GEOMETRY IN NONLINEAR CONTROL
AND DIFFERENTIAL INCLUSIONS
BANACH CENTER PUBLICATIONS, VOLUME 32
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1995

CONNECTIONS BETWEEN RECENT OLECH-TYPE LEMMAS AND VISINTIN'S THEOREM

ERIK J. BALDER

Mathematical Institute, University of Utrecht
Utrecht, The Netherlands

Abstract. A recent Olech-type lemma of Artstein-Rzeżuchowski [2] and its generalization in [7] are shown to follow from Visintin's theorem, by exploiting a well-known property of extreme points of the integral of a multifunction.

1. Main results. Let $(\Omega, \mathcal{F}, \mu)$ be a nonatomic finite measure space, and let $F : \Omega \to 2^{\mathbf{R}^d}$ be a given multifunction with measurable graph and closed values. Recall that the integral of the multifunction F over Ω is defined by

$$\int_\Omega F d\mu := \Big\{ \int_\Omega f d\mu : f \in \mathcal{L}^1_F \Big\},$$

where $\mathcal{L}^1_F$ denotes the set of all integrable a.e.-selectors of F [4]. By nonatomicity of the measure space, such an integral is always convex [4]. In this section (f_k) will denote a given sequence in $\mathcal{L}^1_F$. Correspondingly, we define the pointwise Kuratowski limes superior set by

$$L(\omega) := \mathrm{Ls}(f_k(\omega)) \subset F(\omega).$$

By the well-known identity

$$L(\omega) = \bigcap_{p=1}^{\infty} \mathrm{cl}\{f_k(\omega) : k \geq p\},$$

L has a measurable graph and closed values (this would also have been true if the graph of F had been nonmeasurable). In [2] Artstein and Rzeżuchowski gave the following result.

1991 *Mathematics Subject Classification*: 28A20, 28B20.
The paper is in final form and no version of it will be published elsewhere.

Proposition 1.1 ([2]). *Suppose that* (f_k) *is uniformly integrable and such that*

$$\lim_k \int_\Omega f_k d\mu = e,$$

where e *is an extreme point of* $\int F d\mu$. *Then there exists* $f_* \in \mathcal{L}^1_F$ *such that*

$$\lim_k \int_\Omega |f_k - f_*| d\mu = 0. \tag{1}$$

Of course, (1) implies $e = \int f_* d\mu$; for this reason the result by Artstein and Rzeżuchowski can be seen as a variation on a theme started by Olech, who considered extremality of e in the *closure* of $\int F d\mu$ [13, 14, 3]. Recently, the present author obtained the following extension of Proposition 1.1:

Proposition 1.2 ([7]). *Suppose that*

$$\sup_k \int_\Omega |f_k| d\mu < +\infty$$

and

$$\lim_k \int_\Omega f_k d\mu = e,$$

where e *is an extreme point of* $\int_\Omega F d\mu$. *Moreover, suppose that* e *has the following maximality property*:

$$\Big(\int_\Omega L d\mu - e\Big) \cap C^0 = \{0\},$$

where C^0 *is the negative polar of the cone* C *of all* $y \in \mathbf{R}^d$ *satisfying*

$$(\min(y \cdot f_k, 0)) \quad \textit{is uniformly integrable.}$$

Then there exists $f_* \in \mathcal{L}^1_L \subset \mathcal{L}^1_F$ *such that*

$$(f_k) \textit{ converges in measure to } f_* \textit{ and } \int_\Omega f_* d\mu = e.$$

Clearly, the latter proposition extends the former one (which has of course $C^0 = \{0\}$). The proof of Proposition 1.1 given in [2] is very simple, but it uses [1, Theorem A], which has a fairly hard proof. The proof in [7] is possibly even more complicated (depending upon one's degree of familiarity with Young measure theory). Artstein and Rzeżuchowski observe in [2] that the following well-known theorem by Visintin [18, 17] (see also Theorem 1.4 below) can be considered to be a consequence of their result.

Theorem 1.3 ([18]). *Suppose that* (f_k) *converges weakly* (*in* $\sigma(\mathcal{L}^1, \mathcal{L}^\infty)$) *to some function* $f_0 \in \mathcal{L}^1(\Omega; \mathbf{R}^d)$ *such that*

$$f_0(\omega) \quad \textit{is an extreme point of } \operatorname{co} F(\omega) \textit{ a.e.} \ (^1)$$

(1) By Lemma A.2 the weak convergence itself already implies $f_0(\omega) \in \operatorname{co} L(\omega) \subset \operatorname{co} F(\omega)$ a.e.

Then $\lim_k \int_\Omega |f_k - f_0| d\mu = 0$.

The purpose of this note is to stress that the converse is also true: Visintin's theorem immediately implies Proposition 1.1, via a well-known characterization of the extreme points of $\int F d\mu$. Moreover, the following extension of Visintin's result, which is due to the present author and essentially contained in [6] (cf. [17, 7]), can be used similarly to obtain Proposition 1.2.

THEOREM 1.4. *Suppose that* (f_k) *converges weakly* (*in* $\sigma(\mathcal{L}^1, \mathcal{L}^\infty)$) *to some function* $f_0 \in \mathcal{L}^1(\Omega; \mathbf{R}^d)$ *such that*

$$f_0(\omega) \text{ is an extreme point of } \operatorname{co} L(\omega) \text{ a.e.}(^1)$$

Then $\lim_k \int_\Omega |f_k - f_0| d\mu = 0$.

Note that *closed* convex hulls appear in the original results in [18] and [6] that correspond to in Theorems 1.3, 1.4 (observe that [6] specifically deals with an infinite-dimensional case, of which the present paper considers the finite-dimensional variant). As was briefly indicated in [8, p. 28], the strengthening in terms of convex hulls, as presented in the two theorems above, follows by an obvious adaptation of the arguments in [6, 8], based on the fact that barycenters of probability measures on a finite-dimensional Banach space already lie in the convex hull – and not just the closed convex hull – of their support [15].

The connection between Propositions 1.1, 1.2 on the one side and Theorems 1.3, 1.4 on the other side is provided by the following well-known result, which will be applied to both F and L.

LEMMA 1.5. *Let* $G : \Omega \to 2^{\mathbf{R}^d}$ *be a multifunction with measurable graph and closed values. Suppose that* $e \in \mathbf{R}^d$ *is an extreme point of* $\int G d\mu$. *Then there exists an essentially unique* $f \in \mathcal{L}^1_G$ *such that* $e = \int_\Omega f d\mu$ *and*

$$f(\omega) \text{ is an extreme point of } \operatorname{co} G(\omega) \text{ a.e.} \tag{2}$$

P r o o f. By definition of the set $\int G$, there exists at least one $f \in \mathcal{L}^1_G$ with $\int f = e$. Suppose that $f,\ f' \in \mathcal{L}^1_G$ both satisfy $e = \int f = \int f'$. For any $B \in \mathcal{F}$ both $g := f + 1_B(f' - f)$ and $g' := f' + 1_B(f - f')$ belong to $\mathcal{L}^1_G$, and $\int (g + g') = 2e$. Hence, it follows by the the extreme point property of e that $\int_B (f - f') = 0$. So by arbitrariness of the set B, we conclude that $f = f'$ a.e.

Next, suppose that there exists a nonnull set $B \in \mathcal{F}$ such that for every $\omega \in B$ the property (2) does not hold. For this reason, there exist for each $\omega \in B$ a number N_ω of points $x_{1,\omega}, \cdots, x_{N_\omega,\omega}$ in $G(\omega)$, all of which are *distinct* from $f(\omega)$, and corresponding scalars $\lambda_{1,\omega}, \cdots, \lambda_{N_\omega,\omega} \geq 0$ such that $\sum_i \lambda_{i,\omega} x_{i,\omega} = f(\omega)$ and $\sum_i \lambda_{i,\omega} = 1$. By reducing for affine dependence, the number N_ω can be reduced to so as to be at most $d + 1$ (just as in the proof of Carathéodory's theorem). Of course, by adding arbitrary points $x_{i,\omega} \neq f(\omega)$ with corresponding $\lambda_{i,\omega}$'s set equal to zero, we can ensure $N_\omega = d + 1$. By an obvious measurable selection argument (see the proof of [10, IV.11]) we find that there exist $d + 1$ measurable functions $g_1, \cdots, g_{d+1}$ from B into $\mathbf{R}^d$ and $d + 1$ measurable scalar functions $\alpha_1, \cdots, \alpha_{d+1}$

from B into [0,1] such that for a.e. ω in B: (i) $g_1(\omega), \cdots, g_{d+1}(\omega)$ lie in $G(\omega)$ and are all distinct from $f(\omega)$, (ii) $\sum_i \alpha_i(\omega) g_i(\omega) = f(\omega)$, and (iii) $\sum_i \alpha_i(\omega) = 1$. For $n \in \mathbf{N}$ define B_n to be the set of all $\omega \in B$ for which $\max_{1 \leq i \leq k} |g_i(\omega)| \leq n$. The B_n increase monotonically to B, so there exists n – fixed from now on – with $\mu(B_n) > 0$. Let us define $h_i := 1_{\Omega \setminus B_n} f + 1_{B_n} g_i$, $i = 1, \cdots, d+1$. Clearly, the functions $h_1, \cdots, h_{d+1}$ belong to $\mathcal{L}^1_G$. Further, from (ii)-(iii) above it follows that $\sum_i \alpha_i h_i = f$ a.e. By Lyapunov's theorem [10, IV.17] it follows that there exists a measurable partition $\{C_1, \cdots, C_{d+1}\}$ of Ω such that $e = \int f = \int \sum_i \alpha_i h_i = \sum_i \int_{C_i} h_i$. By the essential uniqueness of f, established above, we conclude that $f = \sum_i 1_{C_i} h_i$ a.e., which amounts to having $f = \sum 1_{C_i} g_i$ a.e. on B_n. But there must be i with $\mu(B_n \cap C_i) > 0$, and then we have a contradiction with the fact that a.e. on B_n the values $g_i(\omega)$ are distinct from $f(\omega)$. ∎

Let us now prove the Artstein-Rzeżuchowski result by means of Theorem 1.3.

P r o o f o f P r o p o s i t i o n 1.1. By Lemma 1.5 there exists an essentially unique $f_* \in \mathcal{L}^1_F$ with $e = \int f_* d\mu$. Define $\alpha := \limsup_k \int |f_k - f_*|$. Then there exists a subsequence (f_{k_j}) with $\lim_k \int |f_{k_j} - f_*| = \alpha$. By the Dunford-Pettis theorem there exists a further subsequence (f_{k_n}) of (f_{k_j}) which converges weakly (in $\sigma(\mathcal{L}^1, \mathcal{L}^\infty)$) to some function $f_0 \in \mathcal{L}^1_{\mathbf{R}^d}$. But then also $e = \int f_0$, so $f_* = f_0$ a.e. by the essential uniqueness of f_*. Further, by Lemma 1.5 the extreme point condition of Theorem 1.3 is precisely fulfilled. So this theorem gives $\lim_n \int |f_{k_n} - f_*| = 0$, which proves that $\alpha = 0$. ∎

Next, let us deduce Proposition 1.2 in a slightly more involved way from Theorem 1.4 by means of the same Lemma 1.5. Here we shall use the biting lemma and facts about w^2-convergence that have been gathered in the appendix.

P r o o f o f P r o p o s i t i o n 1.2. Again, there exists $f_* \in \mathcal{L}^1_F$ with $e = \int f_* d\mu$, and f_* is essentially unique by Lemma 1.5. Let (f_{k_j}) be an arbitrary subsequence of (f_k). By Lemma A.1 (f_{k_j}) has a further subsequence (f_{k_n}) which w^2-converges to some $f_0 \in \mathcal{L}^1_{\mathbf{R}^d}$. Let (B_p) denote the corresponding sequence of "bites", which decreases monotonically to a null set. Fix any y in the cone C. Then

$$y \cdot e = \int_{B_p^c} y \cdot f_0 + \liminf_n \int_{B_p} y \cdot f_{k_n}$$

for any p. So by definition of the cone C it follows easily that $y \cdot e \geq \int_\Omega y \cdot f_0$. Hence, we conclude that $\int f_0 - e$ belongs to C^0; by Lemma A.3 the same vector also belongs to $\int L - e$. So our maximality hypothesis implies that $\int f_0 = e$, which gives $f_* = f_0$ a.e., in view of the essential uniqueness of f_*.

Now we apply Lemma 1.5. This gives that the extreme point condition of Theorem 1.4 is precisely met. So the latter theorem gives for any p

$$\lim_n \int_{\Omega \setminus B_p} |f_{k_n} - f_*| d\mu = 0.$$

Since the bites B_p decrease to a null set, this gives that (f_{k_n}) converges in measure to f_*. Now an arbitrary subsequence of (f_k) has been shown to possess a further subsequence which converges to f_* in measure. Therefore, we conclude that (f_k) itself converges in measure to f_*. ∎

R e m a r k 1.6. By Lemma A.3 and the above proof, we have $e = \int f_* = \int f_0 \in \int L$ in Proposition 1.2. So a slightly sharper formulation [7] would have been to require e to be an extreme point of $\int L d\mu$, rather than of $\int F d\mu$. This observation also signifies that it is not really necessary to work with the hypothesis that the graph of F is measurable, for, by an earlier observation, the graph of L is always measurable, irrespective of the measurability of the graph of F.

Appendix. Here we gather some facts related to the biting lemma and w^2-convergence. First, recall the following definition [9], which weakens the notion of weak convergence: a sequence (f_n) in $\mathcal{L}^1_{\mathbf{R}^d}$ is said to *w^2-converge* to $f_0 \in \mathcal{L}^1_{\mathbf{R}^d}$ if there exists a sequence (B_p) of "bites" in $\mathcal{F}$, monotonically decreasing to a null set (i.e., $B_{p+1} \subset B_p$ for all p and $\mu(\cap_p B_p) = 0$), such that for every p

$$(f_n \mid_{B_p^c})_n \text{ converges weakly (in } \sigma(\mathcal{L}^1(B_p^c), \mathcal{L}^\infty(B_p^c))) \text{ to } f_0 \mid_{B_p^c} .$$

The following result seems due to Gaposhkin [11]; it has been independently rediscovered by many other authors (e.g., see [9, 16]).

LEMMA A.1 (biting lemma). *Suppose that (f_k) is a sequence in $\mathcal{L}^1_{\mathbf{R}^d}$ such that*

$$\sup_k \int_\Omega |f_k| d\mu < +\infty.$$

Then (f_k) has a subsequence which w^2-converges to some function in $\mathcal{L}^1_{\mathbf{R}^d}$.

The following fact, which is essentially Proposition C in [1], is certainly not elementary. Another proof follows by applying [15] to Example 2.3 of [5]([2]).

LEMMA A.2 ([1]). *Suppose that (f_n) is a sequence in $\mathcal{L}^1_{\mathbf{R}^d}$ which w^2-converges to $f_0 \in \mathcal{L}^1_{\mathbf{R}^d}$. Then*

$$f_0(\omega) \in \operatorname{co} \operatorname{Ls}(f_n(\omega)) \qquad \text{a.e.}$$

The next fact comes from [7, Theorem 2.2] and the observation in the last footnote; whether it could also be proven by Aumann's well-known identity [4] and the previous lemma is an open question to the present author.

LEMMA A.3 ([7]). *Suppose that (f_n) is a sequence in $\mathcal{L}^1_{\mathbf{R}^d}$ which w^2-converges to $f_0 \in \mathcal{L}^1_{\mathbf{R}^d}$ and is such that*

$$\sup_n \int_\Omega |f_n| d\mu < +\infty.$$

([2]) By Example 2.2 of [5] it is easy to check that η_* on its p. 574 coincides a.e. with our present f_0.

Then

$$\int_{\Omega} f_0 d\mu \in \int_{\Omega} L d\mu.$$

References

[1] Z. Artstein, *A note on Fatou's lemma in several dimensions*, J. Math. Econom. 6 (1979), 277–282.

[2] Z. Artstein and T. Rzeżuchowski, *A note on Olech's lemma*, Studia Math. 98 (1991), 91–94.

[3] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[4] R. J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl. 12 (1965), 1–12.

[5] E. J. Balder, *A general approach to lower semicontinuity and lower closure in optimal control theory*, SIAM J. Control Optim. 22 (1984), 570–599.

[6] —, *On weak convergence implying strong convergence in L_1-spaces*, Bull Austral. Math. Soc. 33 (1986), 363–368.

[7] —, *A unified approach to several results involving integrals of multifunctions*, Set-Valued Anal. 2 (1994), 63–75.

[8] —, *On equivalence of strong and weak convergence in L_1-spaces under extreme point conditions*, Israel J. Math. 75 (1991), 21–47.

[9] J. K. Brooks and R. V. Chacon, *Continuity and compactness of measures*, Adv. in Math. 37 (1980), 16–26.

[10] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. 580, Springer, Berlin, 1977.

[11] V. F. Gaposhkin, *Convergence and limit theorems for sequences of random variables*, Theory Probab. Appl. 17 (3) (1972), 379–400.

[12] J. Neveu, *Bases Mathématiques du Calcul des Probabilités*, Masson, Paris, 1964.

[13] —, *Extremal solutions of a control system*, J. Differential Equations 2 (1966), 74–101.

[14] —, *Existence theory in optimal control*, in: Control Theory and Topics in Functional Analysis, IAEA, Vienna, 1976, 291–328.

[15] J. Pfanzagl, *Convexity and conditional expectations*, Ann. Probab. 2 (1974), 490-494.

[16] M. Slaby, *Strong convergence of vector-valued pramarts and subpramarts*, Probab. Math. Statist. 5 (1985), 187–196.

[17] M. Valadier, *Young measures, weak and strong convergence and the Visintin-Balder theorem*, Set-Valued Anal. 2 (1994), 357–367.

[18] A. Visintin, *Strong convergence results related to strict convexity*, Comm. Partial Differential Equations 9 (1984), 439–466.