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SINGULAR PERTURBATIONS FOR SYSTEMS OF DIFFERENTIAL INCLUSIONS

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Abstract. We study a system of two differential inclusions such that there is a singular perturbation in the second one. We state new convergence results of solutions under assumptions concerning *contingent derivative* of the perturbed inclusion. These results state that there exists at least one family of solutions which converges to some solution of the reduced system. We extend this result to perturbed systems with state constraints.

1. Introduction. We shall study the following singular system of différential inclusions

$$(1)\qquad x'_\varepsilon(t) \in F(x_\varepsilon(t), y_\varepsilon(t)),\quad \varepsilon y'_\varepsilon(t) \in G(x_\varepsilon(t), y_\varepsilon(t)) \quad \text{for almost all } t \in [0, T].$$

The state-variables x and y belong to some finite dimensional vector-spaces X and Y. These equations are used to model a system with a slow variable $x(\cdot)$ and a fast variable $y(\cdot)$ (cf. [15] for more details). We refer to [12] for numerous examples and applications in control theory.

The convergence of solution of (1) (as $\varepsilon \to 0$) is the main problem in this field. When solutions converge it is interesting to prove that the limit is a solution to the following reduced systems:

$$(2)\qquad x'(t) \in F(x(t), y(t)),\quad 0 \in G(x(t), y(t)).$$

This problem has been extensively studied in the literature since the pioneer work of Tikhonov for differential equations (the reader can refer to [13] for a clear presentation of results obtained by Tikhonov). There also exist papers concerning linear control systems [5], [6], [9] and nonlinear problems [3]. For nonlinear differential inclusions, we refer to [4], for convex set-valued maps to [14], [17]. The reader can find in [11] a more extensive bibliography in this field.

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The purpose of the paper is to give conditions such that there exists *at least one* sequence of solutions to (1) which converges in a suitable topology. We are not interested in convergence of all solutions. In applications, for instance, if we consider the problem of minimization of some functional associated with (1), we are only interested in the convergence of optimal solutions and not in the behaviour of other solutions.

In the last section, we prove some results for singular perturbed systems with state constraints.

2. Limits of solutions to perturbed systems

2.1. *Assumptions and notations.* We assume that set-valued maps F and G are l-Lipschitz (1) with convex compact nonempty values and linear growth (with constant a).

We define the following set-valued map: $R(x) := \{y \mid 0 \in G(x,y)\}$ and its inverse $R^{-1}(y) := \{x \mid 0 \in G(x,y)\}$.

2.2. *Limits-solutions.* We shall prove that *if solutions to perturbed systems converge* then they are solutions to the reduced systems.

PROPOSITION 2.1. *Assume* (2) *that the hypotheses of section* 2.1 *hold true. Consider a sequence* $(x_{\varepsilon_n}(\cdot), y_{\varepsilon_n}(\cdot)) \in S_{F\times G}(\varepsilon_n, x_0, y_0)(T)$. *If there exist some functions* $x(\cdot)$ *and* $y(\cdot)$ *such that*:

- $x_{\varepsilon_n}(\cdot) \to x(\cdot)$ *in* $L^1[0,T]$,
- $x'_{\varepsilon_n}(\cdot) \to x'(\cdot)$ *in* $L^1_{weak}[0,T]$,
- $y_{\varepsilon_n}(\cdot) \to y(\cdot)$ *in* $L^\infty[0,T]$,

then

$$y(t) \in R(x(t)) \qquad \text{for almost all } t \in [0,T] \tag{3}$$

and furthermore, *there exists a subsequence* ε_{n_i} *such that* $\varepsilon_{n_i} y'_{\varepsilon_{n_i}}(\cdot) \to 0$ *in* $L^1_{weak}[0,T]$.

Proof. This proof is very classical (see [4] for instance). In the first step, we prove the convergence of x_{ε_i}. Consider a sequence $\varepsilon_i \to 0$ such that $(x_{\varepsilon_i}(t), y_{\varepsilon_i}(t))$ converges to $(x(t), y(t))$. Thanks to the convergence theorem for differential inclusions (cf. [1], Th. 3.6.5) applied with $F_{\varepsilon_i} := F(\cdot, y_{\varepsilon_i}(\cdot))$, we prove that $x'(t) \in F(x(t), y(t))$ because F is upper semicontinuous with convex closed values.

In the second step, we study (3) the convergence of y_ε. Let us prove that $\varepsilon y'_\varepsilon(\cdot)$ converges to 0 in $L^1_{weak}[0,T]$. It is enough to prove that for any sequence $\varepsilon_i \to 0$, the sequence $\varepsilon_i y'_{\varepsilon_i}(\cdot)$ has a sequence converging weakly to 0. Fix $\varepsilon_i \to 0$.

(1) For sake of simplicity, we assume that the Lipschitz constants of F, G are bounded by the same l.

(2) Let us denote by $S_{F\times G}(\varepsilon_n, x_0, y_0)(T)$ the set of absolutely continuous solutions of (1) starting from (x_0, y_0) at time t_0.

(3) In this part of the proof, we follow [4].

Thanks to the linear growth condition, we know that $\|\varepsilon_i y'_{\varepsilon_i}(t)\| \leq a(1+\|x_{\varepsilon_i}(t)\|+\|y_{\varepsilon_i}(t)\|)$ for almost every t. Since x_{ε_i} and y_{ε_i} converge they are bounded. So there exists some $E > 0$ such that for every $\varepsilon_i > 0$, $\|\varepsilon_i y'_{\varepsilon_i}(\cdot)\|_{L^\infty[0,T]} \leq E$. Thanks to Dunford-Pettis' criterion, there exists a subsequence (again similarly denoted) which converges in $L^1_{weak}[0,T]$ to some $z(\cdot)$. Let us prove that $z(t) = 0$ for almost every $t \in [0,T]$.

We know that $\int_0^t \varepsilon_i y'_{\varepsilon_i}(s)ds \to \int_0^t z(s)ds$. On the other hand,

$$\int_0^T \left| \int_0^t \varepsilon_i y'_{\varepsilon_i}(s)ds \right| dt = \int_0^T \varepsilon_i \|y_{\varepsilon_i}(t) - y_{\varepsilon_i}(0)\| dt$$

which converges to 0 by Lebesgue's Theorem (E is an upper bound). Hence $\int_0^T \| \int_0^t z(s)ds\| dt = 0$ and consequently $\int_0^t z(s)ds = 0$ for almost every t and finally $z(s) = 0$ for almost every s. We can conclude, thanks to the upper semi-continuity of G, that (3) holds true almost everywhere. ∎

3. Existence of convergent solutions

3.1. *Convergence of solutions associated to a convergent sequence of initial conditions.* We shall state our first main result using the contingent derivative ([4]) of the set-valued map R.

THEOREM 3.1. *We keep the assumptions of section* 2.1, *and furthermore we assume that*

$$\forall (x,y) \in Graph(R), \exists u \in F(x,y), DR(x,y)(u,0) \ni 0. \tag{4}$$

Consider a sequence (x_n^0, y_n^0) *converging to* $(x_0, y_0) \in Graph(R)$. *Then there exists a sequence* $\varepsilon_n > 0$ *which converges to* 0 *and solutions* $(x_n(\cdot), y_n(\cdot)) \in S_{F\times G}(\varepsilon_n, x_n^0, y_n^0)(T)$ *which converge to some solution to* (2) *in* $W^{1,1}[0,T] \times W^{1,1}[0,T]$.

This theorem follows from the proposition proved in [16]:

PROPOSITION 3.2. *We keep the assumptions of theorem* 3.1. *Consider* $(x_\varepsilon^0, y_\varepsilon^0)$ *converging to some* $(x_0, y_0) \in Graph(R)$. *If*

$$A_\varepsilon := \frac{1}{\varepsilon} e^{lT/\varepsilon} [\|x_\varepsilon^0 - x_0\| + \|y_\varepsilon^0 - y_0\|] \to 0 \tag{5}$$

then there exist solutions $(x_\varepsilon(\cdot), y_\varepsilon(\cdot)) \in S(\varepsilon, x_\varepsilon^0, y_\varepsilon^0)(T)$ *to* (1) *which converge to some solution of* (2) *in* $W^{1,1}[0,T] \times W^{1,1}[0,T]$. *Furthermore,* $y_\varepsilon(\cdot)$ *converge to the constant function* y_0.

To prove the theorem, it is enough to notice that if some sequence a_n converges to 0, it is possible to find a sequence ε_n such that $a_n(1/\varepsilon_n)e^{lT/\varepsilon_n}$ converges to 0.

([4]) See [1].

3.2. *Convergence results for absolutely continuous solutions.* We prove a convergence result under assumptions concerning the contingent derivative of G.

THEOREM 3.3. *We keep the assumptions of section* 2.1. *Furthermore we assume that for every x the multivalued map $y \mapsto G(x,y)$ has a convex graph. We assume also*

$$\begin{cases} \exists \gamma > 0, \forall (x,y,z) \in Graph(G), \forall (u,v) \text{ such that } v \neq 0 \\ \inf \{\langle v, w\rangle | w \in DG(x,y,z)(u,v) \cap \gamma(1+\|u\|+\|v\|)B\} < 0. \end{cases} \tag{6}$$

If there exists a solution $(\bar{x}(\cdot), \bar{y}(\cdot)) \in W^{1,1}[0,T] \times W^{1,1}[0,T]$ to the reduced system (2) *starting from a given (x_0, y_0), then there exists a sequence $(x_{\varepsilon_i}(\cdot), y_{\varepsilon_i}(\cdot)) \in S_{F\times G}(\varepsilon_i, x_0, y_0)(T)$ such that, as $\varepsilon_i \to 0^+$,*

$$\begin{aligned} x_{\varepsilon_i}(\cdot) &\to x(\cdot) \quad \text{in } L^1[0,T] \times L^1_{weak}[0,T], \\ y_{\varepsilon_i}(\cdot) &\to y(\cdot) \quad \text{in } L^1_{weak}[0,T], \end{aligned}$$

where $(x(\cdot), y(\cdot))$ is a solution to (2).

This theorem is based on the following more precise proposition:

PROPOSITION 3.4. *We keep the assumptions of section* 2.1 *and furthermore we assume that* (6) *holds true. If there exists a solution $(\bar{x}(\cdot), \bar{y}(\cdot)) \in W^{1,1}[0,T] \times W^{1,1}[0,T]$ to the reduced system* (2) *starting from a given (x_0, y_0), then there exists $M > 0$ such that for any $\varepsilon > 0$, there exists $(x_\varepsilon(\cdot), y_\varepsilon(\cdot)) \in S_{F\times G}(\varepsilon, x_0, y_0)(T)$ such that*

$$\|y_\varepsilon(\cdot) - \bar{y}(\cdot)\|_{L^\infty[0,T]} \leq M.$$

Proof of Theorem 3.3. We shall prove that for any sequence $\varepsilon_i \to 0$, there exists some subsequence such that $(x_{\varepsilon_i}(\cdot), y_{\varepsilon_i}(\cdot))$ has a subsequence converging in the suitable topology. We know from Proposition 3.4 that $y_{\varepsilon_i}(\cdot)$ is bounded in $L^\infty[0,T]$ by some constant $c_1 > 0$. Thanks to Dunford-Pettis' criterion, there exists a subsequence (again similarly denoted) such that $y_{\varepsilon_i}(\cdot)$ converges in $L^1_{weak}[0,T]$ to some $y(\cdot)$.

On the other hand, because F has a linear growth,

$$\frac{d}{dt}\|x_{\varepsilon_i}(t)\| \leq \|x'_{\varepsilon_i}(t)\| \leq a(1 + \|x_{\varepsilon_i}(t)\| + \|y_{\varepsilon_i}(t)\|).$$

Thanks to Gronwall's lemma, $\|x_{\varepsilon_i}(t)\|_{L^\infty[0,T]}$ is bounded. Because F is upper semicontinuous, $F(x_{\varepsilon_i}(\cdot), y_{\varepsilon_i}(\cdot))$ lies in a ball of radius Q.

We claim that $x_{\varepsilon_i}(\cdot)$ has a subsequence which converges to some $x(\cdot)$ in $L^1[0,T]$. In fact, to prove this, thanks to Theorem 20, p. 298 in [7], it is enough to prove that $\|x_{\varepsilon_i}(\cdot + s) - x_{\varepsilon_i}\|_{L^1[0,T]}$ converges to 0 as $s \to 0$, uniformly with respect to ε_i. But

$$x_{\varepsilon_i}(t+s) \in x_{\varepsilon_i}(t) + \int_t^{t+s} F(x_{\varepsilon_i}(u), y_{\varepsilon_i}(u))du$$

$$\subset x_{\varepsilon_i}(t) + \int_t^{t+s} QBdu \subset x_{\varepsilon_i}(t) + QsB.$$

Hence $\|x_{\varepsilon_i}(\cdot + s) - x_{\varepsilon_i}\|_{L^1[0,T]} \leq sM$.

Since G is Lipschitz, we have a subsequence of $\varepsilon_i y'_{\varepsilon_i}(\cdot)$ which converges to some $z(\cdot)$ in $L^1_{weak}[0,T]$. By similar arguments to those in Proposition 2.1, $z = 0$ almost everywhere.

For the last step of the proof, we consider a sequence ε_n such that x_{ε_n} converges in L^1-strong topology and $\varepsilon_n y'_{\varepsilon_n}$ and y_{ε_n} converge in L^1-weak topology. Because $G(x,\cdot)$ has a convex graph and since G is Lipschitz, thanks to Theorem 8.4.1 in [2], we get $0 \in G(x(t), y(t))$ for almost every $t \in [0,T]$. ∎

We shall deduce Proposition 3.4 from the following lemma proved in [16]:

LEMMA 3.5. *Under the assumptions of Proposition* 3.4, *for any* (x_0, y_0) *satisfying* $0 \in G(x_0, y_0)$, *the set-valued map*

$$P(x,y) := \{v \in G(x,y) \mid \langle v, y - y_0\rangle \leq 0\}$$

is upper semicontinuous with nonempty convex compact values.

Proof of Proposition 3.4. Let us define the following set-valued map:

$$P(t,x,y) := \{v \in G(x,y) \mid \langle v, y - \bar{y}(t)\rangle \leq 0\},$$

which is upper semicontinuous with convex compact nonempty values thanks to Lemma 3.5. Consider $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ which are solutions to

$$\begin{cases} x'_\varepsilon(t) \in F(x_\varepsilon(t), y_\varepsilon(t)), \\ \varepsilon y'_\varepsilon(t) \in P(t, x_\varepsilon(t), y_\varepsilon(t)) \quad \text{for almost every } t \in [0,T], \\ x_\varepsilon(0) = \bar{x}(0), \qquad y_\varepsilon(0) = \bar{y}(0). \end{cases} \tag{7}$$

Obviously $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ are solutions to (1). Thus, for almost every $t \in [0,T]$, $\langle \varepsilon y'_\varepsilon(t), y_\varepsilon(t) - \bar{y}(t)\rangle \leq 0$ hence $\langle y'_\varepsilon(t), y_\varepsilon(t) - \bar{y}(t)\rangle \leq 0$. So

$$\langle y'_\varepsilon(t) - \bar{y}'(t), y_\varepsilon(t) - \bar{y}(t)\rangle \leq -\langle \bar{y}'(t), y_\varepsilon(t) - \bar{y}(t)\rangle.$$

By integrating on $[0,t]$, we get

$$\|y_\varepsilon(t) - \bar{y}(t)\|^2 \leq \int_0^t \langle y_\varepsilon(s) - \bar{y}(s), \bar{y}'(s)\rangle ds$$

Then $\|y_\varepsilon(t) - \bar{y}(t)\| \leq 1 + \|y_\varepsilon(t) - \bar{y}(t)\|^2 \leq \int_0^t \|y_\varepsilon(s) - \bar{y}(s)\| \|\bar{y}'(s)\| ds + 1$. Thanks to Gronwall's lemma,

$$\|y_\varepsilon(t) - \bar{y}(t)\| \leq \int_0^t \|\bar{y}'(s)\| e^{\int_s^t \|\bar{y}'(\sigma)\| d\sigma} ds.$$

Hence, because $\bar{y}'(\cdot) \in L^1[0,T]$, there exists a nonnegative number M such that $\|y_\varepsilon(t) - \bar{y}(t)\| \leq M$ where M does not depend on ε. ∎

These results assume that there exist absolutely continuous solutions to the reduced system. We shall provide a sufficient ([5]) condition for this.

PROPOSITION 3.6. *Let the assumptions of section* 2.1 *hold true. Then*

$$\sup_{(x,y)\in Graph(R)} \inf_{v\in DR(x,y)(F(x,y))} \|v\| < \infty$$

if and only if starting from any initial condition $(x_0, y_0) \in Graph(R)$ *there exists at least one solution* $(x(\cdot), y(\cdot)) \in W^{1,1}[0,T] \times W^{1,1}[0,T]$ *to* (2).

Proof. Let us denote

$$c := \sup_{(x,y)\in Graph(R)} \inf_{v\in DR(x,y)(F(x,y))} \|v\|.$$

This means that

$$\forall (x,y) \in Graph(R), (F(x,y) \times cB) \cap T_{Graph(R)}(x,y) \neq \emptyset.$$

Hence, thanks to the Viability Theorem, the differential inclusion with constraints

$$x'(t) \in F(x(t), y(t)), \quad y'(t) \in cB, \quad 0 \in G(x(t), y(t))$$

has at least one absolutely continuous solution starting from any initial condition of $Graph(R)$. Consequently, (2) has an absolutely continuous solution. ■

4. Singular perturbations for a system of differential inclusions with state-constraints. We shall study behaviour of solutions of (1) and (2) which stay forever (i.e. on $[0, +\infty[$) in a given subset $K \subset X \times Y$.

PROPOSITION 4.1. *Let* K *be a compact subset of* $X \times Y$. *We keep the assumptions of Proposition* 3.6. *If for any* $\varepsilon > 0$ *and for any initial condition* $(x_0, y_0) \in K$, *there exists a solution to* (1) *which is viable in* K *then*

1) $K \subset Graph(R)$,

2) *for all* $T > 0$ *and* $(x_0, y_0) \in K$, *there exists* $y(\cdot) \in L^1_{weak}[0,T]$ *starting at* y_0 *and a solution* $x(\cdot)$ *to* $x'(t) \in F(x(t), y(t))$ *such that*

$$(x(t), y(t)) \in K, \quad 0 \in G(x(t), y(t)) \quad \textit{for almost all } t \in [0,T],$$

3) *for any* $(x_0, y_0) \in K$ *there exists a solution* $y(\cdot)$ *to* $y'(t) \in G(x_0, y(t))$ *such that* $t \mapsto (x_0, y(t))$ *is viable in* K.

Proof. Because K is compact, there exists some $A > 0$ such that $K \subset B(0, A)$. Since F and G are Lipschitz maps and K is compact, we can consider

$$M := \sup\{\|u\|, \|v\| \mid (u,v) \in F(x,y) \times G(x,y), (x,y) \in K\} < \infty.$$

By the results of section 3, we know in advance that there exists a solution of (2). In fact, without using Proposition 3.4, we shall prove that there exists a viable solution on $[0,T]$ as in the proof of Theorem 3.3. Hence, we deduce results 1) and 2). Let us prove the last one.

([5]) Cf. [15] or [16] for the detailed proof.

Thanks to the viability theorem, we know that for every $(x,y)\in K$ and for every *nonnegative* number ε,

$$\left(F(x,y)\times\frac{1}{\varepsilon}G(x,y)\right)\cap T_K(x,y)\neq\emptyset.$$

Because $T_K(x,y)$ is a cone, we have for every $\varepsilon>0$

$$(\varepsilon F(x,y)\times G(x,y))\cap T_K(x,y)\neq\emptyset$$

and since F is bounded by M,

$$(\{0\}\times G(x,y))\cap T_K(x,y)\neq\emptyset.$$

Hence thanks to the viability theorem, starting from any $(x_0,y_0)\in K$ there exists a solution to $t\mapsto(x_0,y(t))$ which is viable. ∎

We can state our last theorem which uses results (6) of Viability Theory.

THEOREM 4.2. *Let the assumptions of Proposition* 4.1 *hold true*, *and assume furthermore that R is Lipschitz. Then*

$$\begin{aligned}
&\operatorname*{Limsup}_{\varepsilon\to 0} Viab_{F\times\frac{1}{\varepsilon}G}(K)\subset K\cap Graph(R),\\
(9)\qquad &\Pi_X(\operatorname*{Limsup}_{\varepsilon\to 0} Viab_{F\times\frac{1}{\varepsilon}G}(K))\subset Viab_{F(\cdot,R(\cdot))}(\Pi_X(K\cap Graph(R)).
\end{aligned}$$

To prove this it is enough to notice that when R is Lipschitz, for every absolutely continuous solution to (2) its first coordinate x is a solution to

$$x'(t)\in F(x(t),R(x(t)). \tag{10}$$

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References

[1] J.-P. Aubin, *Viability Theory*, Birkhäuser. Boston, Basel, 1992.
[2] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1991.
[3] P. Binding, *Singularly perturbed optimal control problems.I. Convergence*, SIAM J. Control Optim. 14 (1976), 591–612.
[4] A. L. Dontchev and I. I. Slavov, *Singular perturbation in a class of nonlinear differential inclusions*, Proceedings IFIP Conference, Leipzig, 1989, Lecture Notes in Inform. Sci. 143, Springer, Berlin, 1990, 273–280.

(6) Let us recall that a closed set is *viable* for a differential inclusion $x'(t)\in F(x(t))$ when starting from any point of K there exists at least one solution $x(\cdot)$ such that for every $t>0$, $x(t)\in K$. When F is upper semicontinuous with compact convex nonempty values, this property is equivalent to the following contingent condition (this is the Viability Theorem):

$$\forall x\in K,\quad F(x)\cap T_K(x)\neq\emptyset.$$

When the closed set K is not viable, we denote by $Viab_F(K)$ the largest closed viable set contained in K. The reader can refer to [1] for more detailed statements and applications.

[5] A. L. Dontchev and V. M. Veliov, *Singular perturbations in linear control systems with weakly coupled stable and unstable fast subsystems*, J. Math. Anal. Appl. 110 (1985), 1–130.

[6] —, —, *Continuity of a family of trajectories of linear control systems with respect to singular perturbations*, Soviet Math. Dokl. 35 (1987), 283–286.

[7] N. Dunford and J. Schwartz, *Linear Operators, Part I*, Wiley, New York.

[8] A. F. Filippov, *On some problems of optimal control theory*, Vestnik Moskov. Univ. Mat. 1958 (2), 25–32 (in Russian); English transl.: SIAM J. Control 1 (1962), 76–84.

[9] T. F. Filippova and A. B. Kurzhanskiĭ, *Methods of singular perturbations for differential inclusions*, Dokl. Akad. Nauk SSSR 321 (1991), 454–460 (in Russian).

[10] H. Frankowska, S. Plaskacz and T. Rzeżuchowski, *Measurable viability theorems and Hamilton-Jacobi-Bellman equation*, J. Differential Equations 116 (1995), 265–305.

[11] P. V. Kokotović, *Applications of singular perturbation techniques to control problems*, SIAM Rev. 26 (1984), 501–550.

[12] R. O'Malley, *Introduction to Singular Perturbation*, Academic Press, 1974.

[13] A. N. Tikhonov, A. B. Vassilieva and A. G. Sveshnikov, *Differential Equations*, Springer, 1985.

[14] H. Tuan, *Asymptotical solution of differential systems with multivalued right-hand side*, Ph.D. Thesis, University of Odessa, 1990, in Russian.

[15] M. Quincampoix, *Contribution à l'étude des perturbations singulières pour les systèmes contrôlés et les inclusions différentielles*, C. R. Acad. Sci. Paris Sér. I 316 (1993), 133–138.

[16] —, *Singular perturbations for control systems and for differential inclusions*, in: Cahiers Mathématiques de la Décision, Université Paris-Dauphine, 1994.

[17] V. M. Veliov, *Differential inclusions with stable subinclusions*, Nonlinear Anal. 23 (1994), 1027–1038.