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CONTROLLABILITY OF NILPOTENT SYSTEMS

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Abstract. In this paper we study the controllability property of invariant control systems on Lie groups. In [1], the authors state: "If there exists a real function strictly increasing on the positive trajectories, then the system cannot be controllable". To develop this idea, the authors define the concept of symplectic vector via the co-adjoint representation. We are interested in finding algebraic conditions to determine the existence of symplectic vectors in nilpotent Lie algebras. In particular, we state a necessary and sufficient condition for controllability in the simply connected nilpotent case.

1. Introduction. The aim of this paper is to find algebraic conditions which give information about the controllability for a particular class of systems, invariant control systems $\Sigma = (G, \mathcal{D})$ for which the state space is a connected Lie group G and the dynamics $\mathcal{D}$, which is a subset of the Lie algebra $\widetilde{g}$ of G, is determined by the specification of the following data:

$$\dot{g} = X(g) + \sum_{j=1}^{k} \mu_j Y^j(g)$$

where $g \in G$ and X, $Y^j \in \widetilde{g}$, $1 \le j \le k$. We consider the elements of $\widetilde{g}$ as left-invariant vector fields on G and without loss of generality we require that Σ satisfies the rank condition, [9], i.e.,

$$\operatorname{span}_{\mathcal{L.A.}}\{X, Y^1, \ldots, Y^k\} = \widetilde{g}.$$

The class $\mathcal{U} = \mathcal{U}(k)$ of unrestricted admissible controls is the class of all piecewise constant functions $\mu : [0, \infty) \to \mathbb{R}^k$ and $\mathcal{D}$ is the family of vector fields associated

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with Σ, i.e.

$$\mathcal{D} = \Big\{ X + \sum_{j=1}^{k} \mu_j Y^j \mid \mu \in \mathbb{R}^k \Big\}.$$

The systems $\Sigma = (G, \mathcal{D})$ are interesting not only from the theoretical point of view but also for their applications, [3], [7].

For each $Z \in \mathcal{D}$, we denote by $(Z_t)_{t\in\mathbb{R}}$ the 1-parameter group of diffeomorphisms on G generated by the vector field Z.

The rank condition means that Σ is transitive, i.e. the group

$$G_\Sigma = \{Z^1_{t_1} \circ Z^2_{t_2} \circ \ldots \circ Z^r_{t_r} \mid Z^j \in \mathcal{D}, t_j \in \mathbb{R}, r \in \mathbb{N}\}$$

acts transitively on G. Since $\mathcal{D}$ is a family of invariant vector fields, Σ is controllable if and only if the semigroup

$$S_\Sigma = \{Z^1_{t_1} \circ Z^2_{t_2} \circ \ldots \circ Z^r_{t_r} \mid Z^j \in \mathcal{D}, t_j \geq 0, r \in \mathbb{N}\}$$

satisfies $S_\Sigma(e) = G$.

Many people have dealt with this problem under various assumptions on G and $\mathcal{D}$, [2], [5], [6], [8].

In [1], the following idea is given:

"If there exists a function $f : G \to \mathbb{R}$ which is strictly increasing on the positive trajectories of Σ, i.e., on each $\varphi \in S_\Sigma$, then Σ cannot be controllable".

To develop this idea the authors define the concept of symplectic vector for invariant vector fields by using the co-adjoint representation of $\widetilde{g}$ obtaining a necessary condition for the controllability of Σ. This idea works because, if Σ is controllable and $g \in G$ there exist $\varphi, \psi \in S_\Sigma$ such that $g = \varphi(e)$ and $e = \psi(g)$. But, the function f must be strictly increasing on $\psi \circ \varphi$. Therefore, the existence of this type of functions is an obstruction to the controllability of invariant control systems. In section 2, we review some of the standard facts on co-adjoint orbits and we look for algebraic conditions that guarantee the existence of symplectic vectors. An important class that fit in this situation is the class of nilpotent systems, i.e. invariant control systems on nilpotent Lie groups. In the third section, we analize the controllability of nilpotent systems and give a characterization for the simply connected case. In section 4, we compute an example on the Heisenberg group.

1.1. *Main results.* We obtain the following results:

I. *Existence of symplectic vectors*

THEOREM 2.2 *Let G be a nilpotent simply connected Lie group with Lie algebra $\widetilde{g}$ and let $\widetilde{h}$ be an ideal of $\widetilde{g}$ such that $\widetilde{g}/\widetilde{h}$ is not an Abelian algebra. If $\pi : \widetilde{g} \to \widetilde{g}/\widetilde{h}$ is the canonical projection and there exists $Z \in \widetilde{g}$ such that $\pi(Z) \in Z(\widetilde{g}/\widetilde{h})$ is not a null vector field, then there exists a symplectic vector λ for Z.* ■

II. *Controllability.* Given any invariant control system $\Sigma = (G, \mathcal{D})$ we denote by $Z(\widetilde{g})$ the center of $\widetilde{g}$, by $\widetilde{h}$ the Lie subalgebra generated by the control vectors, i.e.

$$\widetilde{h} = \text{span}_{\mathcal{L}.A.}\{Y^1, Y^2, \dots, Y^k\}$$

and by

$$\text{Zt}(\widetilde{h}) = \{w \in \widetilde{g} \mid [w, \xi] = 0, \forall \xi \in \widetilde{h}\}$$

the centralizer of $\widetilde{h}$.

Let G be a nilpotent simply connected Lie group. Then we prove:

THEOREM 3.1. *If $Z(\widetilde{g}) \subsetneq \text{Zt}(\widetilde{h})$ then Σ cannot be controllable.* ■

THEOREM 3.6 *Σ is controllable $\Leftrightarrow \widetilde{h} = \widetilde{g}$.* ■

Moreover, we give in Proposition 3.3 a result closely related to Theorem 7.3 in [6].

2. Existence of symplectic vectors. Let G be a Lie group with Lie algebra $\widetilde{g}$. The adjoint representation ϱ of G is the homomorphism $\varrho : G \to \text{Aut}(\widetilde{g})$ defined as follows: for each $g \in G$, the analytic map $i_g : G \to G$, $i_g(h) = ghg^{-1}$, is an automorphism on G, and its derivative at e is an automorphism of $\widetilde{g}$. Then

$$\varrho : G \to \text{Aut}(\widetilde{g}), \qquad g \to \varrho(g) = di_g|_e,$$

is a linear representation of G in $\widetilde{g}$.

The derivative $d\varrho$ at each $w \in \widetilde{g}$ is given by

$$d\varrho(w)(\cdot) = [w, \cdot].$$

The co-adjoint representation ϱ^* (the contragradient representation of ϱ) is the linear representation of G in the dual space $\widetilde{g}^*$ of $\widetilde{g}$ and may be obtained by the action

$$G \times \widetilde{g}^* \to \widetilde{g}^*, \qquad (g, \lambda) \to \varrho^*(g)(\lambda),$$

defined by

$$\varrho^*(g)(\lambda)(w) = \lambda(\varrho(g^{-1})w), \qquad w \in \widetilde{g},$$

and its derivative is given by bracket evaluation. In fact, the diagram

$$\begin{array}{ccc} \widetilde{g} & \xrightarrow{d\varrho} & \text{End}(\widetilde{g}) \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle e} \\ G & \xrightarrow{\varrho} & \text{Aut}(\widetilde{g}) \end{array}$$

is commutative, and $d\varrho(w) \in \text{End}(\widetilde{g})$ for every $w \in \widetilde{g}$, thus

$$\varrho(\exp(tw)) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(d\varrho(w))^n.$$

Then, if $\xi \in \widetilde{g}$ and $\lambda \in \widetilde{g}^*$ we have

$$d\varrho^*(w)(\lambda)(\xi) = \frac{d}{dt}\bigg|_{t=0} \lambda(\varrho(\exp(-tw)))(\xi).$$

Taking the term by term derivative of the series $\varrho(\exp(-tw))$ it follows that the linear map $d\varrho^*: \widetilde{g} \to \mathrm{End}(\widetilde{g}^*)$ is given by

$$d\varrho^*(w)(\lambda)(\xi) = -\lambda[w,\xi], \qquad \xi \in \widetilde{g}.$$

Let $\lambda \in \widetilde{g}^*$. Then the co-adjoint orbit θ_λ of λ by the ϱ^* action, i.e.

$$\theta_\lambda = \varrho^*(G)(\lambda)$$

is a submanifold of $\widetilde{g}$, [4]. In fact, consider the analytic map

$$\varrho^*_\lambda : G \to \theta_\lambda, \qquad \varrho^*_\lambda(g) = \varrho^*(g)(\lambda).$$

For every $w \in \widetilde{g}^*$ and $g \in G$,

$$d\varrho^*_\lambda|_g(w) = \left.\frac{d}{dt}\right|_{t=0} \varrho^*_\lambda(g\exp(tw)) = \varrho^*(g)(d\varrho^*(w))(\lambda).$$

Since $\varrho^*(g) \in \mathrm{Aut}(\widetilde{g}^*)$, the subalgebra

$$e_\lambda = \{w \in \widetilde{g} | d\varrho^*(\omega)(\lambda) = 0\}$$

satisfies $\mathrm{rank}(d\varrho^*_\lambda|_g) = \dim(\widetilde{g}) - \dim(e_\lambda)$ for any $g \in G$. In particular, the rank of $d\varrho^*_\lambda$ is constant on G. On the other hand, the stabilizer by the ϱ^* action,

$$E_\lambda = \{g \in G \mid \varrho^*(g)(\lambda) = \lambda\},$$

is a closed Lie subgroup of G with Lie algebra e_λ.

Therefore, ϱ^*_λ induces a diffeomorphism $\widetilde{\varrho}^*_\lambda$ on the homogeneous space G/E_λ such that the diagram

$$\begin{tikzcd} G \arrow[rr, "\varrho^*_\lambda"] \arrow[dr, "\pi"'] & & \theta_\lambda \subset \widetilde{g}^* \\ & G/E_\lambda \arrow[ur, "\widetilde{\varrho}^*_\lambda"'] & \end{tikzcd}$$

is commutative. It is clear that $\widetilde{d\varrho}^*_\lambda|_{gE_\lambda}$ is an isomorphism between the tangent spaces $T_{gE_\lambda}(G/E_\lambda)$ and $T_{\varrho^*_\lambda(g)}\theta_\lambda$ for every $g \in G$. In particular, $d\varrho^*(\widetilde{g})(\lambda) \cong T_\lambda\theta_\lambda$. If $\Sigma = (G, \mathcal{D})$ is an invariant control system, the co-adjoint representation ϱ^* induces a system $\varrho^*(\Sigma)$ defined by

$$\varrho^*(\Sigma) := (\varrho^*(G), d\varrho^*(\mathcal{D})) \quad \text{where} \quad d\varrho^*(\mathcal{D}) = \{d\varrho^*(Z) \mid Z \in \mathcal{D}\}.$$

If we fix an initial condition $\lambda \in \widetilde{g}^*$ the systems:

1. $\Sigma_\lambda := (G/E_\lambda, d\pi(\mathcal{D}))$, where $\pi : G \to G/E_\lambda$ is the canonical projection, and
2. $\varrho^*_\lambda(\Sigma) := (\theta_\lambda, d\varrho^*_\lambda(\mathcal{D}))$

are equivalent.

In other words, these systems have the same dynamics for every admissible control $\mu \in U$. In particular,

$$S_{\Sigma_\lambda}(E_\lambda) = G/E_\lambda \Leftrightarrow S_{\varrho^*_\lambda(\Sigma)}(\lambda) = \theta_\lambda.$$

In [1], the authors give the following definition:

"$\lambda \in \widetilde{g}^*$ is a symplectic vector for $\omega \in \widetilde{g}$ if the co-adjoint orbit θ_λ is not trivial and $\beta(w) > 0$, $\forall \beta \in \theta_\lambda$".

The authors use this concept on invariant systems $\Sigma = (G, \mathcal{D})$ with

$$\dot{g} = X(g) + \sum_{j=1}^{k} \mu_j Y^j(g)$$

and obtain a necessary condition for the controllability of this type of systems via the following result:

"If there is a vector field ξ belonging to the centralizer of the subalgebra $\widetilde{h}$ such that the non-null vector field $Z = [X, \xi]$ has a symplectic vector, then Σ cannot be controllable".

In fact, the existence of a symplectic vector λ for Z allows us to construct the function

$$f_\xi : \theta_\lambda \to \mathbb{R}, \qquad \beta \to f_\xi(\beta) = -\beta(\xi),$$

such that for every $j = 1, 2, \ldots, k$ the directional derivatives of f_ξ related to the vector fields generating $d\varrho^*(\mathcal{D})$ satisfy for each $\beta \in \theta_\lambda$ and $j = 1, 2, \ldots, k$,

$$\begin{aligned} d\varrho^*(Y^j).f_\xi(\beta) &= -\frac{d}{dt}\bigg|_{t=0} (\varrho^*(\exp(tY^j)(\beta))(\xi) \\ &= -\beta\bigg(\frac{d}{dt}\bigg|_{t=0} \varrho(\exp(-tY^j))(\xi)\bigg) = \beta[Y^j, \xi] = 0. \end{aligned}$$

Analogously,

$$d\varrho^*(X).f_\xi(\beta) = \beta(Z) > 0.$$

Therefore, for each $\mu \in U$ and $\beta \in \theta_\lambda$,

$$d\varrho^*\Big(X + \sum_{j=1}^{k} \mu_j Y^j\Big).f_\xi(\beta) > 0.$$

In particular, f_ξ is strictly increasing on each $\varphi \in S_{\varrho^*_\lambda(\Sigma)}$. Thus, the system $\varrho^*_\lambda(\Sigma)$ cannot be controllable on θ_λ and consequently Σ is not controllable on G. In fact, the controllability of Σ on G implies the controllability of Σ_λ on the homogeneous space G/E_λ.

Now we analyze the existence of symplectic vectors.

PROPOSITION 2.1. *If $\widetilde{g}$ is not an Abelian algebra and $Z \in Z(\widetilde{g})$ is not a null vector field, then there exists a symplectic vector for Z.*

Proof. By definition, for each $\omega \in \widetilde{g}$ and $g \in G$, we have

$$\varrho(g)(\omega) = \frac{d}{dt}\bigg|_{t=0} g \exp(t\omega) g^{-1}.$$

In particular, every $\lambda \in \widetilde{g}^*$ is constant over the adjoint orbit of elements w belonging to the center of $\widetilde{g}$. We denote by $\mathcal{A}$ the union of the family of non-trivial co-adjoint orbits θ_λ with $\lambda \in \widetilde{g}^*$.

We claim that $\mathcal{A}$ is an non-empty open subset of $\widetilde{g}^*$. In fact, if $\Delta \in \mathrm{End}(\widetilde{g}^*)$ and $\Delta(\lambda) \neq 0$, then by a continuity argument there is a neighbourhood $V = V(\lambda)$ such that $\Delta(\beta) \neq 0$, $\forall \beta \in V$. If $\widetilde{g}$ is not Abelian, there exist $w \in \widetilde{g}$ such that the endomorphism $d\varrho(w)$ is not trivial. Since $\widetilde{g}^*$ separates points, there exists an element $\lambda \in \widetilde{g}^*$ such that the map

$$\beta(\cdot) = \lambda[w, \cdot] \in \widetilde{g}^*$$

is not null. In particular, the orbit θ_λ is not trivial, and this property is valid in a neighbourhood of λ. This proves our claim.

Now we suppose that for each $\lambda \in \widetilde{g}^*$,

$$\{\lambda\} \subsetneq \theta_\lambda \Rightarrow \beta(Z) = 0, \forall \beta \in \theta_\lambda.$$

We consider Z as a linear map defined on the dual of $\widetilde{g}$ by evaluation, i.e.

$$Z : \widetilde{g}^* \to \mathbb{R}, \qquad Z(\lambda) = \lambda(Z).$$

Then it is clear that $\mathcal{A} \subset \ker(Z)$, but this is a contradiction since $Z \neq 0$ and therefore $\mathrm{Ker}(Z)$ is a hyperplane in $\widetilde{g}^*$. So, $\exists \lambda \in \widetilde{g}^*$ such that $\beta(Z) \neq 0$, for some $\beta \in \theta_\lambda$. Now, Z belongs to the centre of $\widetilde{g}$, and we obtain

$$\lambda(\varrho(g)Z) = \lambda(Z), \qquad \forall g \in G.$$

Consequently, the orbit θ_λ must be constant on the vector field Z. Therefore λ (or $-\lambda$) is a symplectic vector for Z. ∎

It is possible to generalize Proposition 2.1 to nilpotent simply connected Lie groups. First, we give some general results on this kind of groups. Let G be a connected nilpotent Lie group and $\widetilde{g}$ its Lie algebra. If $\exp : \widetilde{g} \to G$ denotes the exponential map, then

$$d = \{w \in Z(\widetilde{g}) \mid \exp(w) = e\}$$

is a discrete additive subgroup of $\widetilde{g}$ and exp induces an analytic diffeomorphism $\widetilde{\exp} : \widetilde{g}/d \to G$ from the manifold $\widetilde{g}/d$ onto G. Moreover, $\widetilde{g}$ is a covering manifold of G, where exp is the covering map and d is the fundamental group of G.

If G is simply connected, we have:

(i) $\exp : \widetilde{g} \to G$ is an analytic diffeomorphism.

(ii) If H is a connected subgroup of G and $\widetilde{h}$ is the corresponding subalgebra of $\widetilde{g}$, then

(a) Since $\widetilde{h}$ is a nilpotent Lie algebra, $H = \exp(\widetilde{h})$.

(b) Since exp is a homeomorphism of $\widetilde{g}$ onto G and $\widetilde{h}$ is a closed simply connected subset of $\widetilde{g}$, H must be a closed simply connected subset of G.

(c) The irrational flow on the torus shows that if G is not simply connected its (normal) Lie subgroups are not necessarily closed.

THEOREM 2.2. *Let G be a nilpotent simply connected Lie group with Lie algebra $\widetilde{g}$ and $\widetilde{h}$ an ideal of $\widetilde{g}$ such that $\widetilde{g}/\widetilde{h}$ is not an Abelian algebra. If $\pi : \widetilde{g} \to \widetilde{g}/\widetilde{h}$ is* *the canonical projection and there exist $Z \in \widetilde{g}$ such that $\pi(Z) \in Z(\widetilde{g}/\widetilde{h})$ is not a null vector field, then there exists a symplectic vector λ for Z.*

Proof. By Proposition 2.1 we obtain a symplectic vector $\widetilde{\lambda}$ for the vector field $\pi(Z)$ in the quotient $\widetilde{g}/\widetilde{h}$. Since $\widetilde{h}$ is an ideal, there exists a connected normal Lie subgroup H of G with Lie algebra $\widetilde{h}$. Since G is a nilpotent simply connected Lie group, the exponential map is a diffeomorphism and H is closed. In particular, G/H is a Lie group. We consider the commutative diagrams

$$\begin{array}{ccc} \widetilde{g} & \xrightarrow{\varrho(g)} & \widetilde{g} \\ \pi\downarrow & & \downarrow\pi \\ \widetilde{g}/\widetilde{h} & \xrightarrow{\widetilde{\varrho}(gH)} & \widetilde{g}/\widetilde{h} \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{g} & \xrightarrow{\lambda} & \mathbb{R} \\ & \pi\searrow \quad \nearrow\widetilde{\lambda} & \\ & \widetilde{g}/\widetilde{h} & \end{array}$$

where $g \in G$ and $\widetilde{\varrho}$ is the adjoint representation of G/H. Then, for any $g \in G$ we have

$$\lambda(\varrho(g)Z) = \widetilde{\lambda}(\widetilde{\varrho}(gH)\pi(Z)) = \widetilde{\lambda} \circ \pi(Z).$$

By definition of $\widetilde{\lambda}$, $\theta_{\widetilde{\lambda}}$ is not a trivial orbit and there exist $g \in G$ such that $\widetilde{\lambda} \circ \widetilde{\varrho}(gH) \neq \widetilde{\lambda}$. Since the map

$$(\widetilde{g}/\widetilde{h})^* \longrightarrow \widetilde{g}^*, \qquad \widetilde{\lambda} \longrightarrow \widetilde{\lambda} \circ \pi,$$

is injective, we obtain $\lambda \circ \varrho(g) \neq \lambda$ and the proof is complete, because $\lambda(\varrho(g)Z) > 0, \forall g \in G$. ∎

3. Nilpotent systems. Let $\Sigma = (G, \mathcal{D})$ be a nilpotent system, i.e. Σ is an invariant control system with

$$\dot{g} = X(g) + \sum_{j=1}^{k} \mu_j Y^j(g)$$

such that the Lie algebra $\widetilde{g}$ of G is nilpotent. Additionally, we assume G is simply connected.

THEOREM 3.1. *If $Z(\widetilde{g}) \subsetneq Zt(\widetilde{h})$ then Σ cannot be controllable.*

Proof. Let ξ belong to $Zt(\widetilde{h}) \setminus Z(\widetilde{g})$ and let us define the vector field $Z = [X, \xi]$. Since $\widetilde{g}$ is nilpotent, the descending central series $\widetilde{g}^{(0)} = \widetilde{g}$ and $\widetilde{g}^{(i+1)} = [\widetilde{g}, \widetilde{g}^{(i)}]$, $i \in \mathbb{N}$, satisfies: there exist $n \in \mathbb{N}$ such that

$$\widetilde{g} = \widetilde{g}^{(0)} \supsetneq \widetilde{g}^{(1)} \supsetneq \ldots \supsetneq \widetilde{g}^{(n-1)} \supsetneq \widetilde{g}^{(n)} = 0.$$

Let

$$i_0 = \min\{i \mid Z \in \widetilde{g}^{(i)} \setminus \widetilde{g}^{(i+1)}\}.$$

Since Σ satisfies the rank condition, Z is not a null vector field. Thus $Z \in g^{(1)} \setminus g^{(n)}$, and then $1 \leq i_0 \leq n-1$. Moreover, by the definition of the descending central series, for every $i \in \mathbb{N}$, $\widetilde{g}^{(i)}$ is an ideal of $\widetilde{g}$ and the canonical projection

$\pi : \widetilde{g} \to \widetilde{g}/\widetilde{g}^{(i_0+1)}$ satisfies

$$\widetilde{g}^{(i_0)}/\widetilde{g}^{(i_0+1)} = Z(\widetilde{g}/\widetilde{g}^{(i_0+1)}).$$

In fact, fix w_1 and let $w_2 \in \widetilde{g}$. We have

$$[w_1 + \widetilde{g}^{(i_0+1)}, w_2 + \widetilde{g}^{(i_0+1)}] = \widetilde{g}^{(i_0+1)} \Leftrightarrow [w_1, w_2] \in \widetilde{g}^{(i_0+1)} \Leftrightarrow w_1 \in \widetilde{g}^{(i_0)}.$$

Since $Z \in \widetilde{g}^{(i_0)} \backslash \widetilde{g}^{(i_0+1)}$ the Lie algebra $\widetilde{g}/\widetilde{g}^{(i_0+1)}$ is not Abelian. Moreover, $\pi(Z) \neq \widetilde{g}^{(i_0+1)}$, so Theorem 2.2 gives the existence of a symplectic vector for Z and Σ is not controllable. ∎

Remark 3.2. Now we study the case when $\widetilde{h}$ has codimension 1 in the nilpotent algebra $\widetilde{g}$. For the sake of completeness we give a general result closely related to Theorem 7.3 in [6].

PROPOSITION 3.3. *Let $\Sigma = (G, \mathcal{D})$ be an invariant control system such that $\widetilde{h}$ is an ideal of codimension 1 in $\widetilde{g}$ and let H be the connected subgroup of G with Lie algebra $\widetilde{h}$.*

1. *If H is closed, then Σ is controllable $\Leftrightarrow G/H \simeq S^1$.*
2. *If H is not closed, then Σ is controllable.*

Proof. The subalgebra $\widetilde{h}$ is an ideal, therefore H is a normal subgroup of G.

1. If H is closed, the homogeneous space G/H is a Lie group and we can project Σ on an invariant control system $\pi(\Sigma) = (G/H, \{X + \widetilde{h}\})$ on the 1-dimensional manifold G/H. Suppose $X \in \widetilde{h}$. Thus, since Σ is transitive, it will also be controllable on the group $G = H$.

If $X \notin \widetilde{h}$, we separate the analysis in two cases:

(a) Compact case: $G/H \cong S^1$. In this case Σ is controllable, [6].

(b) Non-compact case: $G/H \cong \mathbb{R}^+$. Here $S_{\pi(\Sigma)}(1) = [1, +\infty)$ and hence Σ cannot be controllable.

2. Now suppose H is not closed. Therefore, the closure $\bar{H}$ of H is a closed Lie group with Lie algebra $\widetilde{g}$. Then H is a dense subgroup.

Since U is the class of unrestricted admissable controls, we have $\overline{H} \subset \overline{S_\Sigma(e)}$. In fact, for every $j = 1, 2, \ldots, k$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\exp t\left(\frac{1}{n}X + Y^j\right) \in S_\Sigma(e).$$

Then the positive orbit of the neutral point e is also dense in G, and Σ is controllable, [6]. ∎

PROPOSITION 3.4. *Let $\Sigma = (G, D)$ be a nilpotent system. If the subalgebra $\widetilde{h}$ has codimension 1 in $\widetilde{g}$ then the assertion of Proposition 3.3 is true.*

Proof. In this case, it is possible to prove ([4]) that $[\widetilde{g}, \widetilde{g}] \subset \widetilde{h}$. If not, let $w \in \widetilde{g}$ with

(i) $\widetilde{g} = \mathbb{R}w \bigoplus \widetilde{h}$,

(ii) $[\xi, w] = aw + \xi_1$, $a \neq 0$, for some $\xi_1 \in \widetilde{h}$. By induction on $j = 1, 2, \ldots$,

$$d\varrho^j(\xi)(w) = a^j w + \xi_j,$$

for some $\xi_j \in \widetilde{h}$. But $\widetilde{g}$ is a nilpotent Lie algebra, and thus there exist $m \in \mathbb{N}$ such that $-a^m w \in \widetilde{h}$, contrary to hypothesis. So, for every $w \in \widetilde{g}$, $[w, \widetilde{h}] \subset \widetilde{h}$. Since $\widetilde{h}$ has codimension 1, the proof is complete. ∎

Remark 3.5. If the drift vector field X belongs to the centre of $\widetilde{g}$ then $\widetilde{h}$ satisfies the hypothesis of Proposition 3.4.

Now we characterize the controllability of nilpotent systems when G is a (connected) simply connected Lie group.

THEOREM 3.6. *Let Σ be a nilpotent system on the simply connected Lie group G. Then Σ is controllable $\Leftrightarrow \widetilde{h} = \widetilde{g}$.*

Proof. It is evident that $\widetilde{h} = \widetilde{g}$ is a sufficient condition for the controllability of Σ. Conversely, we consider the ascending central series $(\widetilde{h}_i)_{i=0}^k$ of $\widetilde{h}$ defined by

$$\widetilde{h}_{(0)} = 0, \quad \widetilde{h}_{(j)} = \{w \in \widetilde{h} \mid [w, \widetilde{h}] \subset \widetilde{h}_{(j-1)}\} \quad \text{for } j = 1, 2, \ldots$$

Since G is a nilpotent Lie group, $\widetilde{h}$ is a nilpotent Lie algebra and there is $n \in \mathbb{N}$ such that

$$0 = \widetilde{h}_{(0)} \subsetneq \widetilde{h}_{(1)} \subsetneq \ldots \subsetneq \widetilde{h}_{(n-1)} = \widetilde{h}.$$

For every $j \in \{1, 2, \ldots, n\}$, $\widetilde{h}_{(j)}$ is an ideal for $\widetilde{h}$.

We claim that if any $\widetilde{h}_{(j)}$ is not an ideal of $\widetilde{g}$, then Σ cannot be controllable. In fact, if we denote

$$j_0 = \min\{j \mid \widetilde{h}_{(j)} \text{ is not an ideal of } \widetilde{g}\}$$

then $\widetilde{h}_{(j_0-1)}$ is an ideal of $\widetilde{g}$ and since $\widetilde{h}_{(0)} = 0$, we obtain $j_0 \geq 1$. Let H_0 be the closed connected normal subgroup of G whose Lie algebra is $\widetilde{h}_{(j_0-1)}$. By hypothesis G is simply connected and therefore $P := G/H_0$ is a simply connected Lie group with Lie algebra $\widetilde{p} = \widetilde{g}/\widetilde{h}_{(j_0-1)}$. By the canonical projection $\pi : G \to P$ we can project Σ on an invariant system $\pi(\Sigma)$ over P. So, the family of vector fields $d\pi(\mathcal{D})$ generates $\widetilde{p}$ and the subalgebra of the control vectors of $\pi(\Sigma)$ is $\widetilde{h}/\widetilde{h}_{(j_0-1)}$.

By the construction of the ascending central series,

$$Z(\widetilde{h}/\widetilde{h}_{(j_0-1)}) = \widetilde{h}_{(j_0)}/\widetilde{h}_{(j_0-1)}$$

is not an ideal of $\widetilde{p}$. This shows that

$$Z(\widetilde{p}) \subsetneq Zt(\widetilde{h}/\widetilde{h}_{(j_0-1)}).$$

In fact,

$$Z(\widetilde{h}/\widetilde{h}_{(j_0-1)}) \subset \mathrm{Zt}(\widetilde{h}/\widetilde{h}_{(j_0-1)})$$

but

$$Z(\widetilde{h}/\widetilde{h}_{(j_0-1)}) \subset Z(\widetilde{p})$$

is not possible by the construction of $\widetilde{h}$. Therefore Proposition 3.1 applies and thus $\pi(\Sigma)$ is not controllable on P. Hence, Σ cannot be controllable on G and this contradicts our hypothesis.

Therefore, for each $j = 0, 1, \ldots, n$, $\widetilde{h}(j)$ is an ideal for $\widetilde{g}$, in particular for $j = n$. Since Σ is a transitive system, there are two possible cases:

1. $\widetilde{h} = \widetilde{g}$, or
2. $\widetilde{h}$ is an ideal of codimension 1 in $\widetilde{g}$.

Proposition 3.4 shows that Σ is controllable $\Leftrightarrow G/H \simeq S^1$. But G/H is a simply connected Lie group. Thus, $\widetilde{h} = \widetilde{g}$. ∎

Remark 3.7. (i) In this work, we consider the elements of $\widetilde{g}$ as left-invariant vector fields (it is possible to obtain the same results for right-invariant control systems). If $x_0 \in \mathbb{R}^n$ and $P \in M_n(\mathbb{R})$, the linear equation

(a) $\dot{x} = Px$, $x(0) = x_0$

induces a matrix equation

(b) $\dot{X} = PX$, $X(0) = \mathrm{Id}$.

The solution e^{tP} of (b) gives the solution $e^{tP} \cdot x_0$ of (a) by the action on the initial condition. Therefore, it is possible to study controllability of bilinear systems B,

$$B = \begin{cases} \dot{x} = Ax + \sum_{j=1}^{k} \mu_j A_j x, \\ x \in \mathbb{R}^n - \{0\} \end{cases}$$

via the right-invariant control system

$$\Sigma = \begin{cases} \dot{g} = Ag + \sum_{j=1}^{k} \mu_j A_j g, \\ g \in G \end{cases}$$

where G is the connected subgroup of the group of non-singular real matrices, with Lie algebra

$$\widetilde{g} = \operatorname*{span}_{\mathcal{L}.A.}\{A, A_1, \ldots, A_k\}.$$

In fact,

$$S_B(x_0) = S_\Sigma(\mathrm{Id}) \cdot x_0.$$

Therefore, when G is a nilpotent simply connected Lie group, the controllability results of this paper can be used for bilinear systems.

4. An example. Let G be the Heisenberg group of dimension $2p+1$. The Lie algebra $\widetilde{g}$ of G is generated by the elements

$$X_1, \ldots, X_p, Y_1, \ldots, Y_p, Z$$

with the following rules for non-null brackets:

$$[X_i, Y_i] = Z, \qquad 1 \le i \le p.$$

It is well known that this algebra has a realization over the vector space of strictly superior matrices of order $p+2$ with the commutator $[A, B] = AB - BA$. If $\delta(i, j)$

is the matrix of order $p+2$ with 1 in the (i,j)-coordinate and zeros elsewhere, we can identify for $i,j \in \{1,2,\ldots,p+2\}$:

$$X_i = \delta(1,i+1), \quad Y_j = \delta(j+1,p+2) \quad \text{and} \quad Z = \delta(1,p+2).$$

In this way we can identify the elements of G with linear combinations of X_i, Y_j and Z having 1's on the diagonal.

So, $g \in G$ has coordinates $g = (x,y,z)$, $x,y \in \mathbb{R}^p$, $z \in \mathbb{R}$.

Now, we consider the dual of $\widetilde{g}$,

$$\widetilde{g}^* = \operatorname{span}_{\mathcal{L}.A.}\{X_1^*, X_2^*, \ldots, X_p^*, Y_1^*, Y_2^*, \ldots, Y_p^*, Z^*\}.$$

Each $\lambda \in \widetilde{g}^*$ has coordinates $\lambda = (a,b,c)$, $a,b \in \mathbb{R}^p$, $c \in \mathbb{R}$. A straightforward calculation shows that the orbit of λ by the co-adjoint representation is

$$\theta_\lambda = \{(a+cy, b-cx, c) \mid x,y \in \mathbb{R}^p\}.$$

In particular, for every $a,b \in \mathbb{R}$, we have:

1. $c = 0 \Rightarrow \theta_{(a,b,0)}$ is a trivial orbit.
2. $c \neq 0 \Rightarrow \theta_{(a,b,c)} = \{\beta \in \widetilde{g}^* \mid \beta(Z) = 0\} \oplus c \cdot Z^*$.

Therefore, no invariant system Σ of the type

$$\Sigma = \begin{cases} \dot{g} = X_{i_0}(g) + \sum_{i \neq i_0} \mu_i X_i(g) + \sum_{j=1}^p \mu_j Y_j(g), \\ u \in \mathcal{U} = \mathcal{U}(2p-1) \end{cases}$$

can be controllable on G. In fact, $Y_{i_0} \in \operatorname{Zt}(\widetilde{h})$ and $Z = [X_{i_0}, Y_{i_0}]$ is the centre of $\widetilde{g}$ and therefore Theorem 3.1 is applicable. We have

1. Each vector $\lambda = (a,b,c)$ with $c > 0$ is a symplectic vector for Z.
2. Since G is a simply connected Lie group, Theorem 3.6 can be applied directly.

Remark 4.1. Let G be a connected and simply connected Lie group. Then Theorem 3.6 allows us to construct all the controllable systems on G.

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