CONNECTIONS BETWEEN RECENT OLECH-TYPE LEMMAS AND VISINTIN’S THEOREM

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Abstract. A recent Olech-type lemma of Artstein-Rzeżuchowski [2] and its generalization in [7] are shown to follow from Visintin’s theorem, by exploiting a well-known property of extreme points of the integral of a multifunction.

1. Main results. Let $(\Omega, \mathcal{F}, \mu)$ be a nonatomic finite measure space, and let $F : \Omega \to 2^{\mathbb{R}^d}$ be a given multifunction with measurable graph and closed values. Recall that the integral of the multifunction $F$ over $\Omega$ is defined by

$$
\int\Omega F d\mu := \left\{ \int\Omega f d\mu : f \in \mathcal{L}^1_F \right\},
$$

where $\mathcal{L}^1_F$ denotes the set of all integrable a.e.-selectors of $F$ [4]. By nonatomicity of the measure space, such an integral is always convex [4]. In this section $(f_k)$ will denote a given sequence in $\mathcal{L}^1_F$. Correspondingly, we define the pointwise Kuratowski limes superior set by

$$
L(\omega) := Ls(f_k(\omega)) \subset F(\omega).
$$

By the well-known identity

$$
L(\omega) = \bigcap_{p=1}^{\infty} \text{cl}\{f_k(\omega) : k \geq p\},
$$

$L$ has a measurable graph and closed values (this would also have been true if the graph of $F$ had been nonmeasurable). In [2] Artstein and Rzeżuchowski gave the following result.

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Proposition 1.1 ([2]). Suppose that \((f_k)\) is uniformly integrable and such that
\[
\lim_k \int_{\Omega} f_k \, d\mu = e,
\]
where \(e\) is an extreme point of \(\int F \, d\mu\). Then there exists \(f_* \in L^1_F\) such that
\[
(1) \quad \lim_k \int_{\Omega} |f_k - f_*| \, d\mu = 0.
\]
Of course, (1) implies \(e = \int f_* \, d\mu\); for this reason the result by Artstein and Rzeżuchowski can be seen as a variation on a theme started by Olech, who considered extremality of \(e\) in the closure of \(\int F \, d\mu\) [13, 14, 3]. Recently, the present author obtained the following extension of Proposition 1.1:

Proposition 1.2 ([7]). Suppose that
\[
\sup_k \int_{\Omega} |f_k| \, d\mu < +\infty
\]
and
\[
\lim_k \int_{\Omega} f_k \, d\mu = e,
\]
where \(e\) is an extreme point of \(\int_{\Omega} F \, d\mu\). Moreover, suppose that \(e\) has the following maximality property:
\[
\left( \int_{\Omega} L \, d\mu - e \right) \cap C^0 = \{0\},
\]
where \(C^0\) is the negative polar of the cone \(C\) of all \(y \in \mathbb{R}^d\) satisfying
\[
(\min(y : f_k, 0)) \text{ is uniformly integrable.}
\]
Then there exists \(f_* \in L^1_L \subset L^1_F\) such that
\[(f_k)\text{ converges in measure to } f_* \text{ and } \int_{\Omega} f_* \, d\mu = e.\]

Clearly, the latter proposition extends the former one (which has of course \(C^0 = \{0\}\)). The proof of Proposition 1.1 given in [2] is very simple, but it uses [1, Theorem A], which has a fairly hard proof. The proof in [7] is possibly even more complicated (depending upon one's degree of familiarity with Young measure theory). Artstein and Rzeżuchowski observe in [2] that the following well-known theorem by Visintin [18, 17] (see also Theorem 1.4 below) can be considered to be a consequence of their result.

Theorem 1.3 ([18]). Suppose that \((f_k)\) converges weakly (in \(\sigma(L^1, L^\infty)\)) to some function \(f_0 \in L^1(\Omega; \mathbb{R}^d)\) such that
\[
f_0(\omega) \text{ is an extreme point of } \text{co } F(\omega) \text{ a.e. (1)}
\]

(1) By Lemma A.2 the weak convergence itself already implies \(f_0(\omega) \in \text{co } L(\omega) \subset \text{co } F(\omega)\) a.e.
Then \( \lim_k \int_{\Omega} |f_k - f_0| d\mu = 0 \).

The purpose of this note is to stress that the converse is also true: Visintin’s theorem immediately implies Proposition 1.1, via a well-known characterization of the extreme points of \( \int F d\mu \). Moreover, the following extension of Visintin’s result, which is due to the present author and essentially contained in [6] (cf. [17, 7]), can be used similarly to obtain Proposition 1.2.

**Theorem 1.4.** Suppose that \( (f_k) \) converges weakly (in \( \sigma(\mathcal{L}^1, \mathcal{L}^\infty) \)) to some function \( f_0 \in \mathcal{L}^1(\Omega; \mathbb{R}^d) \) such that

\[
    f_0(\omega) \text{ is an extreme point of } \text{co} \, L(\omega) \text{ a.e.}(1)
\]

Then \( \lim_k \int_{\Omega} |f_k - f_0| d\mu = 0 \).

Note that closed convex hulls appear in the original results in [18] and [6] that correspond to in Theorems 1.3, 1.4 (observe that [6] specifically deals with an infinite-dimensional case, of which the present paper considers the finite-dimensional variant). As was briefly indicated in [8, p. 28], the strengthening in terms of convex hulls, as presented in the two theorems above, follows by an obvious adaptation of the arguments in [6, 8], based on the fact that barycenters of probability measures on a finite-dimensional Banach space already lie in the convex hull – and not just the closed convex hull – of their support [15].

The connection between Propositions 1.1, 1.2 on the one side and Theorems 1.3, 1.4 on the other side is provided by the following well-known result, which will be applied to both \( F \) and \( L \).

**Lemma 1.5.** Let \( G : \Omega \to 2^{\mathbb{R}^d} \) be a multifunction with measurable graph and closed values. Suppose that \( e \in \mathbb{R}^d \) is an extreme point of \( \int Gd\mu \). Then there exists an essentially unique \( f \in \mathcal{L}_G^1 \) such that \( e = \int_G f d\mu \) and

\[
    f(\omega) \text{ is an extreme point of } \text{co} \, G(\omega) \text{ a.e.}(2)
\]

**Proof.** By definition of the set \( \int G \), there exists at least one \( f \in \mathcal{L}_G^1 \) with \( \int f = e \). Suppose that \( f, f' \in \mathcal{L}_G^1 \) both satisfy \( e = \int f = \int f' \). For any \( B \in \mathcal{F} \) both \( g := f + 1_B(f' - f) \) and \( g' := f' + 1_B(f - f') \) belong to \( \mathcal{L}_G^1 \), and \( \int (g + g') = 2e \). Hence, it follows by the the extreme point property of \( e \) that \( \int_B (f - f') = 0 \). So by arbitrariness of the set \( B \), we conclude that \( f = f' \) a.e.

Next, suppose that there exists a nonnull set \( B \in \mathcal{F} \) such that for every \( \omega \in B \) the property (2) does not hold. For this reason, there exist for each \( \omega \in B \) a number \( N_\omega \) of points \( x_{1,\omega}, \ldots, x_{N_\omega,\omega} \) in \( G(\omega) \), all of which are distinct from \( f(\omega) \), and corresponding scalars \( \lambda_{1,\omega}, \ldots, \lambda_{N_\omega,\omega} \geq 0 \) such that \( \sum_{i} \lambda_{i,\omega} x_{i,\omega} = f(\omega) \) and \( \sum_{i} \lambda_{i,\omega} = 1 \). By reducing for affine dependence, the number \( N_\omega \) can be reduced to so as to be at most \( d + 1 \) (just as in the proof of Carathéodory’s theorem). Of course, by adding arbitrary points \( x_{i,\omega} \neq f(\omega) \) with corresponding \( \lambda_{i,\omega} \)’s set equal to zero, we can ensure \( N_\omega = d + 1 \). By an obvious measurable selection argument (see the proof of [10, IV.11]) we find that there exist \( d + 1 \) measurable functions \( g_1, \ldots, g_{d+1} \) from \( B \) into \( \mathbb{R}^d \) and \( d + 1 \) measurable scalar functions \( \alpha_1, \ldots, \alpha_{d+1} \).
from $B$ into $[0,1]$ such that for a.e. $\omega$ in $B$: (i) $g_i(\omega), \ldots, g_{d+1}(\omega)$ lie in $G(\omega)$ and are all distinct from $f(\omega)$, (ii) $\sum_i \alpha_i(\omega) g_i(\omega) = f(\omega)$, and (iii) $\sum_i \alpha_i(\omega) = 1$. For $n \in \mathbb{N}$ define $B_n$ to be the set of all $\omega \in B$ for which $\max_{1 \leq i \leq k} |g_i(\omega)| \leq n$. The $B_n$ increase monotonically to $B$, so there exists $n$—fixed from now on—with $\mu(B_n) > 0$. Let us define $h_i := 1_{B_n} f + 1_{B_n} g_i$, $i = 1, \ldots, d+1$. Clearly, the functions $h_1, \ldots, h_{d+1}$ belong to $L_0^1$. Further, from (ii)-(iii) above it follows that

$$\sum \alpha_i h_i = f \ a.e.$$ By Lyapunov’s theorem [10, IV.17] it follows that there exists a measurable partition $\{C_1, \ldots, C_{d+1}\}$ of $\Omega$ such that $e = \int f = \int \sum_i \alpha_i h_i = \sum \int_{C_i} h_i$. By the essential uniqueness of $f$, established above, we conclude that $f = \sum 1_{C_i} h_i$ a.e., which amounts to having $f = \sum 1_{C_i} g_i$ a.e. on $B_n$. But there must be $i$ with $\mu(B_n \cap C_i) > 0$, and then we have a contradiction with the fact that a.e. on $B_n$ the values $g_i(\omega)$ are distinct from $f(\omega)$. ■

Let us now prove the Artstein-Rzeżuchowski result by means of Theorem 1.3.

Proof of Proposition 1.1. By Lemma 1.5 there exists an essentially unique $f_* \in L_1^1$ with $e = \int f_* d\mu$. Define $\alpha := \limsup_k \int |f_k - f_*|$. Then there exists a subsequence $(f_k_j)$ with $\limsup_j \int |f_k_j - f_*| = \alpha$. By the Dunford-Pettis theorem there exists a further subsequence $(f_k_{j\ell})$ of $(f_k_j)$ which converges weakly (in $\sigma(L^1, L^\infty)$) to some function $f_0 \in L_1^1$. But then also $e = \int f_0$, so $f_* = f_0$ a.e. by the essential uniqueness of $f_*$. Further, by Lemma 1.5 the extreme point condition of Theorem 1.3 is precisely fulfilled. So this theorem gives $\lim_n \int |f_k_n - f_*| = 0$, which proves that $\alpha = 0$. ■

Next, let us deduce Proposition 1.2 in a slightly more involved way from Theorem 1.4 by means of the same Lemma 1.5. Here we shall use the biting lemma and facts about $w^2$-convergence that have been gathered in the appendix.

Proof of Proposition 1.2. Again, there exists $f_* \in L_1^1$ with $e = \int f_* d\mu$, and $f_k$ is essentially unique by Lemma 1.5. Let $(f_k_j)$ be an arbitrary subsequence of $(f_k)$. By Lemma A.1 $(f_k_j)$ has a further subsequence $(f_k_{j\ell})$ which $w^2$-converges to some $f_0 \in L_1^1$. Let $(B_p)$ denote the corresponding sequence of “bites”, which decreases monotonically to a null set. Fix any $y$ in the cone $C$. Then

$$y \cdot e = \int \left. y \cdot f_0 + \liminf_{\Omega \setminus B_p} \int y \cdot f_{k_n} \right|_{B_p}$$

for any $p$. So by definition of the cone $C$ it follows easily that $y \cdot e \geq \int_{\Omega \setminus B_p} y \cdot f_0$. Hence, we conclude that $\int f_0 - e$ belongs to $C^0$; by Lemma A.3 the same vector also belongs to $\int L - e$. So our maximality hypothesis implies that $\int f_0 = e$, which gives $f_* = f_0$ a.e., in view of the essential uniqueness of $f_*$. Now we apply Lemma 1.5. This gives that the extreme point condition of Theorem 1.4 is precisely met. So the latter theorem gives for any $p$

$$\lim_n \int \left. |f_{k_n} - f_*| d\mu \right|_{\Omega \setminus B_p} = 0.$$
Since the bites $B_p$ decrease to a null set, this gives that $(f_{k_n})$ converges in measure to $f_*$. Now an arbitrary subsequence of $(f_k)$ has been shown to possess a further subsequence which converges to $f_*$ in measure. Therefore, we conclude that $(f_k)$ itself converges in measure to $f_*$. 

Remark 1.6. By Lemma A.3 and the above proof, we have $e = \int f_* = \int f_0 \in \int L$ in Proposition 1.2. So a slightly sharper formulation [7] would have been to require $e$ to be an extreme point of $\int Ld\mu$, rather than of $\int Fd\mu$. This observation also signifies that it is not really necessary to work with the hypothesis that the graph of $F$ is measurable, for, by an earlier observation, the graph of $L$ is always measurable, irrespective of the measurability of the graph of $F$.

Appendix. Here we gather some facts related to the biting lemma and $w^2$-convergence. First, recall the following definition [9], which weakens the notion of weak convergence: a sequence $(f_n)$ in $L^1(\mathbb{R}^d)$ is said to $w^2$-converge to $f_0 \in L^1(\mathbb{R}^d)$ if there exists a sequence $(B_p)$ of “bites” in $F$, monotonically decreasing to a null set (i.e., $B_{p+1} \subset B_p$ for all $p$ and $\mu(\cap_p B_p) = 0$), such that for every $p$

$$(f_n |_{B_p})_n$$

converges weakly (in $\sigma(L^1(B_p), L^{\infty}(B_p)))$ to $f_0 |_{B_p}$.

The following result seems due to Gaposhkin [11]; it has been independently rediscovered by many other authors (e.g., see [9, 16]).

**Lemma A.1 (biting lemma).** Suppose that $(f_k)$ is a sequence in $L^1(\mathbb{R}^d)$ such that

$$\sup_k \int \Omega |f_k|d\mu < +\infty.$$ 

Then $(f_k)$ has a subsequence which $w^2$-converges to some function in $L^1(\mathbb{R}^d)$.

The following fact, which is essentially Proposition C in [1], is certainly not elementary. Another proof follows by applying [15] to Example 2.3 of [5](2).

**Lemma A.2 ([1]).** Suppose that $(f_n)$ is a sequence in $L^1(\mathbb{R}^d)$ which $w^2$-converges to $f_0 \in L^1(\mathbb{R}^d)$. Then

$$f_0(\omega) \in \text{co} \text{Ls}(f_n(\omega)) \quad \text{a.e.}$$

The next fact comes from [7, Theorem 2.2] and the observation in the last footnote; whether it could also be proven by Aumann’s well-known identity [4] and the previous lemma is an open question to the present author.

**Lemma A.3 ([7]).** Suppose that $(f_n)$ is a sequence in $L^1(\mathbb{R}^d)$ which $w^2$-converges to $f_0 \in L^1(\mathbb{R}^d)$ and is such that

$$\sup_n \int \Omega |f_n|d\mu < +\infty.$$ 

(2) By Example 2.2 of [5] it is easy to check that $\eta_*$ on its p. 574 coincides a.e. with our present $f_0$. 

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Then
\[ \int_\Omega f_0 \, d\mu \in \int_\Omega L \, d\mu. \]

References