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ABNORMALITY OF TRAJECTORY IN SUB-RIEMANNIAN STRUCTURE

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Abstract. In the sub-Riemannian framework, we give geometric necessary and sufficient conditions for the existence of abnormal extremals of the Maximum Principle. We give relations between abnormality, C^1-rigidity and length minimizing. In particular, in the case of three dimensional manifolds we show that, if there exist abnormal extremals, generically, they are locally length minimizing and in the case of four dimensional manifolds we exhibit abnormal extremals which are not C^1-rigid and which can be minimizing or non minimizing according to different metrics.

1. Introduction. Let M be a connected complete n-dimensional manifold, TM its tangent bundle and T^*M its cotangent bundle. A sub-Riemannian structure on M, $(\mathcal{E}, G, M)$, is a locally free submodule $\mathcal{E}$, of finite rank p of the C^∞-module $\Gamma(M)$ of vector fields on M, satisfying the generating Hörmander condition (see section 2), together with a positive quadratic form G on $\mathcal{E}$. It is now well known that:

1°) any two points of M can be joined by a C^1-piecewise, so called "admissible" curve (that is, a curve whose velocity vector, where defined, lies in $\mathcal{E}$),

2°) if two points are joined by an admissible curve, by a classical functional argument, these two points can be joined by a G-length minimizing curve (i.e. a curve realizing the G-distance between the two points).

Consequently, in particular locally, the distance between two points is attained on some admissible curve.

It is natural to look for a characterization of admissible curves with minimal G-length. Applying the Maximum Principle one finds two kinds of extremals, namely Hamiltonian geodesics which are always G-length minimizing, and abnor-

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mal extremals. We know now plenty of examples of sub-Riemannian structures having abnormal extremals, which **are not** Hamiltonian geodesics but are yet G-length minimizing ([Mo], [K1], [L-S], [Su]). Moreover, admissible curves which are locally "geometrically" isolated (called locally rigid in [B-H]) can also be G-length minimizing. We know that admissible locally rigid curves are abnormal extremals ([A]) and can be G-length minimizing too ([Su]). The purpose of this work is to study the problem of existence of abnormal extremals and to elucidate some relations between abnormality, rigidity and the property of being G-length minimizing.

In section 2 we recall how Hamiltonian geodesics and abnormal extremals are defined in the context of regular and singular sub-Riemannian structures. In section 3 we give a characterization of Hamiltonian geodesics and abnormal extremals in terms of the Lie derivative. Using a new intrinsic derivative related to sub-Riemanniann stucture, we give, in section 4, an alternative characterization of Hamiltonian geodesics. In section 5, in the context of a local analytic regular module, we give necessary and sufficient conditions for non existence of abnormal extremals. In section 6, we show that, if a 2-dimensional generic distribution on a 3-dimensional manifold has abnormal admissible extremals, then they are G-length minimizing for any metric. This result includes all examples of 2-dimensional distributions on $\mathbb{R}^3$ which were known before. In the last section, we construct an example of a 3-dimensional distribution in a 4-dimensional manifold, having a field of abnormal curves which are not locally rigid, and which are locally G-length minimizing or not, according to the choice of the metric G.

2. Geodesics in sub-Riemannian structures. Starting with a locally free submodule $\mathcal{E}$, of rank p, of $\Gamma(M)$, we define a sequence of modules $(\mathcal{E}_k)$:

$$\mathcal{E}_1 = \mathcal{E}, \quad \text{and} \quad \mathcal{E}_k = \mathcal{E}_1 + \sum_{X \in \mathcal{E}} [X, \mathcal{E}_{k-1}], \quad k \geq 2$$

(where $[\,,\,]$ denotes the Lie bracket), and a nondecreasing sequence of integers, $r_k(x) = \dim(\mathcal{E}_k)_x$ called the growth vector of $\mathcal{E}$ at x.

The submodule $\mathcal{E}$ is said to satisfy the generating Hörmander condition if, for any x in M, there exists an integer $k_0(x)$ such that $r_k(x) = n$ for all $k \geq k_0(x)$. Under this assomption, there exists, up to a vector bundle isomorphism, a unique p-dimensional vector bundle $\pi : C \to M$, and a morphism $H : C \to TM$, such that $\operatorname{Im} H = \mathcal{E}$. Let h be any Riemannian metric on C, and let us denote by $h^\sharp_x : C^*_x \to C_x$ the duality bundle isomorphism induced by the quadratic form h. We put $g = H \circ h^\sharp \circ H^* : T^*M \to TM$. The same symbol g will denote also the corresponding symmetric bilinear form on T^*M given by

$$\langle \xi_x, g_x \eta_x \rangle = \langle \xi_x, (H \circ h^\sharp \circ H^*)_x \cdot \eta_x \rangle = \langle H^*_x \xi_x, (h^\sharp \circ H^*_x) \cdot \eta_x \rangle$$

(where $\langle\ ,\ \rangle$ is the duality product). For any h as above there exists a unique quadratic C^∞-mapping $G : \mathcal{E} \to \mathbb{R}^+$ such that, for any local C^∞-vector field X

in $\mathcal{E}$, and for any x in M,

$$G_x(X_x, X_x) = h_x(s(x), s(x)) = \inf_\sigma\{h_x\big(\sigma(x), \sigma(x)\big)\ ;\ H_x \cdot \sigma(x) = X_x\} < +\infty,$$

where $g_x(\xi_x) = X_x$, and $s_x = (h_x^\sharp \circ H_x^*) \cdot \xi_x$. Conversely, to any such mapping G is associated a unique metric h. Moreover, the image of $h^\sharp \circ H^*$ in C_x lies in $(\operatorname{Ker} H_x)^{\perp_{h_x}}$, for

$$\forall \sigma \in \operatorname{Ker} H_x, \quad h_x((h^\sharp \circ H^*)_x \cdot \xi_x\ ,\ \sigma_x) = \langle H_x^* \cdot \xi_x\ ,\ \sigma_x\rangle = \langle \xi_x\ ,\ H_x \cdot \sigma\rangle = 0.$$

From now on, we shall denote by either $(\mathcal{E}, G, M)$ or $(\mathcal{E}, g, M)$ the sub-Riemannian structure. Further, the sub-Riemannian structure $(\mathcal{E}, g, M)$ on M will be called regular (resp. singular) if for any $x \in M$, $\operatorname{Ker} H_x = 0$ (resp. $\operatorname{Ker} H_x \neq 0$ for some x in M). In the singular case there exists no Riemannian metric on M whose restriction is G. On the other hand, in the regular case, G is the restriction of infinitely many Riemannian metrics on M ([P-V 1,2]).

A curve $\gamma : [a, b] \to M$ is called "admissible" if there exists an L^2-section σ over γ of the fibre bundle C such that

$$\forall t \in [a, b], \qquad \gamma(t) = \pi \circ \sigma(t) = \pi \circ \sigma(a) + \int_a^t (H \circ \pi \circ \sigma) \cdot \sigma(t) dt.$$

Such a curve is absolutely continuous and, for any metric G on $\mathcal{E}$, its G-length and its G-energy are given respectively by

$$l(\gamma) = \int_a^b |\dot\gamma(t)|_G\, dt = \int_a^b \sqrt{h_x(s(x), s(x))}\, dt,$$

$$E(\gamma) = \frac{1}{2} \int_a^b |\dot\gamma(t)|_G^2\, dt.$$

When M is connected and complete, the Hörmander generating condition implies that any two points can be joined by a piecewise C^1-admissible curve. In both regular and singular case, we have been able to show that any two points can be joined by a G-length minimizing curve ([P-V 1,2]):

(2-1) Theorem. *Let G denote any sub-Riemannian metric, let x_0 and x_1 denote two points in M, $I = [a, b]$ a closed interval in $\mathbb{R}$, $\mathcal{H}_{x_0,x_1}(I; A)$ denote the set of absolutely continuous curves such that*

$$\mathcal{H}_{x_0,x_1}(I; A) = \{\gamma : I \to M; \gamma(a) = x_0, \gamma(b) = x_1, \dot\gamma \in \mathcal{E}, E(\gamma) \leq A\}.$$

Then, among the curves of $\mathcal{H}_{x_0,x_1}(I; A)$, there exists at least one admissible curve γ such that the infimum of energy is achieved on γ. Furthermore, if a curve is G-energy minimizing, then it is G-length minimizing too.

So, it makes sense to speak of a G-distance on M. This G-distance is usually called the Carnot-Carathéodory distance. An admissible curve $\gamma : [a, b] \to M$ will be called a locally minimizing curve (GLM-curve) if for any point $\gamma(t)$, there exists an interval $]t-\alpha, t+\alpha[\subset [a,b]$ such that γ realizes the distance between $\gamma(t_1)$ and $\gamma(t_2)$, for all $[t_1,t_2]\subset]t-\alpha,t+\alpha[$.

It is natural to look for an explicit characterization of GLM-curves. This is a local problem, so we can limit ourselves to the domain of a chart U, making $\mathcal{E}$ trivial. Let us denote $(g^{\alpha\beta})$ $(\alpha,\beta=1,\dots,n)$ the matrix of $g: T^*M\to TM$ in these local coordinates. A H^1-curve $\gamma:[a,b]\to U$ is admissible if there exists a L^2-curve $\xi:[a,b]\to\mathbb{R}^n$ such that

$$\dot\gamma^\alpha = g^{\alpha\beta}\xi_\beta, \qquad \alpha=1,\dots,n.$$

Any such 1-form ξ will be called a "lift" of γ. In this notation the G-energy of γ is

$$E_G(\gamma)=\frac{1}{2}\int_a^b g^{\alpha\beta}\xi_\alpha\xi_\beta.$$

So, in U, admissible H^1-curves which are locally minimizing are solutions of the differential equations $\dot x = g\xi$, and are extremals of the Pontryagin Maximum Principle with respect to the generalized "Hamiltonian function":

$$\mathcal{H}(x,\xi,\lambda,\lambda_0)=\langle\lambda,g\xi\rangle-\lambda_0\langle\xi,g\xi\rangle.$$

This principle states that if a curve γ is a GLM-curve, and a 1-form ξ is any lift of γ, then there exists an absolutely continuous curve $\lambda:[a,b]\to\mathbb{R}^n$ and a constant $\lambda_0\geq 0$ such that

$$\text{(M-P)}\qquad \begin{cases} \text{a) } (\lambda(t),\lambda_0)\neq 0 \qquad \forall t\in[a,b],\\ \text{b) } \dot\gamma=\dfrac{\partial\mathcal{H}}{\partial\lambda}=g\xi,\\ \text{c) } \dot\lambda=-\dfrac{\partial\mathcal{H}}{\partial x}=-\Big\langle\lambda,\dfrac{\partial g}{\partial x}\xi\Big\rangle-\dfrac{\lambda_0}{2}\Big\langle\xi,\dfrac{\partial g}{\partial x}\xi\Big\rangle,\\ \text{d) } \mathcal{H}(\gamma(t),\xi(t),\lambda(t),\lambda_0)=\sup\limits_{\eta\in\mathbb{R}^n}\mathcal{H}(\gamma(t),\eta,\lambda(t),\lambda_0)=\text{const.}\end{cases}$$

If $\lambda_0\neq 0$, we may replace λ by λ/λ_0, and get $g\xi=g\lambda$; γ is, thus, a solution of the system

$$\dot x = g\lambda, \qquad \dot\lambda=-\Big\langle\lambda,\frac{\partial g}{\partial x}\lambda\Big\rangle.$$

But these are just the classical Hamiltonian system equations on T^*M with its canonical symplectic structure for the Hamiltonian

$$\langle\lambda,g\lambda\rangle: T^*M\cong\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}.$$

So, if γ is a GLM-curve with $\lambda_0\neq 0$, (γ,ξ) is an integral curve of the Hamiltonian vector field of $\langle\lambda,g\lambda\rangle$, and consequently is C^∞.

If, in equations (M-P), $\lambda_0=0$, then $\mathcal{H}(x,\xi,\lambda,0)$ is linear with respect to λ, and the condition ((M-P) d) implies $\langle\lambda,g\eta\rangle=0$ along γ, for any $\eta\in\mathbb{R}^n$.

(2-2) Definitions. 1°) A curve $\gamma : [a,b] \to M$ is called a *Hamiltonian geodesic* if there exists a lift ξ of γ such that (γ, ξ) is an integral curve of the Hamiltonian vector field of the Hamiltonian function

$$\langle \lambda, g\lambda \rangle : T^*M \to \mathbb{R}.$$

2°) An admissible curve $\gamma : [a,b] \to M$ is called an *abnormal extremal* if it satisfies the Maximum Principle with $\lambda_0 = 0$, that is, there exist curves $(\gamma, \xi) : [a,b] \to T^*M$ which are solutions of (M-P) with $\lambda_0 = 0$ in any chart whose domain contains $\gamma([a,b])$.

3°) A GLM-curve which is an abnormal extremal is called an *abnormal geodesic*.

4°) A GLM-curve which is not a Hamiltonian geodesic is called a *strictly abnormal geodesic*.

As is shown in section 7 a GLM-curve may be at the same time a Hamiltonian geodesic and an abnormal extremal. However, we get:

(2-3) Theorem [P-V 1,2]. *A GLM-curve $\gamma : [a,b] \to M$ is either a Hamiltonian geodesic or is strictly abnormal. If it is Hamiltonian, then it is C^∞.*

3. A characterization of Hamiltonian geodesics and abnormal extremals. Let X, Z be two vector fields and ω a 1-form on M, the following identity is well known:

$$\langle L_X\omega, Z\rangle = X\langle \omega, Z\rangle - \langle L_Z\omega, X\rangle + Z\langle \omega, X\rangle, \tag{3-1}$$

where $L_X\omega$ and $L_Z\omega$ denote the Lie derivatives.

Now, if $\gamma : [a,b] \to M$ is any simple curve of class H^1 and ω is a H^1 1-form such that $\langle \omega, \dot\gamma\rangle = 0$, a.e., then we can extend $\dot\gamma$ to an L^2-vector field X on a neighbourhood U of $\gamma([a,b])$ and ω to an H^1 1-form Ω on U such that $\langle \Omega, X\rangle = 0$. Using formula (3-1) for any C^∞ vector field Z on U, we see that the expression $\langle L_X\Omega, Z\rangle$ is well defined.

Moreover, it is easy to show that the value of $\langle L_X\Omega, Z\rangle$ along the curve γ depends only on ω and $\dot\gamma$. From now on, let $L_{\dot\gamma}\omega$ denote the 1-form which is defined just above. Similarly, for any C^k simple curve $(k\gamma 2)$ $\gamma : [a,b] \to M$ and for any H^1 1-form ω along γ we can define $L_{\dot\gamma}\,\omega$ using a C^{k-1} vector field X extending $\dot\gamma$, a 1-form Ω of class H^1, extending ω and formula (3-1).

(3-2) Theorem [P-V 1,2]. *Let $(\mathcal{E}, g, M)$ be a sub-Riemannian structure on M.*

1°) *A simple admissible curve $\gamma : [a,b] \to M$ of class H^1 is an abnormal extremal if and only if there exists a nonzero 1-form ν of class H^1 along γ such that, for any $t \in [a,b]$, $\nu(t) \in \operatorname{Ker} g_{\gamma(t)}$, and $L_{\dot\gamma}\nu = 0$.*

2°) *A simple admissible curve $\gamma : [a,b] \to M$ of class C^2 is a Hamiltonian geodesic if and only if there exists a lift $\xi : [a,b] \to T^*M$ such that $\langle \xi, \dot\gamma\rangle = \text{const.}$ and $L_{\dot\gamma}\xi = 0$.*

Proof. Since the problem is local, we can suppose that $M = \mathbb{R}^n$.

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a simple admissible curve of class H^1, ν a 1-form along γ, of class H^1, such that $\nu(t) \in \operatorname{Ker} g_{\gamma(t)}$. It is easy to show that, for any lift $\xi : [a, b] \to \mathbb{R}^n$ of γ, we have

$$L_{\dot\gamma}\nu = \dot\nu + \left\langle \nu, \frac{\partial g}{\partial x}\xi \right\rangle. \tag{3-3}$$

We know that γ is an abnormal extremal if and only if there exists a lift $\xi : [a, b] \to \mathbb{R}^n$ and an absolutely continuous curve $\lambda : [a, b] \to \mathbb{R}^n$ such that

$$\dot\gamma = g\xi, \quad \dot\lambda = \left\langle \lambda, \frac{\partial g}{\partial x}\xi \right\rangle, \quad \forall t \in [a, b],\ \lambda(t) \neq 0,$$

and

$$\forall \eta \in \mathbb{R}^n, \qquad \langle \lambda, g\eta \rangle = 0 \text{ along } \gamma.$$

So the nonzero 1-form λ is of class H^1 and satisfies $\lambda(t) \in \operatorname{Ker} g_{\gamma(t)}$ and $L_{\dot\gamma}\lambda = 0$.

Let $\gamma : [a, b] \to M$ be a C^2 curve. After a possible reparametrization of γ leaving $[a, b]$ unchanged, we can assume that there exists a lift ξ of γ such that $\langle \xi, \dot\gamma \rangle = \text{const}$. By a straightforward calculation, one gets

$$L_{\dot\gamma}\xi = \dot\xi + \frac{1}{2}\left\langle \xi, \frac{\partial g}{\partial x}\xi \right\rangle.$$

Thus γ is a Hamiltonian geodesic if and only if

$$\langle \xi, \dot\gamma \rangle = \text{const.}, \qquad L_{\dot\gamma}\xi = 0. \tag{3-4}$$

(3-5) Remark. This characterization of abnormal extremals has been already proved in [He], [K2], [P-V 1,2].

For any admissible simple curve γ of class H^1, let $\mathcal{A}(\gamma)$ denote the vector space of 1-forms ν, of class H^1, such that

$$\nu(t) \in \operatorname{Ker} g_{\gamma(t)} \quad \text{and} \quad L_{\dot\gamma}\nu = 0. \tag{3-6}$$

Theorem (3-2) implies that γ is *regular* if and only if $\mathcal{A}(\gamma) = 0$.

The vector space $\mathcal{A}(\gamma)$ will be called the abnormality set of γ.

4. Intrinsic derivative. Let us consider a sub-Riemannian structure $(\mathcal{E}, G, M)$. Recall that, in classical Riemannian geometry, where $\mathcal{E}$ is the module $\Gamma(M)$ of all C^∞ vector fields on M there exists a natural covariant derivative ∇ related to G giving a nice characterization of GLM-curves, they are just the solutions of the equations

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

(4-1) The derivative ∇, inducing the Levi-Civita connection, is characterized by vanishing of its torsion and by the following identity:

$$\begin{aligned} G(\nabla_X Y, Z) = \frac{1}{2}(&X(G(Y,Z)) + Y(G(Z,X)) - Z(G(X,Y)) \\ &- G(X,[Y,Z]) + G(Y,[Z,X]) + G(Z,[X,Y])). \end{aligned}$$

(4-2) In a sub-Riemannian structure, let ΛM denote the module of 1-forms on M, and define $D : \Lambda M \times \Lambda M \to \Gamma(M)$, $(\alpha, \beta) \to D_\alpha \beta$, as the unique map satisfying the following relation:

$$\langle \gamma, D_\alpha \beta \rangle = \frac{1}{2}(g\alpha\langle \beta, g\gamma \rangle + g\beta\langle \gamma, g\alpha \rangle - g\gamma\langle \alpha, g\beta \rangle - \langle \alpha, [g\beta, g\gamma]\rangle + \langle \beta, [g\gamma, g\alpha]\rangle + \langle \gamma, [g\alpha, g\beta]\rangle).$$

(4-3) Lemma. *The map D is $\mathbb{R}$-bilinear and satisfies the following properties, for any function f of class C^1 on M*:

(i) $D_{(f\alpha)}\beta = f(D_\alpha \beta)$,
(ii) $D_\alpha(f\beta) = f(D_\alpha \beta) + ((g\alpha)f)g\beta$,
(iii) $(g\alpha)\langle \beta, g\gamma \rangle = \langle \beta, D_\alpha \gamma \rangle + \langle \gamma, D_\alpha \beta \rangle$,
(iv) $D_\alpha \beta - D_\beta \alpha = [g\alpha, g\beta]$.

Let $\gamma : [a, b] \to M$ be an H^1 admissible simple curve, ξ a lift of γ, η a 1-form of class H^1 along γ. Let $\hat{\xi}$ and $\hat{\eta}$ denote some local 1-forms of class H^1, extending respectively ξ and η in a neighbourhood of $\gamma[a, b]$. Then $D_{\hat{\xi}}\hat{\eta}$ is well defined almost everywhere and its values on γ do not depend on the choice of the extensions. Thus, the expression $D_\xi \eta$ is well defined.

Let $\operatorname{Ker} g$ denote the distribution of vector spaces of 1-forms α such that

$$\alpha(x) \in \operatorname{Ker} g_x, \qquad \forall x \in M,$$

and $\mathcal{N}$ denote the module of 1-forms which annihilates $\mathcal{E}$. The following lemma, along the curve γ, is proved by a straightforward calculation:

(4-4) Lemma. 1°) *If $\eta \in \operatorname{Ker} g$, then $D_\xi \eta = \frac{1}{2} g(L_{\dot{\gamma}} \eta)$.*

2°) *If γ is of class C^2 and $\langle \xi, \dot{\gamma} \rangle = $ const., then $D_\xi \xi = g(L_{\dot{\gamma}} \xi)$.*

Let $F : \operatorname{Ker} g \times \mathcal{E} \to \mathcal{E}$ denote the map defined in the following way:

$$\text{for any } \eta \text{ such that } Y = g\eta, \qquad \langle \eta, F(\nu, X) \rangle = \langle \nu, [X, Y] \rangle.$$

Clearly, the vector $F(\nu, X)_x$ depends only on the values of ν and X at x. Moreover, we have $D_\xi \nu = D_\nu \xi = -\frac{1}{2} F(\nu, X)$, for every ξ such that $g\xi = X$. As the map $\nu \to F(\nu, X)$ is linear, we put

$$\text{for any } u \in T_x M, \qquad F(u) = \operatorname{Im}\{\nu \to F(\nu, X); X_x = u\}.$$

Then $F(u)$ is a linear subspace of $\mathcal{E}_x$ satisfying the following properties:

(4-5) Lemma. 1°) *Let X and Y be vector fields in $\mathcal{E}$. Then the Lie bracket $[X, Y]$ lies in $\mathcal{E}$ if and only if, for all x in M, $Y(x)$ is G-orthogonal to $F(X_x)$.*

2°) *Let ξ, ξ' be 1-forms such that $g\xi = g\xi'$, and η be any 1-form. Then*

$$D_\xi \eta - D_{\xi'} \eta \in F(g\eta).$$

5. On the nonexistence of abnormal extremals. Let us consider a sub-Riemannian structure in the regular case. A classical geometric sufficient condition for the nonexistence of abnormal extremals is the so called strong Hörmander condition ([St], [B], [BA], [L]), namely: *for every vector field X in $\mathcal{E}$, the linear space defined by evaluating $\mathcal{E}+[X,\mathcal{E}]$ at any point x in M is T_xM.* In this section, using an adaptation of an argument of Hsu [Hs], we will give geometric sufficient conditions—generalizing the previous one—for the nonexistence of abnormal extremals in the general case.

We will say that the module $\mathcal{E}$ is locally analytic if, for any point $x \in M$, there exists a local coordinate system in a neighbourhood of x in which $\mathcal{E}$ is generated by analytic vector fields. Until the end of this section we will assume that $\mathcal{E}$ is regular and locally analytic. Let $\mathcal{N} = \mathcal{E}^{\perp}$ denote the annihilator of $\mathcal{E}$. Clearly it is a submodule of T^*M. The cotangent bundle T^*M is endowed with the canonical symplectic 2-form Ω inducing a closed 2-form $\bar{\Omega}$ on $\mathcal{N}$ which is not symplectic in general. Let us put $\Sigma = \{z \in \mathcal{N};\ \text{rank of } \bar{\Omega}_z < 2n-p\}$, $C_z = \operatorname{Ker} \bar{\Omega}_z \subset T_zT^*M$. If p is odd, we have necessarily $\Sigma = \mathcal{N}$. If p is even, then in the general case, Σ and $\mathcal{N}\backslash\Sigma$ are not empty. Furthermore, if $\mathcal{N}\backslash\Sigma \neq \emptyset$, Σ is locally defined by analytic equations; thus, at any point z of Σ, the tangent space to Σ is well defined.

Let Σ_0 be the subset of points $z \in \Sigma$ such that $C_z \cap T_z\Sigma = \{0\}$, and $\Sigma_1 = \Sigma\backslash\Sigma_0$; thus for $z \in \Sigma_1$ we have $\dim(C_z \cap T_z\Sigma) \geq 1$.

(5-1) THEOREM. 1°) *Let $\gamma : [a,b] \to M$ be a simple admissible curve of class H^1. Then a 1-form ν of class H^1 along γ is in the abnormality set $\mathcal{A}(\gamma)$ of γ if and only if the curve (γ,ν) lies in Σ, and its tangent vector is in $C = \bigcup_z C_z$.*

2°) *Abnormal curves don't exist if and only if Σ_1 is a totally discontinuous subset of T^*M.*

Proof. Since the problem is local, we can replace M by a domain U of a chart. Let $(g^{\alpha\beta})$ denote the matrix of g in the chosen coordinates. The equation of $\mathcal{N}$ is then

$$\Phi^{\alpha} = g^{\alpha\beta}\nu_{\beta} = 0, \qquad \alpha = 1,\ldots,n.$$

Let us consider the restriction of $\bar{\Omega}$ to $\mathcal{N}$, its kernel is $T_z\mathcal{N} \cap (T_z\mathcal{N})^{\perp\Omega}$ (if V denotes a linear subspace of T_zT^*M, then let $V^{\perp\Omega}$ denote its Ω-orthogonal complement). Moreover, $(T_z\mathcal{N})^{\perp\Omega}$ is generated by the values at z of the Hamiltonian vector fields X_{Φ_α} of the functions Φ_α. A 1-form ν along γ lies in $\mathcal{A}(\gamma)$ if and only if there exists an L^2-curve $\xi : [a,b] \to \mathbb{R}^n$ such that

$$\text{(i) } \dot{\nu}_{\alpha} = -\sum_{\rho,\sigma=1}^{n} \frac{\partial g^{\rho\sigma}}{\partial x^{\alpha}} \xi_{\rho}\nu_{\sigma},$$

$$\text{(ii) } \dot{\gamma}^{\alpha} = \sum_{\rho=1}^{n} g^{\alpha\rho}\xi_{\rho},$$

$$\text{(iii) } g^{\alpha\rho}\nu_{\rho} = 0, \qquad \alpha = 1,\ldots,n.$$

Equation (iii) says that the curve (γ,ν) lies in $\mathcal{N}$ and equations (i) and (ii) mean that $(\dot{\gamma},\dot{\nu})$ is a linear combination of the Hamiltonian vector fields $X_{\Phi_1},\ldots,X_{\Phi_n}$ with coefficients $\xi_1,\ldots,\xi_n$, respectively; this proves 1°).

If $\mathcal{A}(\gamma) \neq \{0\}$, then there exists a curve (γ,ν) in Σ_1 of class H^1. Suppose that

Σ_1 is not totally discontinuous. Since Σ_1 is an analytic set (in a good coordinate system), its regular part has dimension greater than 1. Thus there must exist somewhere a smooth vector field X on the regular part of Σ_1, tangent to C. Then, necessarily, the projection on M of any integral curve (γ, ν) of X is an abnormal curve of class H^1. ■

(5-2) Remark. In the singular case, if the module $\mathcal{E}$ is locally analytic, we can give analogous, but more technical conditions, which are equivalent to the nonexistence of abnormal curves.

These results will be given in a forthcoming paper.

Let us consider a local basis $\{\nu_1, \ldots, \nu_q\}$ of $\mathcal{N}$. If $\pi : T^*M \to M$ is the natural projection, a straightforward local calculation gives the following result (see, for instance [Hs]):

$$\pi(\Sigma) = \{x \in M; \exists \nu \in \mathcal{N}; \operatorname{Ker} d\nu_x \cap \mathcal{E}_x \neq \{0\}\}.$$

The strong Hörmander condition is then equivalent to $\Sigma = \emptyset$. Similarly,

$$\pi(\Sigma_1) = \{x \in \pi(\Sigma); \exists \nu \in \mathcal{N}; \operatorname{Ker} d\nu_x \cap \mathcal{E}_x \subset T_x\pi(\Sigma)\}.$$

Thus, we get:

(5-3) Theorem. *Let $\mathcal{E}$ be a locally analytic and regular module. The set*

$$S = \{x \in M; \exists \nu \in \mathcal{N}; \operatorname{Ker} d\nu_x \cap \mathcal{E}_x \neq \{0\}\}.$$

is an analytic set, possibly empty. There exist no abnormal extremals if and only if either S is empty, or the set

$$S_1 = \{x \in S; \exists \nu \in \mathcal{N}; \operatorname{Ker} d\nu_x \cap \mathcal{E}_x \subset T_x S\}$$

is a totally discontinuous subset of M.

(5-4) Corollary. *Let $\mathcal{E}$ be a locally analytic and regular module of codimension 1. Abnormal extremals don't exist if and only if:*

1°) *n is odd $(n = 2r + 1)$, and*
2°) *one of the following conditions is satisfied, where ν is a generator of $\mathcal{N}$:*

$$\text{(i) } S = \{x \in M; \nu \wedge d\nu^{2r}(x) = 0\} = \emptyset$$
$$\text{(ii) } S = \{x \in M; \nu \wedge d\nu^{2r}(x) = 0\} \neq \emptyset$$

but $S_1 = \{x \in S; \operatorname{Ker} d\nu_x \cap \mathcal{E}_x \cap T_x S\}$ is a totally discontinuous subset of M.

(5-5) Remark. If $S_1 \neq \emptyset$, as happens in the general situation, then from a result of B. Jakubczyk and F. Przytycki, it follows that the condition "S_1 is a totally discontinuous subset of M" is not generic. This is true in particular if $\dim \mathcal{N} = 1$.

6. On strictly abnormal G-energy minimizing curves. The problem of existence of abnormal minimizing curves which are not solutions of Euler-Lagrange equations was already known to E. Cartan and has been studied by many authors: for instance [He], [Bi]. The first published paper which contained an explicit example of a strictly abnormal, locally G-energy minimizing curve in sub-Riemannian geometry was, as far as we know, Montgomery's example (see [Mo]). Analogous examples were given by Kupka [K], Liu and Sussmann [L-S] (see also [P-V 1,2]). All these examples are constructed in $M = \mathbb{R}^3$, and can be conceived as generic by the following argument: Let $\mathcal{E}$ be a regular two-dimensional distribution on a three-dimensional manifold M. The distribution $\mathcal{E}$ is locally the kernel of a 1-form ν on M. The set $S(\mathcal{E}) = \{x \in M; \nu \wedge d\nu(x) = 0\}$ does not depend on the choice of ν. In the generic case ([Mar], [J-P]), $\Sigma(E)$ is either empty or is a two-dimensional submanifold of M. In this latter case, $\mathcal{E}$ is transversal to $S(\mathcal{E})$ on an open dense subset $S_0(\mathcal{E})$ of M, and, if the complement $S_1(\mathcal{E}) = S(\mathcal{E}) \backslash S_0(\mathcal{E})$ is not empty, then $S_1(\mathcal{E})$ is a one-dimensional submanifold. Furthermore, in the generic situation, $\mathcal{E}$ fulfills the Hörmander condition. On $S_0(\mathcal{E})$, the intersection $\mathcal{E} \cap TS(\mathcal{E})$ gives rise to a field of directions, say Δ. Let Δ-*curves* be the integral curves of Δ. From theorem (5-2), it follows that they are abnormal extremals. We will show that any arc of a Δ-curve, say γ_0, is a locally GLM-curve **for any metric** G, in a C^0 neighbourhood of γ_0.

(6-1) Remark. Generically in $\mathbb{R}^3$, because of a result proved by J. Martinet, for any point a of $S_0(\mathcal{E})$ there exist local coordinates such that $a = (0,0,0)$ and $\mathcal{E}$ is generated by $(\partial/\partial x, \partial/\partial y + x^2 \partial/\partial z)$, thus, it is the kernel of $\nu = dz - x^2 dy$. So

$$\begin{aligned} \langle dy, \Delta \rangle &\neq 0 \qquad \text{on } S_0(\mathcal{E}) \cap U, \\ \operatorname{Ker} dx &= TS(\mathcal{E}) \qquad \text{on } S_0(\mathcal{E}) \cap U. \end{aligned}$$

(6-2) Theorem. *Let $\mathcal{E}$ be a generic (as described above) regular 2-dimensional distribution on a 3-dimensional manifold M, with nonempty $S_0(\mathcal{E})$. Then there exists an open domain intersecting $S_0(\mathcal{E})$, on which there exists a coordinate system as in remark (6-1). Then, for any sub-Riemannian metric G, any integral curve in $S_0(\mathcal{E})$ of a vector field with constant G-norm tangent to $\Delta = \mathcal{E} \cap TS(\mathcal{E})$ is locally a GLM-curve (G-length minimizing) with respect to a C^0-neighbourhood of admissible H^1-curves.*

Proof. As the problem is local, it consists, first, in choosing a good coordinate system such that Δ is generated by $(\partial/\partial y)$ on $\{(x,y,z); x=0\}$, as in remark (6-1). This is a classical result: see [Mar] and [J-P]. So the y-axis is a Δ-curve inside this coordinate chart. **Let U denote a relatively compact neighbourhood of a inside the coordinate chart**. Then it remains to prove that the G-length of any admissible curve starting at $(0,0,0)$ and ending at $(0, y_1, 0)$, inside U, is bigger than the G-length of the curve $(0,y,0)$ joining $(0,0,0)$ to $(0,y_1,0)$, that is done in the following.

(6-3) Notations. Let G be any sub-Riemannian metric. Then let

$$R_1, R_2, \varphi : U \to \mathbb{R}^+_*$$

denote respectively

$$\left|\frac{\partial}{\partial x}\right|_G, \quad \left|\frac{\partial}{\partial y}+x^2\frac{\partial}{\partial z}\right|_G, \qquad \operatorname{Arc}\cos\frac{G(\partial/\partial x, \partial/\partial y + x^2\partial/\partial z)}{R_1R_2}.$$

As $\mathcal{E}$ is a regular distribution, $\varphi \notin \mathbb{Z}\pi$, and we can suppose without loss of generality that $\varphi \in]0,\pi[$. Let $v = X\partial/\partial x + Y\partial/\partial y + Z\partial/\partial z$ denote any vector field on U. Now, v is in $\mathcal{E}$ if and only if $Z = x^2Y$ and

$$|v|_G^2 = R_1^2X^2 + 2R_1R_2\cos\varphi XY + R_2^2Y^2.$$

The examples described in [Mo], [K1], [L-S] are of this kind with $R_1 = 1$ and $\varphi = \pi/2$. Theorem (6-2) is implied by the following lemma:

(6-4) TECHNICAL LEMMA (1). *Let U be defined as above and $\mathcal{D}_\rho = U \cap \{|x| \leq \rho \leq 1\}$. Within $\mathcal{D}_\rho$, let $\gamma_0 : [0, l_0] \to M$ denote a Δ-curve (with its tangent vector lying entirely in $\mathcal{E} \cap TS(\mathcal{E})$), i.e. an abnormal extremal such that*

$$\forall s_0, \qquad \gamma_0(s_0) = (0, y_{\gamma_0}(s_0), 0),$$
$$\gamma_0(0) = (0,0,0), \quad \gamma_0(l_0) = (0, y_1, 0), \quad |\dot\gamma_0(s_0)|_G = 1.$$

Then, for any sub-Riemannian metric G, there exists a strictly positive bound $B(G,\rho)$, decreasing with respect to ρ, such that if $0 < y_1 < B(G,\rho)$, then γ_0 is strictly shorter than any absolutely continuous admissible curve γ, inside $\mathcal{D}_\rho$, joining $(0,0,0)$ to $(0,y_1,0)$.

The hypothesis and conclusion do not depend on parametrization. Actually, with notations (6-3) and (6-5),

$$B(G,\rho) = \frac{1}{R_2^0}\inf\left\{\frac{r_2^\rho\sin\varphi_\rho}{\rho}; \frac{1-|\cos\varphi_\rho|}{\dfrac{M^\rho}{r_1^\rho r_2^\rho\sqrt{2\sin\varphi_\rho}} + \dfrac{\mu^\rho}{(r_1^\rho)^2}}\right\}.$$

(6-5) NOTATIONS.

$$M^\rho = \sup_{\mathcal{D}_\rho}\left|\frac{\partial R_2}{\partial x}\right| + \sup_{\mathcal{D}_\rho}\left|\frac{\partial R_2}{\partial z}\right| + \sup_{\mathcal{D}_\rho}\left|\frac{\partial(R_1\cos\varphi)}{\partial y}\right| + \sup_{\mathcal{D}_\rho}\left|\frac{\partial(R_1\cos\varphi)}{\partial z}\right|,$$

$$\mu^\rho = \sup_{\mathcal{D}_\rho}\left|\frac{\partial(R_1\cos\varphi)}{\partial x}\right|,$$

$$\inf_{\mathcal{D}_\rho} R_1 = r_1^\rho > 0, \qquad \inf_{\mathcal{D}_\rho} R_2 = r_2^\rho > 0, \qquad \inf_{\mathcal{D}_\rho}\sin\varphi = \sin\varphi_\rho > 0,$$

$$\sup_{\mathcal{D}_\rho} R_1 = R_1^\rho > 0, \quad \sup_{\mathcal{D}_\rho} R_2 = R_2^\rho > 0, \quad \sup_{\mathcal{D}_\rho}|\cos\varphi| = \cos\varphi_\rho > 0.$$

Proof of (6-4). Let $\gamma_0 : [0, l_0] \to \mathcal{D}_\rho$ be an arc of an abnormal extremal

(1) A recent preprint ([A-S]) gives a more general result but no explicit lower bound.

parametrized by its arc length such that

$$\gamma_0(s_0) = (0, y_{\gamma_0}(s_0), 0), \quad \gamma_0(0) = (0,0,0), \quad \gamma_0(l_0) = (0, y_1, 0), \quad |\dot\gamma_0|_G = 1.$$

Let $\gamma : [0, l_0] \to \mathcal{D}_\rho$ denote any other admissible absolutely continuous simple curve parametrized by s_0 such that

$$\gamma(s_0) = (x(s_0), y(s_0), z(s_0)), \quad \gamma(0) = (0,0,0), \quad \gamma(l_0) = (0, y_1, 0), \quad |\dot\gamma|_G = l/l_0.$$

Let us suppose that $l = \int_0^{l_0} |\dot\gamma|_G \leq l_0$. We want to establish a contradiction.

Let us write down explicitly conditions for the end points of γ and some of their consequences (notations are explained below):

$$\text{(6-6)} \qquad \begin{cases} \text{(i)} & \int\limits_0^{l_0} \dot x \, ds = 0 \;\Rightarrow\; \int\limits_0^{l_0} |\dot x| \, ds_0 \geq 2 \sup\limits_\gamma |x|, \\ \text{(ii)} & \int\limits_0^{l_0} \dot y \, ds_0 = y_1 \;\Leftrightarrow\; \int\limits_0^{l_0} |\dot y| = y_1 + 2Y_-, \\ \text{(iii)} & \int\limits_0^{l_0} \dot z \, ds_0 = 0 \;\Leftrightarrow\; \int\limits_0^{l_0} x^2 \dot y \, ds_0 = 0 \\ & \Rightarrow \int\limits_0^{l_0} x^2 |\dot y| \, ds_0 = 2 \int\limits_{S_\pm x}^{2} |\dot y| \, ds_0 \geq 2 \sup\limits_\gamma |z|. \end{cases}$$

The condition (iii) implies that there exists a measurable subset S_- (resp. S_+) of $[0, l_0]$ such that $\dot y$ is strictly negative on S_- (resp. nonnegative on S_+).

In particular that implies the local rigidity of the curve γ_0 among the subset of admissible C^1-curves γ as shown in section 7. We put

$$Y_\pm = \int\limits_{S_\pm} |\dot y| \, ds_0. \tag{6-7}$$

Writing $\dot x = dx_\gamma/ds_0$, $\dot y = dy_\gamma/ds_0$, we get

$$|\dot\gamma(s_0)|_G^2 = (R_1^2 \dot x^2 + 2R_1R_2 \cos\varphi \dot x \dot y + R_2^2 \dot y^2)_{\gamma(s_0)} = \frac{l^2}{l_0^2} \leq 1,$$

where $l = \int_0^{l_0} |\dot\gamma| \, ds_0$. If the curve γ exists, then this polynomial has nonnegative discriminant and we get

$$|\dot y(s_0)| \leq \left(\frac{1}{R_2 \sin\varphi}\right)_{\gamma(s_0)} \leq \frac{1}{r_2^\rho \sin\varphi_\rho}.$$

By integration, we get

$$Y_\pm \leq \frac{l_0}{r_2^\rho \sin\varphi_\rho} \leq \frac{R_2^0 \, y_1}{r_2^\rho \sin\varphi_\rho}. \tag{6-8}$$

Then, the hypothesis $y_1 \leq B(G, \rho)$ implies that

$$\rho^2 Y_+ Y_- \leq 1. \tag{6-9}$$

Let $E(\gamma)$ denote the energy of the curve $\gamma(s_0)$. Then

$$2E(\gamma) = \int_0^{l_0} |\dot\gamma|^2 \, ds_0 = \frac{l^2}{l_0},$$

$$2E(\gamma) = \int_0^{l_0} R_1^2 \dot x^2 + \int_0^{l_0} (R_2 \dot y - 1)^2 + 2 \int_0^{l_0} R_1 \dot x (R_2 \dot y - 1) \cos\varphi$$
$$+ \int_0^{l_0} (2R_2 \dot y - 1) \, ds_0 + 2 \int_0^{l_0} R_1 \dot x \cos\varphi \, ds_0.$$

Put

$$A^2 := \int_0^{l_0} R_1^2 \dot x^2 \, ds_0, \qquad B^2 := \int_0^{l_0} (R_2 \dot y - 1)^2 \, ds_0.$$

Then we can write

$$\frac{l^2}{l_0} = A^2 + B^2 + 2 \int_0^{l_0} R_1 \dot x (R_2 \dot y - 1) \cos\varphi \, ds_0$$
$$+ 2 \int_0^{l_0} \{R_2(x,y,z)_{\gamma(s_0)} - R_2(0, y_\gamma(s_0), 0)\} \dot y_\gamma(s_0) \, ds_0$$
$$+ \int_0^{l_0} \{2R_2(0, y_\gamma(s_0), 0) \dot y_\gamma(s_0) - 1\} \, ds_0 + 2 \int_0^{l_0} R_1 \dot x \cos\varphi \, ds_0. \tag{6-10}$$

The following inequalities are obvious:

$$A\sqrt{l_0} \geq 2r_1^\rho \sup_\gamma |x|, \qquad B^2 \geq \int_{S_-} (R_2 \dot y + 1)^2 > 2r_2^\rho Y_-,$$

and the Cauchy-Schwarz inequality gives

$$\int_0^{l_0} R_1 |\dot x| \, |R_2 |\dot y| - 1| |\cos\varphi| \, ds_0 \leq AB \cos\varphi_\rho.$$

Then, because of (6-8) and (6-9),

$$\begin{aligned} \sup_\gamma |x|^2 Y_+ Y_- &\leq \sup_\gamma |x| \sqrt{Y_+ Y_-} \\ &\leq \frac{A\sqrt{l_0}}{2r_1^\rho} \frac{\sqrt{l_0}}{\sqrt{r_2^\rho \sin\varphi_\rho}} \frac{B}{\sqrt{2r_2^\rho}}. \end{aligned}$$

Using integration by parts and Taylor's formula in (6-10), terms as $\int_0^{l_0} |x\dot y|$, $\int_0^{l_0} |z\dot y|$, $\int_0^{l_0} |x\dot x|$ will appear. These quantities satisfy the following inequalities which are deduced from (6-6):

$$\int_0^{l_0} |x\dot{y}| = \int_{S_+} |x||\dot{y}| + \int_{S_-} |x||\dot{y}|$$

$$\leq \sqrt{Y_+ \int_{S_+} x^2|\dot{y}|} + \sqrt{Y_- \int_{S_-} x^2|\dot{y}|} \leq 2\sup_\gamma |x| \sqrt{Y_+Y_-},$$

$$\int_0^{l_0} |z\dot{y}| = \int_{S_+} |z||\dot{y}| + \int_{S_-} |z||\dot{y}| \leq 2\sup_\gamma |x|^2\, Y_+Y_-$$

$$\int_0^{l_0} |x\dot{x}| \leq \frac{l_0A^2}{2(r_1^\rho)^2},$$

and finally, we get

$$\text{(6-11)} \qquad \frac{l^2-l_0^2}{l_0} > \Big(\cos\varphi_\rho + \frac{M^\rho}{r_1^\rho r_2^\rho\sqrt{2\sin\varphi_\rho}}l_0\Big)(A-B)^2$$
$$+ \Big(1-\cos\varphi_\rho - \frac{M^\rho}{r_1^\rho r_2^\rho\sqrt{2\sin\varphi_\rho}}l_0 - \frac{\mu^\rho}{(r_1^\rho)^2}l_0\Big)A^2$$
$$+ \Big(1-\cos\varphi_\rho - \frac{M^\rho}{r_1^\rho r_2^\rho\sqrt{2\sin\varphi_\rho}}l_0\Big)B^2.$$

The condition $0 < y_1 < B(G,\rho)$ implies that the coefficients of A^2 and B^2 are strictly positive, and then $l > l_0$ gives the desired contradiction.

(6-12) Remark. The explicit lower bound of lemma (6-4) is certainly not the best one; for instance in the Montgomery-Kupka case, an adapted method gives globally $y_1 = 2$ ([V]), when $B(G,\rho)$ takes its values between 0 and $\sqrt{2}$.

7. Rigidity, abnormality and G-energy minimizing. In sections 2 and 6, we studied the minimizing problem for curves of class H^1. In other words, the framework was the Hilbert manifold of admissible curves on M. Similar considerations in the context of the Banach manifold of curves of class C^k, $k \geq 1$, lead to the same definition of Hamiltonian geodesics and show existence of locally minimizing curves "geometrically isolated" in the induced topology, i.e. rigid curves [A], [BH]. More precisely:

(7-1) Definition. 1°) An admissible curve $\gamma_0 : [a,b] \to M$ of class C^k is called *rigid* if there exists a neighbourhood $\mathcal{U}$ of γ_0 in the C^k-Whitney topology ([BH], [Hs]) such that the only admissible curves $\gamma_0' : [a,b] \to M$ in $\mathcal{U}$ joining $\gamma_0(a)$ to $\gamma_0(b)$ are merely reparametrizations of γ_0.

2°) An admissible curve $\gamma_0 : [a,b] \to M$ of class C^k is called *locally rigid* if for any $t \in]a,b[$ there exists a neighbourhood $]t-\alpha, t+\alpha[\subset [a,b]$ such that the restriction of γ_0 to any closed subinterval of $]t-\alpha, t+\alpha[$ is rigid.

For any sub-Riemannian structure $(\mathcal{E}, g, M)$, the following proposition holds:

(7-2) PROPOSITION ([A]). *An admissible curve locally C^1-rigid is an abnormal curve.*

(7-3) PROPOSITION. *In the generic situation of theorem* (6-2), *any integral Δ-curve is locally C^1 rigid.*

Proof. The condition is local and does not depend on the parametrization. Using the tools of the proof of theorem (6-2), replacing the parameter s_0 with t, it is easy to prove that the curve $\gamma_0 : [0, y_1] \to \mathbb{R}^3$, $\gamma_0(t) = (0, t, 0)$ of lemma (6-4) is C^1-rigid. If a C^1-curve $\gamma(t) = (x(t), y(t), z(t))$ defined on $[0, y_1]$, such that $\gamma(0) = (0, 0, 0)$, $\gamma(y_1) = (0, y_1, 0)$, is admissible, and C^1 close to γ_0, then $|\dot{x}(t)|$, $|\dot{y}(t) - 1|$, $|\dot{z}(t)|$ must be small, and thus we can choose a neighbourhood of γ_0 such that for γ in this neighbourhood, $\dot{y}(t) > 0$ for all t. But the condition $\dot{z}(t) = x^2\dot{y}(t)$ implies that the function $t \to z(t)$ is nondecreasing and nonconstant if $x(t)$ is not identically zero, so $z(y_1) > 0$, which is in contradiction with $z(y_1) = 0$.

Alternative examples of distributions with locally rigid trajectories are Engel distributions in 4-dimensional manifolds ([BS]). Recall that a regular two-dimensional distribution in a 4-dimensional manifold is an Engel distribution if, at any point, its growth vector is (2,3,4). Then there exists a nonzero vector field $X \in \mathcal{E}$ such that $[X, \mathcal{E}_2] \subset \mathcal{E}_2$ (notations as in section 2). A result of Bryant and Hsu on the one hand, and by Liu and Sussmann on the other hand, gives:

(7-4) PROPOSITION. *Any integral curve of the direction field defined by X is locally rigid. Moreover*, *it is locally minimizing* (*in the H^1-topology*) *for any metric on $\mathcal{E}$* (Sussmann's result).

We end this paper with an example of a distribution with abnormal non-C^1-rigid trajectories, which can be locally minimizing for some metrics and not locally minimizing for some others.

Let us consider the module $\mathcal{E}$ generated by

$$X_1 = (1+x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = (1+x)\frac{\partial}{\partial z} - y\frac{\partial}{\partial t},$$

on

$$M = \{(x, y, z, t) \in \mathbb{R}^4 : x > -1, y > 0\}.$$

Let G be the metric on $\mathcal{E}$ given by

$$ds^2 = \left(\frac{R_1}{1+x}\right)^2 dx^2 + R_2^2 dy^2 + \left(\frac{R_3}{1+x}\right)^2 dz^2,$$

with $R_i \geq 0$, $i = 1, 2, 3$, such that the above generators are orthogonal. At any point x, the annihilator of $\mathcal{E}_x$ is collinear to $\nu = ydz + (1+x)dt$ and X_1 is in the kernel of $\nu \wedge d\nu$. Thus, any curve which is tangent to the direction defined by X_1 is an abnormal extremal (corollary (5-4)). Let us consider now the G-orthogonal moving frame

$$e_i = \frac{X_i}{R_i}, \quad i = 1,2,3 \quad \text{and} \quad \theta^1 = \frac{R_1}{1+x}dx \quad (\text{so } \langle \theta^1, e_i \rangle = \delta_i^1).$$

But, since we have $\langle L_{e_1}\theta^1, \partial/\partial t\rangle = 0$, from lemma (4-4) it follows that an integral curve of e_1 is a Hamiltonian geodesic if and only if $D_{\theta^1}\theta^1 = 0$ (we have $F(e_1, \mathcal{N}) = 0$). But a simple calculation shows that

$$D_{\theta^1}\theta^1 = \sum_i \langle \theta^1, [e_i, e_1]\rangle e_i = -\frac{e_2(R_1)}{R_1}e_2 - \frac{e_3(R_1)}{R_1}e_3.$$

Now, the hypothesis that R_1 is a function of y only, for instance, implies that, in this case, integral curves of e_1 are never Hamiltonian geodesics.

(7-5) PROPOSITION. 1°) *No integral curve of* e_1 *is locally* C^1*-rigid.*

2°) *No integral curve of* e_1 *is locally* G*-energy minimizing if, for instance,* R_1 *does not depend on* z *and* $\partial R_1/\partial y$ *is nowhere zero.*

Proof. The leaves of the foliation defined by $dz = dt = 0$ are admissible (any curve within a leaf is admissible) and e_1 is tangent to this foliation, thus any integral curve of e_1, $\gamma_0 : [a,b] \to M$, is contained in a leaf, say $\mathcal{L}$, of this foliation. It is now clear that in any C^1-neighbourhood of γ_0, there exists an alternative C^1-curve $\gamma : [a,b] \to \mathcal{L}$ joining $\gamma_0(a)$ to $\gamma_0(b)$, and, evidently γ_0 is then not C^1-rigid. But a necessary and sufficient condition for γ_0 to be G-length minimizing, among all admissible curves joining $\gamma_0(a)$ to $\gamma_0(b)$, is that γ_0 be length minimizing among all curves joining $\gamma_0(a)$ to $\gamma_0(b)$ in the leaf $\mathcal{L}$, with respect to the metric induced by G on $\mathcal{L}$. Thus γ_0 is characterized as a Riemannian geodesic in this leaf. But in such a leaf all Riemannian geodesics are Hamiltonian, thus restricting to $\mathcal{L}$, we must have

$$D_{\theta^1}\theta^1 = -\frac{e_2(R_1)}{R_1}e_2 = 0,$$

which contradicts the hypothesis $e_2(R_1) \neq 0$.

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