\delta]) \\ &\subset \bigcup_{n \geq p} \overline{\text{co}} \Big[\bigcup_{s \in [t, t+\delta]} (F(\theta_n(s), X_n(\theta_n(s))) \cap c\overline{B}_E) + \varepsilon_n \overline{B}_E \Big] \end{aligned}$$

according to condition (c). Note that for $n \geq p$ and $s \in [t, t+\delta]$, we have

$$\varepsilon_n \leq \delta \quad \text{and} \quad t \leq \theta_n(t) \leq \theta_n(s) \leq \theta_n(t+\delta) \leq t + \delta + \varepsilon_n \leq t + 2\delta.$$

Hence, for $n \geq p$,

$$\begin{aligned} &\bigcup_{s \in [t, t+\delta]} (F(\theta_n(s), X_n(\theta_n(s))) \cap c\overline{B}_E) + \varepsilon_n \overline{B}_E \\ &\qquad \subset [F(G \cap (I^+_{t,2\delta} \times B_{t,h})) \cap c\overline{B}_E] + \delta \overline{B}_E \end{aligned}$$

because $X_n(\theta_n(s)) \in B_{t,2\delta} \subset B_{t,h}$ for all $s \in [t, t+\delta]$. This shows that

$$\mathcal{X}_p(t) \subset \overline{\text{co}}[(F(G \cap (I^+_{t,2\delta} \times B_{t,h})) \cap c\overline{B}_E) + \delta \overline{B}_E].$$

Hence

$$\alpha(\mathcal{X}_p(t)) \le \alpha[F(G \cap (I^+_{t,2\delta} \times B_{t,h})) \cap c\overline{B}_E] + 2\delta. \tag{5.6.9}$$

Then by (5.6.4) and (5.6.9) we get

$$\alpha(\mathcal{X}_p(t)) \le \omega(t, \alpha(B_{t,h})) + \varepsilon + 2\delta. \tag{5.6.10}$$

Also by (5.6.8) and (5.6.10), we have

$$\dot{\varrho}(t) \le \omega(t, \alpha(B_{t,h})) + 2\varepsilon + h. \tag{5.6.11}$$

Now by the definition of $B_{t,h}$ and (5.6.1), it is easily seen that

$$A_0(t) \subset B_{t,h} \subset A_0(t) + (c+1)\mu(]t, t+h])\overline{B}_E.$$

Hence

$$\varrho(t) \le \alpha(B_{t,h}) \le \varrho(t) + 2(c+1)\mu(]t, t+h])$$

so that $\lim_{h\to 0^+} \alpha(B_{t,h}) = \varrho(t)$. Finally, by (5.6.11) we obtain

$$\dot{\varrho}(t) \le \omega(t, \varrho(t))$$

as $\varepsilon \to 0^+$ and $h \to 0^+$. Now it remains to prove that (5.6.3) is valid for $t = T$ if $\mu(\{T\}) > 0$, that is, $\dot{\varrho}(T) \le \omega(T, \varrho(T))$. By condition (C.2) we have

$$\inf_{\delta>0} \alpha[F(G \cap (I^+_{T,\delta} \times A_0(T))) \cap c\overline{B}_E] \le \omega(T, \alpha(A_0(T))).$$

Since $I^+_{T,\delta} = \{T\}$ and $\alpha(A_0(T)) = \varrho(T)$, we have

$$\alpha[F(\{T\} \times A_0(T)) \cap c\overline{B}_E] \le \omega(T, \alpha(A_0(T))) = \omega(T, \varrho(T)). \tag{5.6.12}$$

Recall that $\dot{\varrho}(T) = (\varrho(T) - \varrho^-(T))/\mu(\{T\})$. Moreover, for all $t \in [0, T[$, we have

$$\begin{aligned}\varrho(T) \le \alpha\{&X_n(T) - X_n^-(T) : n \in \mathbb{N}\} \\ &+ \alpha\{X_n^-(T) - X_n(t) : n \in \mathbb{N}\} + \alpha\{X_n(t) : n \in \mathbb{N}\}\end{aligned}$$

so that

$$\varrho(T) \le \alpha\{\mu(\{T\})X_n'(T) : n \in \mathbb{N}\} + 2(c+1)\mu(]t, T[) + \varrho(t).$$

Then as $t \to T^-$, it follows that

$$\varrho(T) \le \alpha\{\mu(\{T\})X_n'(T) : n \in \mathbb{N}\} + \varrho^-(T).$$

This implies

$$\dot{\varrho}(T) = \frac{\varrho(T) - \varrho^-(T)}{\mu(\{T\})} \le \alpha\{X_n^-(T) : n \in \mathbb{N}\}. \tag{5.6.13}$$

By (c) for any integer $m \in \mathbb{N}$, we have

$$\{X_n'(T) : n \ge m\} \subset \bigcup_{n\ge m} [(F(T, X_n(T)) \cap c\overline{B}_E) + \varepsilon_n \overline{B}_E].$$

Set $\varepsilon'_m = \sup_{n\geq m} \varepsilon_n$. Then

$$\{X'_n(T) : n \geq m\} \subset [F(\{T\} \times A_0(T)) \cap \overline{B}_E] + \varepsilon'_m B_E. \tag{5.6.14}$$

Then by (5.6.12), (5.6.13) and (5.6.14) we obtain

$$\dot{\varrho}(T) \leq \alpha[F(\{T\} \times A_0(T)) \cap c\overline{B}_E] + 2\varepsilon'_n \leq \omega(T, \alpha(A_0(T))) + 2\varepsilon'_m.$$

As $\varepsilon'_m \to 0$, $\dot{\varrho}(T) \leq \omega(T, \alpha(A_0(T))$ as desired. Since $\varrho(0) = \alpha(\{x_0\}) = 0$ and since ω is a Kamke function, we have $\varrho \equiv 0$. This shows that $A_0(t) = \{X_n(t) : n \in \mathbb{N}\}$ is relatively compact for all $t \in I$. Now we check that $(X'_n)_{n\in\mathbb{N}}$ is relatively $\sigma(L^1, L^\infty)$-compact. By (5.6.1), we have

$$X_n(\theta_n(t)) \in A_0(t) + (c+1)\mu(]t, \theta_n(t)]\overline{B}_E$$

for all $n \in \mathbb{N}$ and all $t \in I$. Since $\lim_{n\to\infty} \mu(]t, \theta_n(t)] = 0$ for all $t \in I$, and $A_0(t)$ is relatively compact, it is immediate that $\{X_n(\theta_n(t)) : n \in \mathbb{N}\}$ is relatively compact too, for all $t \in I$. For each $t \in I$, set $K(t) = \{\theta_n(t), X_n(\theta_n(t))) : n \in \mathbb{N}\}$. Then the multifunction $t \mapsto \overline{K(t)}$ is obviously measurable with nonempty compact values in G. Since $F : G \to ck(E)$ is upper semicontinuous, $t \mapsto F(\overline{K(t)})$ is a measurable multifunction from I to $k(E)$. Now by (c) we have $X'_n(t) \in F(\overline{K(t)}) \cap c\overline{B}_E) + \varepsilon_n \overline{B}_E$ μ-a.e.; then a fortiori we have

$$X'_n(t) \in \overline{\text{co}}[F(\overline{K(t)}) \cap c\overline{B}_E] + \varepsilon_n \overline{B}_E \qquad \mu\text{-a.e.}$$

Since $\overline{\text{co}}F(\overline{K(\cdot)})$ is measurable too, it follows from Theorem 4.4 that (X'_n) is relatively $\sigma(L^1, L^\infty)$-compact. Now it is easy to finish the proof. We can suppose that (X'_n) converges to X' for $\sigma(L^1, L^\infty)$ so that, for all $t \in I$, we have

$$\lim_{n\to\infty} x_0 + \int_{]0,t]} X'_n(s)\, ds = x_0 + \int_{]0,t]} X'(s)\, \mu(ds)$$

for $\sigma(E, E')$. Since $A_0(t) = \{X_n(t) : n \in \mathbb{N}\}$ is relatively compact for all $t \in I$, it follows that, $\forall t \in I$, $\lim_{n\to\infty} X_n(t) = X(t)$ for the norm topology where $X(t) = x_0 + \int_{]0,t]} X'(s)\, \mu(ds)$. Now for (t, x) in G, set

$$\widetilde{F}(t, x) = F(t, x) \cap c\overline{B}_E.$$

Then $\widetilde{F}$ is upper semicontinuous on its domain $\widetilde{D} \subset G$ and by (c), for μ-a.e. $t \in I$, $K(t) \subset \widetilde{D}$. Note that for all t in I, $\lim_{n\to\infty} \theta_n(t) = t$ and $\lim_{n\to\infty} X_n(\theta_n(t)) = \lim_{n\to\infty} X_n(t) = X(t)$ since $\|X_n(\theta_n(t)) - X_n(t)\| \leq (c+1)\mu(]t, \theta_n(t)]$. Then $\lim_{n\to\infty}(\theta_n(t), X_n(\theta_n(t))) = (t, X(t)) \in G$ since G is right closed. For any measurable set $A \subset I$ and any $x' \in E$, we have

$$\begin{aligned}\int_A \langle x', X'(t)\rangle\, \mu(dt) &= \lim_{n\to\infty} \int_A \langle x', X'_n(s)\rangle\, \mu(ds) \\ &\leq \limsup_{n\to\infty} \int_A \delta^*(x', \widetilde{F}(\theta_n(t), X_n(\theta_n(t))))\, \mu(dt)\end{aligned}$$

$$\le \int_A \limsup_{n\to\infty} \delta^*(x', \widetilde{F}(\theta_n(t), X_n(\theta_n(t))))\, \mu(dt)$$

$$\le \int_A \delta^*(x', \widetilde{F}(t, X(t)))\, \mu(dt).$$

This inequality shows that $X'(t) \in \widetilde{F}(t, X(t))$ μ-a.e. and finishes the proof.

5.b. *Absolutely continuous solutions.* In this section λ is the Lebesgue measure on I. There are analoguous results for the existence of absolutely continuous solutions for (1.1). We need first a preliminary lemma.

LEMMA 5.7. *Let $g : I \to \mathbb{R}^+$ be an integrable function and let J be a measurable subset in I. Then for λ-a.e. t in J, one has*

$$\lim_{\delta\to 0^+} \frac{1}{\delta} \int_{[t-\delta,t]\cap(I\setminus J)} g(s)\, ds = 0.$$

Proof. Let us consider the function $f : t \mapsto 1_{I\setminus J}(t)g(t)$ on I and the measure $\nu := f\lambda$. In view of a result due to Jeffery [J], for λ-a.e. t in I, we have

$$f(t) = \frac{d\nu}{d\lambda}(t) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{\nu[t-\varepsilon, t]}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{[t-\varepsilon,t]\cap(I\setminus J)} g(s)\, ds.$$

Hence the desired result follows by noting that $f(t) = 0$ for all t in J.

PROPOSITION 5.8. *Let $C : I \to c(E)$ be a multifunction such that its graph G is left closed. Let $F : I \times E \to ck(E)$ be a scalarly $\mathcal{T}_\lambda(I) \otimes \mathcal{B}(E)$-measurable multifunction such that for any $t \in I$, $F(t,\cdot)$ is upper semicontinuous on E. Let $\omega \in \mathrm{Kam}(I, \lambda)$. Suppose that the following three conditions are satisfied:*

(C.1) *There is $c \in L^1_{\mathbb{R}^+}(I, \lambda)$ such that*

$$\forall (t,x) \in I \times E, \qquad |F(t,x)| \le c(t)(1 + \|x\|)$$

(C.2) *For each $\varepsilon > 0$, there is a closed set $J_\varepsilon \subset I$ with $\lambda(I \setminus J_\varepsilon) \le \varepsilon$ such that for λ-a.e. t in J_ε and for any nonempty bounded subset B of E, one has*

$$\inf_{\delta>0} \alpha[F(I^-_{t,\delta} \times B)] \le \omega(t, \alpha(B))$$

where $I^-_{t,\delta} = J_\varepsilon \cap [t-\delta, t]$.

(C.3) *For each $(t,x) \in G^\bullet$ and each $\varepsilon > 0$, there is $(t_\varepsilon, x_\varepsilon) \in G$ such that $0 < t_\varepsilon - t \le \varepsilon$ and that $x_\varepsilon - x \in \int_t^{t_\varepsilon} F(s,x)\, ds + (t_\varepsilon - t)\varepsilon \overline{B}_E$.*

Then for any $x_0 \in C(0)$, there is an absolutely continuous solution X of (1.1) with $X(0) = x_0$.

Proof. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $]0,1]$ such that $\lim_{n\to\infty} \varepsilon_n = 0$. In view of Proposition 5.1, there are $m > 1$, a sequence $(\theta_n)_{n\in\mathbb{N}}$ with $\theta_n \in J^-_{\varepsilon_n}(I)$ and a sequence $(X_n)_{n\in\mathbb{N}}$ of absolutely continuous functions with the following properties:

(5.8.1) $$\forall t \in I, \qquad X_n(t) = x_0 + \int_0^t (X'_n(s)\,ds \text{ where } X'_n \in L^1_E(I,\lambda),$$

(5.8.2) $$\forall t \in I, \qquad X_n(\theta_n(t)) \in C(\theta_n(t)),$$

(5.8.3) $$X'_n(t) \in F(t, X_n(\theta_n(t)) + \varepsilon_n \overline{B}_n \qquad \lambda\text{-a.e.},$$

(5.8.4) $$\|X'_n(t)\| \le mc(t) + 1 \qquad \lambda\text{-a.e.}$$

Then $(X_n)_{n\in\mathbb{N}}$ is an equicontinuous subset of $\mathcal{C}_E([0,T]$. For each $t \in I$, set

$$A_0(t) = \{X_n(t) : n \in \mathbb{N}\} \quad \text{and} \quad \varrho(t) = \alpha(A_0(t)).$$

We shall prove that $(X_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{C}_E([0,T])$. By Ascoli's theorem, it is enough to show that $A_0(t)$ is relatively compact for any t in I. So it is equivalent to show that the function ϱ is identically equal to zero. For $(t,t') \in I\times I$ with $t \le t'$, we have

$$A_0(t') \subset A_0(t) + \{X_n(t') - X_n(t) : n \in \mathbb{N}\}.$$

By (5.8.4), we have

$$\{X_n(t') - X_n(t) : n \in \mathbb{N}\} \subset \Big(\int_t^{t'} g(s)\,ds\Big)\overline{B}_E$$

where $g(t) = mc(t) + 1$, $\forall t \in I$. Hence $A_0(t') \subset A_0(t) + (\int_t^{t'} gs\,ds)\overline{B}_E$. It follows that $\alpha(A_0(t')) \le \alpha(A_0(t)) + 2\int_t^{t'} g(s)\,ds$. Then $\varrho(t') \le \varrho(t) + 2\int_t^{t'} g(s)\,ds$. Consequently,

$$\forall (t,t') \in I \times I,\ t \le t', \qquad |\varrho(t') - \varrho(t)| \le 2\int_t^{t'} g(s)\,ds.$$

It follows that ϱ is absolutely continuous. Let $\dot{\varrho}$ be the Radon–Nikodym density of ϱ with respect to the Lebesgue measure λ. Take $\eta > 0$. Then by our assumption (C.2), Jeffery's theorem [J] and Lemma 5.7, there are a closed set $J_\eta \subset I$ with $\lambda(I - J_\eta) \le \eta$ and a negligible set $N_\eta \subset J_\eta$ such that

(5.8.5) $$\inf_{\delta>0} \alpha[F(J_\eta \cap [t-\delta, t] \times B])] \le \omega(t, \alpha(B))$$

whenever $(t,B) \in (J_\eta \setminus N_\eta) \times b(E)$.

(5.8.6) $$\forall t \in J_\eta \setminus N_\eta, \qquad \dot{\varrho}(t) = \lim_{\delta\to 0^+} \frac{\varrho(t) - \varrho(t-\delta)}{\delta},$$

(5.8.7) $$\forall t \in J_\eta \setminus N_\eta, \qquad \lim_{\delta\to 0^+} \frac{1}{\delta} \int_{[t-\delta,\delta]\cap(I\setminus J_\eta)} g(s)\,ds = 0.$$

Now let $t \in J_\eta \setminus N_\eta$ with $t \ne 0$. Let $h > 0$ with $[t-h,h] \subset I$. Set $B_{t,h} = \bigcup_{s\in[t-h,h]} A_0(s)$ and note that $B_{t,h}$ is bounded by (5.8.4). Let $\varepsilon > 0$. By (5.8.5), (5.8.6) and (5.8.7) there is $\delta_\varepsilon \in]0, h/2]$ such that

(5.8.8) $\alpha[F(I^-_{t,\delta} \times B_{t,h})] \le \omega(t, \alpha(B_{t,h})) + \varepsilon \quad$ where $I^-_{t,\delta} = J_\eta \cap [t-\delta, t]$,

(5.8.9) $$\dot{\varrho}(t) \le \frac{1}{\delta}(\varrho(t) - \varrho(t-\delta)) + \varepsilon$$

(5.8.10) $$\frac{1}{\delta} \int\limits_{[t-\delta,t]\cap(I\setminus J_\eta)} g(s)\,ds \le \varepsilon$$

whenever $0 < \delta \le \delta_\varepsilon$. Now take $\delta \in]0, \delta_\varepsilon]$ and choose $n_0 \in \mathbb{N}$ such that $n \ge n_0$ implies $\varepsilon_n \le \delta$. Set

$$\mathcal{X}_n(t) = \left\{ \frac{1}{\delta}(X_k(t) - X_k(t-\delta)) : k \ge n \right\}, \qquad n \in \mathbb{N}.$$

Then $A_0(t) \subset A_0(t-\delta) + \delta\mathcal{X}_0(t)$ and

$$\alpha(A_0(t)) \le \alpha(A_0(t-\delta)) + \delta\alpha(\mathcal{X}_0(t)) = \alpha(A_0(t-\delta)) + \delta\alpha(\mathcal{X}_{n0}(t)).$$

This implies $\varrho(t) \le \varrho(t-\delta) + \delta\alpha(\mathcal{X}_{n_0}(t))$. By (5.8.9) we get

(5.8.11) $$\dot{\varrho}(t) \le \alpha(\mathcal{X}_{n_0}(t)) + \varepsilon.$$

Further, by the mean value theorem, we have

$$\begin{aligned}\mathcal{X}_{n_0}(t) &\subset \Big(\bigcup_{n\ge n_0} \overline{\mathrm{co}} X'_n([t-\delta,t]\cap J_\eta) \Big) + \Big\{ \frac{1}{\delta} \int\limits_{[t-\delta,t]\cap(I\setminus J_\eta)} X'_n(s)\,ds : n \ge N_0 \Big\} \\ &\subset \Big(\bigcup_{n\ge n_0} \overline{\mathrm{co}} X'_n([t-\delta,t]\cap J_\eta) \Big) + \Big(\frac{1}{\delta} \int\limits_{[t-\delta,t]\cap(I\setminus J_\eta)} g(s)\,ds \Big) \overline{B}_E.\end{aligned}$$

Then by (5.8.10), we have

$$\mathcal{X}_{n_0}(t) \subset \Big(\bigcup_{n\ge n_0} \overline{\mathrm{co}} X'_n([t-\delta,t]\cap J_\eta) \Big) + \varepsilon\overline{B}_E.$$

Hence $\alpha(\mathcal{X}_{n0}(t)) \le \alpha(\bigcup_{n\ge n_0} \overline{\mathrm{co}} X'_n([t-\delta,t]\cap J_\eta) + 2\varepsilon$.

By (5.8.3) we have

$$\forall n \ge n_0, \qquad \overline{\mathrm{co}} X'_n([t-\delta,t]\cap J_\eta) \subset \overline{\mathrm{co}} \Big[\bigcup_{s\in[t-\delta,t]\cap J_\eta} F(s, X_n(\theta_n(s))) + \varepsilon_n \overline{B}_E \Big].$$

Note that $\forall n \ge n_0$, $\forall s \in [t-\delta, t]$, $\varepsilon_n \le \delta$ and

$$t - 2\delta \le t - \delta - \varepsilon_n \le \theta_n(t-\delta) \le \theta_n(s) \le t,$$

so $\theta_n(s) \in [t-2\delta, t] \subset [t-h, t]$ and $X_n(\theta_n(s)) \in B_{t,h}$. Therefore for $n \ge n_0$ we have

$$\overline{\mathrm{co}} X'_n([t-\delta,t]\cap J_\eta) \subset \overline{\mathrm{co}}[F(I^-_{t,\delta} \times B_{t,h}) + \delta\overline{B}_E],$$

so that, using (5.8.8), we get

(5.8.12) $$\begin{aligned}\alpha(\mathcal{X}_{n_0}(t)) &\le \alpha(\overline{\mathrm{co}} F(I^-_{t,\delta} \times B_{t,h})) + 2\delta + 2\varepsilon \\ &= \alpha(F(I^-_{t,\delta} \times B_{t,h})) + 2\delta + 2\varepsilon \le \omega(t, \alpha(B_{t,h})) + h + 3\varepsilon.\end{aligned}$$

Further, it easy to check that

$$A_0(t) \subset B_{t,h} \subset A_0(t) + \Big(\int\limits_{t-h}^{t} g(s)\,ds\Big)\overline{B}_E.$$

Hence

$$\varrho(t) \le \alpha(B_{t,h}) \le \varrho(t) + 2\int\limits_{t-h}^{t} g(s)\,ds.$$

It follows that $\lim_{h\to 0^+} \alpha(B_{t,h}) = \varrho(t)$.

By using (5.8.11) and (5.8.12), we obtain

$$\dot{\varrho}(t) \le \omega(t, \alpha(B_{t,h})) + h + 4\varepsilon.$$

Then as $h \to 0^+$ and $\varepsilon \to 0^+$, we get

$$\dot{\varrho}(t) \le \omega(t, \varrho(t))$$

for all $t \in J_\eta \setminus N_\eta$ and $t \ne 0$.

Let $\Omega = \{t \in [0,T] : \dot{\varrho}(t) \le \omega(t,\varrho(t))\}$. Then Ω is measurable and by previous arguments, for every $\eta > 0$, there are a closed set $J_\eta \subset I$ with $\lambda(I \setminus J_\eta) \le \eta$ and a negligible set $N_\eta \subset I$ such that $J_\eta \setminus (N_\eta \cup \{0\}) \subset \Omega$. Therefore $I \setminus \Omega \subset (I \setminus J_\eta) \cup N_\eta \cup \{0\}$ so that $\lambda(I \setminus \Omega) \le \lambda(I \setminus J_\eta) + \lambda(N_\eta) \le \eta$. Hence $\lambda(I \setminus \Omega) = 0$ and $\dot{\varrho}(t) \le \omega(t, \varrho(t))$ λ-a.e. Since $\varrho(0) = \alpha(\{x_0\}) = 0$ and ω is a Kamke function, ϱ is identically equal to zero. It follows that $(X_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{C}_E([0,T])$, in particular, for each $t \in I$, $A_0(t)$ is relatively compact. Since we have

$$\forall n \in \mathbb{N}, \qquad X_n(\theta_n(t)) \in A_0(t) + \Big(\int\limits_{\theta_n(t)}^{t} g(s)\,ds\Big)\overline{B}_E$$

and $\lim_{n\to\infty} \int_{\theta_n(t)}^t g(s)\,ds = 0$, the set $A'_0(t) := \{X_n(\theta_n(t)) : n \in \mathbb{N}\}$ is relatively compact for all t in I. By our assumption, $F(t,\cdot)$ is upper semicontinuous with convex compact values. It follows that $F(t, \overline{A'_0(t)})$ is compact for all t in I. Further, by (5.8.3), we have

$$\forall n \in \mathbb{N}, \qquad X'_n(t) \in F(t, \overline{A'_0(t)}) + \varepsilon_n \overline{B}_E.$$

Since $(X'_n)_{n\in\mathbb{N}}$ is uniformly integrable, by Theorem 4.4, we see that $(X'_n)_{n\in\mathbb{N}}$ is relatively compact for $\sigma(L^1, L^\infty)$ topology. Therefore there are $X \in \mathcal{C}_E(I)$, $X' \in L^1_E(I,\lambda)$ and a subsequence $(X'_{n_k})_{k\in\mathbb{N}}$ of $(X'_n)_{n\in\mathbb{N}}$ such that $(X_{n_k})_{k\in\mathbb{N}}$ convergs to X in $\mathcal{C}_E(I)$ and $(X'_{n_k})_{k\in\mathbb{N}}$ converges to X' in $L^1_E(I,\lambda)$ for $\sigma(L^1,L^\infty)$ topology with $\forall t \in I$, $X(t) = x_0 + \int_0^t X'(s)\,ds$. But it is already proved that

$$\forall t \in I, \qquad \lim_{n\to\infty} \|X_n(t) - X_n(\theta_n(t))\| = 0.$$

So that $\lim_{k\to\infty} X_{n_k}(\theta_{n_k}(t)) = X(t)$ for all t in I. Since the graph G of C is left closed, by (5.8.2), we have $(t, X(t)) \in G$. Now the inclusion $X'(t) \in F(t, X(t))$ λ-a.e. follows from (5.8.3) and the arguments of the proof of Proposition 5.2. See also Remarks 5.3 and 5.4.

References

[AVC] A. Amrani, C. Castaing et M. Valadier, *Convergence forte dans L^1 impliquée par la convergence faible. Méthodes de troncature*, C. R. Acad. Sci. Paris Sér. I 314 (1992), 37–40.

[AC] J.-P. Aubin and A. Cellina, *Differential Inclusions. Set-Valued Maps and Viability Theory*, Springer, Berlin, 1984.

[BP] C. Bardaro and P. Pucci, *Some contributions to the theory of multivalued differential equations*, Atti Sem. Mat. Fis. Univ. Modena 32 (1983), 175–202.

[B] J. M. Bony, *Principe de maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier 19 (1969), 277–304.

[Br] H. Brezis, *On a characterization of flow invariant sets*, Comm. Pure Appl. Math. 23 (1970), 261–263.

[BCS] P. Brandi, L. Cesari and A. Salvadori, *Existence theorems, converging simple integrals of the calculus of variations for discontinuous solutions*, Arch. Rational Mech. Anal. 98 (1987), 285–396.

[Ca1] C. Castaing, *Quelques résultats de compacité liés à l'intégration*, C. R. Acad. Sci. Paris Sér. A 270 (1970), 1732–1735.

[Ca2] —, *Equation différentielle multivoque avec contrainte sur l'état dans les espaces de Banach*, Sém. d'Analyse Convexe Montpellier (1978), exp. 13 et C. R. Acad. Sci. Paris Sér. A 288 (1979), 507–509.

[Ca3] —, *Quelques résultats de convergence des suites adaptées*, Sém. d'Analyse Convexe Montpellier (1987), exp. 2 (24 pages) et Acta Math. Vietnam. 14 (1989), 51–66.

[CV] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. 580, Springer, Berlin, 1977.

[De1] K. Deimling, *Ordinary Differential Equations in Banach Spaces*, Lecture Notes in Math. 596, Springer, Berlin, 1985.

[De2] —, *Multivalued differential equations on closed sets*, Differential and Integral Equations 1 (1988), 23–30.

[EJ] H. W. Ellis and R. L. Jeffery, *Derivatives and integrals with respect to a base function of generalized bounded variation*, Canad. J. Math. 19 (1967), 225–241.

[G] A. Grothendieck, *Espaces Vectoriels Topologiques*, Publ. Soc. Math. Sao Paulo, 1954.

[J] R. L. Jeffery, *Non-absolutely convergent integrals with respect to functions of bounded variation*, Trans. Amer. Math. Soc. 34 (1932), 645–675.

[L] M. Larrieu, *Invariance d'un fermé pour un champ de vecteurs de Carathéodory*, Publ. Math. Univ. Pau, 1987.

[Ma] R. H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, Krieger, Malabar, Fla., 1987.

[Me] H. Methlouthi, *Equations différentielles multivoques sur un graphe localement compact*, Cahiers Math. de la Décision 7713 (1977).

[MM] M. D. P. Monteiro Marquès, *Perturbations convexes semi-continues supérieurement de problèmes d'évolution dans les espaces de Hilbert*, Sém. d'Analyse Convexe Montpellier (1984), exp. 2 (23 pages).

[M1] J. J. Moreau, *Sur les mesures différentielles des fonctions vectorielles à variation localement bornée*, Sém. d'Analyse Convexe Montpellier (1975), exp. 17 (39 pages).

[M2] —, *Evolution problem associated with a moving convex set in a Hilbert space*, J. Differential Equations 26 (1977), 347–374.

[MV1] J. J. Moreau et M. Valadier, *Dérivation d'une mesure vectorielle sur un intervalle*, Sém. d'Analyse Convexe Montpellier (1984), exp. 1 (14 pages).

[MV2] J. J. Moreau et M. Valadier, *Quelques résultats sur les fonctions vectorielles à variation bornée d'une variable réelle*, Sém. d'Analyse Convexe Montpellier (1984), exp. 2 (16 pages).

[MV3] —, —, *A chain rule involving vector functions of bounded variation*, J. Funct. Anal. 74 (1987), 333-345.

[MO] M. Moussaoui, *Approximations lipschitziennes de multifonctions et application à la relaxation de certains problèmes de contrôle optimal et de calcul de variations*, Thèse, Université Montpellier II, 1990 (159 pages).

[N] M. Nagumo, *Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen*, Proc. Math. Soc. Japan 24 (1942), 555–559.

[T] A. A. Tolstonogov, *On the structure of the solutions set for differential inclusions in a Banach space*, Mat. Sb. 118 (1982), 3–18 (in Russian); English transl. in Math. USSR-Sb 46 (1983).

[V] M. Valadier, *Applications des mesures de Young aux suites uniformément intégrables dans un Banach séparable*, Sém. d'Analyse Convexe Montpellier (1990), exp. 3 (14 pages).