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STRONG AND WEAK SOLUTIONS TO STOCHASTIC INCLUSIONS

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Abstract. Existence of strong and weak solutions to stochastic inclusions $x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau)q(d\tau,dz)$ and $x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{|z|\le 1} H_{\tau,z}(x_\tau)q(d\tau,dz) + \int_s^t \int_{|z|>1} H_{\tau,z}(x_\tau)p(d\tau,dz)$, where p and q are certain random measures, is considered.

1. Notations and definitions. We present here the basic definitions and notations used in the paper. They are taken from [1] and [2], respectively. We assume that there is given a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$, where the family $(\mathcal{F}_t)_{t_0\le t\le T}$, $0 \le t_0 < T < \infty$, of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ is assumed to be increasing: $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \le t$. Throughout we assume that the following usual hypotheses are satisfied: (i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$ and (ii) $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$ for all $t, t_0 \le t \le T$; that is, the filtration $(\mathcal{F}_t)_{t\in[t_0,T]}$ is right continuous. Let X be a set, $\mathcal{B}$ a σ-algebra of subsets. We shall say that a random measure μ is given on $\mathcal{B}$ if for every $A \in \mathcal{B}$, a random variable $\mu(A)$ is defined on $(\Omega, \mathcal{F})$, and if the following condition is met: For any sequence (A_k) of disjoint sets in $\mathcal{B}$ such that $A = \bigcup_k A_k$ the series $\sum_{k=1}^\infty \mu(A_k)$ converges in probability to $\mu(A)$.

If for disjoint sets $A_1, \ldots, A_k$, the variables $\mu(A_1), \ldots, \mu(A_k)$ are mutually independent, then we shall call μ a *measure with independent values.*

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Let $\mathcal{B}^m$ be the ring of all Borel sets $A \subset [t_0, T] \times \mathbb{R}^m$ for which

$$\int_A \frac{dudt}{|u|^{m+1}} < \infty. \tag{1}$$

We shall denote by p a random measure with independent values that is defined on $\mathcal{B}^m$ and for which $p(A)$ has a Poisson distribution with parameter (1) for every $A \in \mathcal{B}^m$. We shall denote by q the random measure defined by the relation

$$q(A) = p(A) - Ep(A). \tag{2}$$

It is possible to construct the measure p such that $p(A)$ is indicated to the number of discontinuities of a given homogeneous process $(\xi)_{t_0 \le t \le T}$ with independent increments (see [1], pp. 34–35).

We suppose that for every $t \in [t_0, T]$, a σ-algebra $\mathcal{F}_t$ of events is defined. Also we suppose that $p(A)$ is measurable with respect to $\mathcal{F}_t$ and that no matter what the sets $A_1, \ldots, A_k$ in $[t, T] \times \mathbb{R}^m$ may be, the quantities $p(A_1), \ldots, p(A_k)$ are independent of any event in $\mathcal{F}_t$.

We denote by $\mathcal{M}(\mathcal{F}_t)$ the collection of all n-dimensional nonanticipative random processes $(f_{t,u})_{t \in [t_0, T], u \in \mathbb{R}^m}$ depending on the parameter $u \in \mathbb{R}^m$ (see [2]). Let $\mathcal{M}_p^1(\mathcal{F}_t)$ be the collection of all $f \in \mathcal{M}(\mathcal{F}_t)$ for which

$$\int_{t_0}^{T} \int_{\mathbb{R}^m} E|f_{t,u}| \frac{dtdu}{|u|^{m+1}} < \infty;$$

let $M_p^1(\mathcal{F}_t)$ be the collection of all $f \in \mathcal{M}(\mathcal{F}_t)$ for which

$$P\left\{ \int_{t_0}^{T} \int_{\mathbb{R}^m} |f_{t,u}| \frac{dtdu}{|u|^{m+1}} < \infty \right\} = 1;$$

let $\mathcal{M}_q^2(\mathcal{F}_t)$ be the collection of all $f \in \mathcal{M}(\mathcal{F}_t)$ for which

$$\int_{t_0}^{T} \int_{\mathbb{R}^m} E|f_{t,u}|^2 \frac{dtdu}{|u|^{m+1}} < \infty;$$

and let $M_q^2(\mathcal{F}_t)$ be the collection of all $f \in \mathcal{M}(\mathcal{F}_t)$ for which

$$P\left\{ \int_{t_0}^{T} \int_{\mathbb{R}^m} |f_{t,u}|^2 \frac{dtdu}{|u|^{m+1}} < \infty \right\} = 1.$$

Given $f \in M_q^2(\mathcal{F}_t)$ we denote by $(\int_{t_0}^T \int_{\mathbb{R}^m} f_{t,u} q(dt, du))_{t \in [t_0, T]}$ its stochastic integral over a random measure q. It is defined in a standard way and has all standard properties (see [1], pp. 36–37). In particular it is a martingale (as a function of t) and with probability 1 does not have discontinuities of the second kind. If $f \in M_q^1(\mathcal{F}_t) \cap M_p^1(\mathcal{F}_t)$, we set

$$\int_{t_0}^{T} \int_{\mathbb{R}^m} f_{t,u} p(dt, du) = \int_{t_0}^{T} \int_{\mathbb{R}^m} f_{t,u} q(dt, du) + \int_{t_0}^{T} \int_{\mathbb{R}^m} f_{t,u} \frac{dtdu}{|u|^{m+1}}. \tag{3}$$

The integral defined possesses quite similar properties to the previous one (see [1], pp. 38–39). In particular, a process $(\int_{t_0}^T \int_{\mathbb{R}^m} f_{t,u} p(dt, du))_{t \in [t_0, T]}$ (as a function of t) with probability 1 does not have discontinuities of the second kind. If for some $\varepsilon > 0$, $f_{t,u} = 0$ when $|u| \le \varepsilon$ then $\int_{t_0}^T \int_{\mathbb{R}^m} f_{t,u} p(dt, du)$ with probability 1 is a piecewise function.

Given a measure space $(X, \mathcal{B}, m)$, a set-valued function $\Re : X \to \mathrm{Cl}(\mathbb{R}^n)$ is said to be $\mathcal{B}$-measurable if $\{x \in X : \Re(x) \cap C \neq \emptyset\} \in \mathcal{B}$ for every closed set $C \subset \mathbb{R}^n$. For such a multifunction we define the subtrajectory integrals to be the set $\mathcal{S}^p(\Re) = \{g \in L^p(X, \mathcal{B}, m, \mathbb{R}^n) : g(x) \in \Re(x) \text{ a.e. }\}$. It is clear that for $\mathcal{S}^p(\Re)$ to be nonempty we must assume more than $\mathcal{B}$-measurability of $\Re$. In what follows we shall assume that the $\mathcal{B}$-measurable set-valued function $\Re : X \to \mathrm{Cl}(\mathbb{R}^n)$ is p-integrable bounded, $p \ge 1$, i.e., that a real-valued mapping : $X \ni x \to \|\Re(x)\| \in \mathbb{R}_+$ belongs to $L^p(X, \mathcal{B}, m, \mathbb{R}_+)$. It can be verified (see [2], Th. 3.2) that a $\mathcal{B}$-measurable set-valued mapping $\Re : X \to \mathrm{Cl}\,(\mathbb{R}^n)$ is p-integrable bounded, $p \ge 1$, if and only if $\mathcal{S}^p(\Re)$ is nonempty and bounded in $L^p(X, \mathcal{B}, m, \mathbb{R}^n)$. Finally, it is easy to see that $\mathcal{S}^p(\Re)$ is decomposable, i.e., $\mathbf{1}_A f_1 + \mathbf{1}_{X \setminus A} f_2 \in \mathcal{S}^p(\Re)$ for $A \in \mathcal{B}$ and $f_1, f_2 \in \mathcal{S}^p(\Re)$.

We have the following general result concerning the properties of subtrajectory integrals (see [2], [3]).

PROPOSITION 1. *Let* $\Re : X \to \mathrm{Cl}(\mathbb{R}^n)$ *be* $\mathcal{B}$*-measurable and* p*-integrable bounded*, $p \ge 1$. *Then* $\mathcal{S}^p(\Re)$ *is a nonempty bounded closed subset of* $L^p(X, \mathcal{B}, m, \mathbb{R}^n)$. *Moreover*, *if* $\Re$ *takes on convex values then* $\mathcal{S}^p(\Re)$ *is convex and weakly compact in* $L^p(X, \mathcal{B}, m, \mathbb{R}^n)$. ∎

We shall also deal with upper and lower semicontinuous set-valued mappings. Recall a set-valued mapping $\Re$ with nonempty values in a topological space $(Y, \mathcal{T}_Y)$ is said to be *upper* (*lower*) *semicontinuous* [u.s.c. (l.s.c.)] on a topological space $(X, \mathcal{T}_X)$ if $\Re^-(C) := \{x \in X : \Re(x) \cap C \neq \emptyset\}$ ($\Re_-(C) := \{x \in X : \Re(x) \subset C\}$) is a closed subset of X for every closed set $C \subset Y$. In particular, for $\Re$ defined on a metric space $(\mathcal{X}, d)$ with values in $\mathrm{Comp}(\mathbb{R}^n)$ this is equivalent (see [2]) to $\lim_{n\to\infty} \overline{h}(\Re(x_n), \Re(x)) = 0$ ($\lim_{n\to\infty} \overline{h}(\Re(x), \Re(x_n)) = 0$) for every $x \in \mathcal{X}$ and every sequence (x_n) in $\mathcal{X}$ converging to x. If moreover $\Re$ takes convex values then this is equivalent to upper (lower) semicontinuity of the real-valued function $s(p, \Re(\cdot))$ on $\mathbb{R}^n$ for every $p \in \mathbb{R}^n$, where $s(\cdot, A)$ denotes the support function of the set $A \in \mathrm{Comp}(\mathbb{R}^n)$. We have the following continuous selection theorem (see [2]).

THEOREM (Michael). *Let* X *be a metric space*, Y *a Banach space and* F *from* X *into the closed convex subsets of* Y *be l.s.c. on* X. *Then there exists a* *continuous function $f : X \to Y$ such that $f(x) \in F(x)$ for $x \in X$.* ∎

In what follows we shall also consider set-valued stochastic processes $(F_t)_{t\geq 0}$, $(\mathcal{G}_t)_{t\geq 0}$ and $(\mathcal{R}_{t,z})_{t\geq 0, z\in\mathbb{R}^n}$ taking values from the space $\mathrm{Comp}(\mathbb{R}^n)$ of all nonempty compact subsets of n-dimensional Euclidean space $\mathbb{R}^n$. They are assumed to be nonanticipative (see [3]) and such that $\int_0^\infty \|F_t\|^p dt < \infty$, $p \geq 1$, $\int_0^\infty \|\mathcal{G}_t\|^2 dt < \infty$ and $\int_0^\infty \int_{\mathbb{R}^n} \|\mathcal{R}_{t,z}\|^2 dtq(dz) < \infty$, a.s., where q is a measure on the Borel σ-algebra $\mathcal{B}^n$ of $\mathbb{R}^n$ and $\|A\| := \sup\{|a| : a \in A\}$, $A \in \mathrm{Comp}(\mathbb{R}^n)$. The space $\mathrm{Comp}(\mathbb{R}^n)$ is endowed with the Hausdorff metric h defined in the usual way, i.e., $h(A,B) = \max\{\overline{h}(A,B), \overline{h}(B,A)\}$ for $A, B \in \mathrm{Comp}(\mathbb{R}^n)$, where $\overline{h}(A,B) = \{\mathrm{dist}(a,B) : a \in A\}$ and $\overline{h}(B,A) = \{\mathrm{dist}(b,A) : b \in B\}$.

Denote by $M^1_{\text{s-v}}(\mathcal{F}_t)$ and $\mathcal{M}^2_{\text{s-v}}(\mathcal{F}_t)$ the families of all nonanticipative set-valued processes $\mathcal{G} = (\mathcal{G}_t)_{t\in[t_0,T]}$ such that $P(\{\int_{t_0}^T \|\mathcal{G}_t\|^2 dt < \infty\} = 1)$ and $\int_{t_0}^T E\|\mathcal{G}_t\|^2 dt < \infty$, respectively. Similarly, we denote by $M^1_{\text{s-v}}(\mathcal{F}_t, p)$, $M^2_{\text{s-v}}(\mathcal{F}_t, q)$, $\mathcal{M}^1_{\text{s-v}}(\mathcal{F}_t, p)$ and $\mathcal{M}^2_{\text{s-v}}(\mathcal{F}_t, q)$ the families of all nonanticipative set-valued processes $\mathcal{R} = (\mathcal{R}_{t,u})_{t\in[t_0,T], u\in\mathbb{R}^m}$ depending on the parameter $u \in \mathbb{R}^m$ such that

$$P\Big(\Big\{\int_{t_0}^{T}\int_{\mathbb{R}^m} \|\mathcal{R}_{t,u}\| dtdu/|u|^{m+1} < \infty\Big\}\Big) = 1,$$

$$P\Big(\Big\{\int_{t_0}^{T}\int_{\mathbb{R}^m} \|\mathcal{R}_{t,u}\|^2 dtdu/|u|^{m+1} < \infty\Big\}\Big) = 1,$$

$$\int_{t_0}^{T}\int_{\mathbb{R}^m} E\|\mathcal{R}_{t,u}\| dtdu/|u|^{m+1} < \infty$$

and

$$\int_{t_0}^{T}\int_{\mathbb{R}^m} E\|\mathcal{R}_{t,u}\|^2 dtdu/|u|^{m+1} < \infty,$$

respectively. Immediately from the Kuratowski and Ryll-Nardzewski measurable selection theorem (see [2]) it follows that for every $F, \mathcal{G} \in M^1_{\text{s-v}}(\mathcal{F}_t)$ and $\mathcal{R} \in M^1_{\text{s-v}}(\mathcal{F}_t, p) \cap M^2_{\text{s-v}}(\mathcal{F}_t, q)$ their subtrajectory integrals (see [3]): $\mathcal{S}^2(F) := \{f \in \mathcal{M}^2(\mathcal{F}_t) : f_t(\omega) \in F_t(\omega), dt \times P\text{-a.e. }\}$, $\mathcal{S}^2(\mathcal{G}) := \{g \in \mathcal{M}^2(\mathcal{F}_t) : g_t(\omega) \in \mathcal{G}_t(\omega), dt \times P\text{-a.e. }\}$, $\mathcal{S}^1_p(\mathcal{R}) := \{h \in M^1(\mathcal{F}_t, p) : h_{t,z}(\omega) \in \mathcal{R}_{t,z}(\omega), P \times p\text{-a.e. }\}$ and $\mathcal{S}^2_q(\mathcal{R}) := \{h \in M^2(\mathcal{F}_t, q) : h_{t,z}(\omega) \in \mathcal{R}_{t,z}(\omega), P \times q\text{-a.e. }\}$ are nonempty. Indeed, let $\Sigma = \{Z \in \mathcal{B}_+ \otimes \mathcal{F} : Z_t \in \mathcal{F}_t \text{ for each } t \geq 0\}$, where Z_t denotes the section of Z determined by $t \geq 0$. It is a σ-algebra on $\mathbb{R}_+ \times \Omega$ and a function $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ (a multifunction $F : \mathbb{R}_+ \times \Omega \to \mathrm{Cl}(\mathbb{R}^n)$) is nonanticipative if and only if it is Σ-measurable. Therefore, by the Kuratowski and Ryll-Nardzewski measurable selection theorem every nonanticipative set-valued function admits a nonanticipative selector. It is clear that for $F \in \mathcal{M}^2_{\text{s-v}}(\mathcal{F}_t)$ such for selector belongs to $\mathcal{M}^2(\mathcal{F}_t)$. Similarly, define on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$ a σ-algebra $\widetilde{\Sigma} = \{Z \in \mathcal{B}_+ \otimes \mathcal{F} \otimes \mathcal{B}^n :$

$Z_t^u \in \mathcal{F}_t$ for each $t \geq 0$ and $u \in \mathbb{R}^n\}$, where $Z_t^u = (Z^u)_t$ and Z^u is the section of Z determined by $u \in \mathbb{R}^n$. We can repeat the foregoing arguments to obtain the above result for nonanticipative set-valued processes depending on a parameter $u \in \mathbb{R}^m$.

Let us denote by D the family of all n-dimensional $\mathcal{F}_t$-adapted cádlág (see [5]) processes $(x_t)_{t\in[t_o,T]}$ such that $E\sup_{t_0\leq t\leq T}|x_t|^2 < \infty$.

Given $F, \mathcal{G} \in M^2_{\text{s-v}}(\mathcal{F}_t)$, $\mathcal{R} \in M^1_{\text{s-v}}(\mathcal{F}_t, p) \cap M^2_{\text{s-v}}(\mathcal{F}_t, q)$, by

$$\Big(\int_{t_0}^{t} F_\tau d\tau\Big)_{t\in[t_0,T]}, \qquad \Big(\int_{t_0}^{t} \mathcal{G}_\tau dw_\tau\Big)_{t\in[t_0,T]},$$

$$\Big(\int_{t_0}^{t}\int_{\mathbb{R}^m} \mathcal{R}_{\tau,u}q(d\tau,du)\Big)_{t\in[t_0,T]} \quad \text{and} \quad \Big(\int_{t_0}^{t}\int_{\mathbb{R}^m} \mathcal{R}_{\tau,u}p(d\tau,du)\Big)_{t\in[t_0,T]},$$

we denote their set-valued stochastic integrals with respect to Lebesgue measure dt, $\mathcal{F}_t$-Brownian motion $(w_t)_{t\geq 0}$ and Poisson measures q and p, respectively. These integrals (see [4]), understood as families of subsets of $M(\mathcal{F}_t)$, are defined by $\int_{t_0}^t F_\tau d\tau = \{\int_{t_0}^t f_\tau d\tau : f \in \mathcal{S}^2(F)\}$, $\int_{t_0}^t \mathcal{G}_\tau dw_\tau = \{\int_{t_0}^t g_\tau dw_\tau : g \in \mathcal{S}^2(\mathcal{G})\}$, $\int_{t_0}^t\int_{\mathbb{R}^m}\mathcal{R}_{\tau,u}q(d\tau,du) = \{\int_{t_0}^t\int_{\mathbb{R}^m} h_{\tau,u}q(d\tau,du) : h \in \mathcal{S}^2_q(\mathcal{R})\}$ and $\int_{t_0}^t\int_{\mathbb{R}^m}\mathcal{R}_{\tau,u}p(d\tau,du) = \{\int_{t_0}^t\int_{\mathbb{R}^m} h_{\tau,u}p(d\tau,du) : h \in \mathcal{S}^2_p(\mathcal{R})\}$. Given $0 \leq \alpha < \beta < \infty$ we also define $\int_\alpha^\beta F_s ds := \{\int_\alpha^\beta f_s ds : f \in \mathcal{S}^2(F)\}$, $\int_\alpha^\beta \mathcal{G}_s dw_s := \{\int_\alpha^\beta g_s dw_s : g \in \mathcal{S}^2(\mathcal{G})\}$, $\int_\alpha^\beta\int_{\mathbb{R}^n}\mathcal{R}_{s,u}q(ds,du) := \{\int_\alpha^\beta\int_{\mathbb{R}^n} h_{s,u}q(ds,du) : h \in \mathcal{S}^2_q(\mathcal{R})\}$ and $\int_\alpha^\beta\int_{\mathbb{R}^n}\mathcal{R}_{s,u}p(ds,du) := \{\int_\alpha^\beta\int_{\mathbb{R}^n} h_{s,u}p(ds,du) : h \in \mathcal{S}^2_p(\mathcal{R})\}$.

2. Existence of strong solutions. Let $F = \{(F_t(x))_{t\in[t_0,T]} : x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t\in[t_0,T]} : x \in \mathbb{R}^n\}$ and $H = \{(H_{t,u}(x))_{t\in[t_0,T],u\in\mathbb{R}^m} : x \in \mathbb{R}^n\}$ be measurable.

Stochastic inclusions corresponding to F, G and H are understood as relations of the form

$$x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t\int_{\mathbb{R}^m} H_{\tau,u}(x_\tau)q(d\tau,du) \tag{4}$$

and

$$\begin{aligned} x_t - x_s \in &\int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t\int_{|u|\leq 1} H_{\tau,u}(x_\tau)q(d\tau,du) \\ &+ \int_s^t\int_{|u|>1} H_{\tau,u}(x_\tau)p(d\tau,du) \end{aligned} \tag{5}$$

that are to be satisfied for every $t_0 \leq s < t \leq T$ by a stochastic process $x = (x_t)_{t\in[t_0,T]} \in D$ such that $F \circ x \in \mathcal{M}^p_{\text{s-v}}(\mathcal{F}_t), G \circ x \in \mathcal{M}^2_{\text{s-v}}(\mathcal{F}_t)$ and $H \circ x \in \mathcal{M}^2_{\text{s-v}}(\mathcal{F}_t, q)$, where $F \circ x = (F_t(x_t))_{t\geq 0}, G \circ x = (G_t(x_t))_{t\geq 0}$ and $H \circ x =$

$(H_{t,z}(x_t))_{t\geq 0, z\in\mathbb{R}^n}$. Every stochastic process $(x_t)_{t\in[t_0,T]}$ without discontinuities of second kind and satisfying the conditions mentioned above is said to be a *strong solution* to stochastic inclusions (3) or (4), respectively.

In what follows we shall assume that some of the following conditions $(\mathcal{A}_1)$–$(\mathcal{A}_2)$ are satisfied.

$(\mathcal{A}_1)$ F, G, H are measurable.

$(\mathcal{A}_2)$ F, G, H are are linearly bounded on D, i.e., there is a number $C > 0$ such that for every x one has $\|F_t(x)\|^2 + \|G_t(x)\|^2 + \int_{\mathbb{R}^m} \|H_{t,z}(x)\|^2 q(dz) \leq C(1 + |x|^2)$ for each $t \in [t_0, T]$.

$(\mathcal{A}_3)$ There are $k, \ell \in L^2(\mathcal{B}_+)$ and $m \in L^2(\mathcal{B}_+ \times \mathcal{B}^n)$ such that $h(F_t(x_2)(\omega), F_t(x_1)(\omega)) \leq k(t)|x_1 - x_2|$, $h(G_t(x_2)(\omega), G_t(x_1)(\omega)) \leq \ell(t)|x_1 - x_2|$ and $h(H_{t,z}(x_2)(\omega), H_{t,z}(x_1)(\omega)) \leq m(t,z)|x_1 - x_2|$ a.e. for each $t \geq 0$ and $x_1, x_2 \in \mathbb{R}^n$.

$(\mathcal{A}_4)$ F, G and H are l.s.c. for fixed $t \in [t_0, T]$.

As in [5], the following existence theorem can be proved.

THEOREM 2. *Let $\varphi \in L^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$. Suppose F, G and H satisfy $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and are such that $E\int_{t_0}^T \|(F\circ 0)_t\|^2 dt < \infty$, $E\int_{t_0}^T \|(G\circ 0)_t\|^2 dt < \infty$ and $E\int_{t_0}^T \int_{\mathbb{R}^n} \|(H\circ 0)_{t,z}\|^2 dt q(dz) < \infty$. Then inclusions* (3) *and* (4) *have at least one strong solutions satisfying an initial condition $x_{t_0} = \varphi$ with probability* 1.

3. Existence of weak solutions. We begin with recalling some notations connected with limit theorems for random processes with values in a metric space (X, ρ). A process $(\xi_t)_{t\in[t_0,T]}$ with values in X is said to be *stochastically continuous* at a point $t_1 \in [t_0, T]$ if for every $\varepsilon > 0$, $\lim_{t\to t_1} P\{\rho(\xi_t, \xi_{t_1}) > \varepsilon\} = 0$. It is said to be *stochastically continuous from the right* at t_1 if for every $\varepsilon > 0$, $\lim_{t\to t_1, t>t_1} P\{\rho(\xi_t, \xi_{t_1}) > \varepsilon\} = 0$. We have the following (see [1], p. 2) property of stochastically continuous processes.

PROPOSITION 2. *For every process $(\xi_t)_{t\in[t_0,T]}$ with values in X, there is a separable process stochastically equivalent to it. If a process $(\xi_t)_{t\in[t_0,T]}$ is stochastically continuous at all points $t \in [t_0, T]$ with possible exception of a finite number of points, then there exists a separable measurable process that is stochastically equivalent to a process $(\xi_t)_{t\in[t_0,T]}$.* ■

We often need to deal with sequences of processes whose finite distributions converge to a particular limiting distribution. It turns out that under rather broad assumptions, we may treat the relevant process as converging to some limiting process in probability. More precisely, we have the following theorem (see [1], p. 9).

THEOREM A. *Supoose that a sequence $(\xi_t^n)_{t\in[t_0,T]}$ of random processes with values in $\mathbb{R}^m$ is stochastically continuous from the right at every point $t \in [t_0, T]$*

such that for every k and $t_1,\ldots,t_k$ in $[t_0,T]$, the joint distribution of the values of $\xi^n_{t_1},\ldots,\xi^n_{t_k}$ converges weakly to a limiting distribution, and that for every $\varepsilon>0$,

$$\lim_{h\to 0}\lim_{n\to\infty}\sup_{|s_1-s_2|\le h} P\{|\xi^n_{s_1}-\xi^n_{s_2}|>\varepsilon\}=0. \tag{6}$$

Then there is a sequence of random processes $(x^n_t)_{t\in[t_0,T]}$, $n=0,1,\ldots$, on the probability space $(\Omega',\mathcal{F}',P')$, where $\Omega'=[0,1]$, $\mathcal{F}'$ is a σ-algebra of Borel sets of the interval $[0,1]$, and P' is the Lebesgue measure on $[0,1]$, such that: x^0_t is stochastically continuous, x^n_t converges in probability to x^0_t for every t, and for $n>0$, the finite distributions of the processes $(x^n_t)-t\in[t_0,T]$ and $(\xi^n_t)_{t\in[t_0,T]}$ coincide. ■

R e m a r k 1. Let be $(\xi^n_t)_{t\in[t_0,T]}$ a sequence of processes that are stochastically continuous from the right and that satisfy the conditions (6). Suppose that

$$\lim_{C\to\infty}\lim_{n\to\infty}\sup_{t_0\le t\le T} P\{|\xi^n_t|>C\}=0. \tag{7}$$

Then it is possible to exhibit a sequence (n_k) of positive integers such that the limiting distributions for finite-dimensional distributions of the processes $\xi^{n_k}_t$ exist as $k\to\infty$. In fact, the condition (7) implies compactness of the sequence of distributions of the quantities $\xi^{n_k}_{t_1},\ldots,\xi^{n_k}_{t_\ell}$ no matter what $t_1,\ldots,t^{n_k}_\ell$ and ℓ are. ■

COROLLARY 1. *If the processes $(\xi^n_t)_{t\in[t_0,T]}$ satisfy the conditions of Remark 1, then for some sequence (n_k) of positive integers it is possible to construct processes $(x^{n_k}_t)_{t\in[t_0,T]}$ on the probability space $(\Omega',\mathcal{F}',P')$ having the same finite-dimensional distributions as $\xi^{n_k}_t$ and converging in probability to some process $(x^0_t)_{t\in[t_0,T]}$ as $k\to\infty$.* ■

R e m a r k 2. The results of Corollary 1 can be extended to r sequences of stochastic processes (see [1], Corollary 2, p. 13). ■

We can now formulate sufficient conditions for the existence of weak solution to the stochastic inclusion (5). Let us note that by a weak solution to (5) we mean a system $[(\Omega',\mathcal{F}',(\mathcal{F}'_t)_{t\in[t_0,T]},P'),(\xi'_t)_{t\in[t_0,T]},(w'_t)_{t\in[t_0,T]},p',q']$ consisting of a probability filtered space $(\Omega',\mathcal{F}',(\mathcal{F}'_t)_{t\in[t_0,T]},P')$, an $\mathcal{F}'_t$-adapted stochastically continuous process $(\xi'_t)_{t\in[t_0,T]}$, an $\mathcal{F}'_t$-Brownian motion $(w'_t)_{t\in[t_0,T]}$ and Poisson measures p', q' such that $(\xi'_t)_{t\in[t_0,T]}$ satisfy (5) if in this inclusion we replace w_t, p, q with w'_t, p', q'. The proof of such an existence theorem will follow from the existence theorem given in [1] (see [1], p. 59–73) and Michael's continuous selection theorem. We shall assume that the following conditions $(\mathcal{A}_3)$–$(\mathcal{A}_6)$ are satisfied.

$(\mathcal{A}_3)$ F,G,H are are linearly bounded, i.e., there is a number $K>0$ such that $\|F_t(x)\|^2+\|G_t(x)\|^2+\int_{|u|\le 1}\|H_{t,u}(x)\|^2du/|u|^{m+1}\le K(1+|x|^2)$.

$(\mathcal{A}_4)$ F, G are l.s.c. on $[t_0,T]\times\mathbb{R}^n$.

($\mathcal{A}_5$) H is bounded in every bounded region of x and u, is measurable with respect to $u\in\mathbb{R}^m$ for fixed $t\in[t_0,T]$ and $x\in\mathbb{R}^n$, and is l.s.c. for almost all u with respect to t and x.

($\mathcal{A}_6$) For all $t_1\in[t_0,T]$ and $x_1\in\mathbb{R}^n$,

$$\lim_{t\to t_1,x\to x_1}\int\limits_{|u|\le 1}\overline{h}^2(H_{t,u}(x),H_{t_1,u}(x_1))\frac{du}{|u|^{m+1}}=0.$$

THEOREM 2. *Suppose F, G and H take compact convex values and are such that conditions ($\mathcal{A}_3$)–($\mathcal{A}_6$) are satisfied and let $\varphi:\Omega\to\mathbb{R}^n$ be $\mathcal{F}_{t_0}$-mesurable and such that $E|\varphi|^2<\infty$. Then there is at least one weak solution*

$$[(\Omega',\mathcal{F}',(\mathcal{F}'_t)_{t\in[t_0,T]},P'),(\xi'_t)_{t\in[t_0,T]},(w'_t)_{t\in[t_0,T]},p',q']$$

to (5) *such that the distribution of ξ'_{t_0} coincides with the distribution of φ.*

Proof. By virtue of Michael's continuous selection theorem there are continuous functions $a,b:[t_0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ such that $a(t,x)\in F_t(x)$ and $b(t,a)\in G_t(x)$ for $(t,x)\in[t_0,T]\times\mathbb{R}^n$. Moreover, Michael's continuous selection theorem implies the existence of a function $f:[t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ that is measurable with respect to $u\in\mathbb{R}^m$ and continuous with respect to $(t,x)\in[t_0,T]\times\mathbb{R}^n$ and such that $f(t,x,u)\in H_{t,u}(x)$ for $(t,x,u)\in[t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m$. Immediately from properties of F, G and H it follows that the following conditions are satisfied:

$$|a(t,x)|^2+|b(t,x)|^2+\int\limits_{|u|\le 1}|f(t,x,u)|^2\frac{du}{|u|^{m+1}}\le K(1+|x|^2)$$

and

$$\lim_{t\to t_1,x\to x_1}\int\limits_{|u|\le 1}|f(t,x,u)-f(t_1,x_1,u)|^2\frac{du}{|u|^{m+1}}=0.$$

Consider now the stochastic equation

$$\begin{aligned}\xi_t=\xi_{t_0}&+\int\limits_{t_0}^{t}a(t,s,\xi_s)ds+\int\limits_{t_0}^{t}b(t,s,\xi_s)dw_s\\&+\int\limits_{t_0}^{t}\int\limits_{|u|\le 1}f(t,s,\xi_s,u)q(ds,du)\\&+\int\limits_{t_0}^{t}\int\limits_{|u|>1}f(t,s,\xi_s,u)p(ds,du).\end{aligned}\tag{8}$$

Let us consider a sequence of subdivisions of the interval $[t_0,T]$: $t_0=t_0^n<t_1^n<\ldots<t_n^n=T$ such that $\lim_{n\to\infty}\max_k(t_{k+1}^n-t_k^n)=0$ and define a random variable ξ_k^n by the relations: $\xi_0^n=\varphi$,

$$\xi^n_{k+1} = \xi_k + a(t^n_k, , \xi^n_k)\Delta t^n_k + b(t^n_k, , \xi^n_k)[w_{t^n_{k+1}} - w_{t^n_k}] \int_{t^n_k}^{t^n_{k+1}} \int_{|u|\le 1} f(t^n_k, s, \xi_{t^n_k}, u) q(ds, du) + \int_{t^n_k}^{t^n_{k+1}} \int_{|u|>1} f(t^n_k, s, \xi_{t^n_k}, u) p(ds, du), \tag{9}$$

where $\Delta t^n_k = t^n_{k+1} - t^n_k$. Define ξ^n_t and ζ^n_t by taking $\xi^n_t = \xi^n_k$ for $t \in [t^n_k, t^n_{k+1}]$ and $\zeta^n_t = \int_{t^t_0} \int_{|u|\le 1} uq_n(ds, du) + \int_{t_0}^t \int_{|u|>1} up_n(ds, du)$, where (p_n) is a sequence of Poisson measures with independent values defined on $[t_0, T] \times \mathbb{R}^m$ for which $Ep_n(A) = \int_A udtdu/|u|^{m+1}$, and $q_n(A) = p_n(A) - Ep_n(A)$. As in [1] (pp. 64–67) we verify that for each of the processes ξ^n_t, ζ^n_t, w_t, the conditions of Remark 1 are fulfilled. Therefore, by Corollary 1 it is possible to choose a sequence (n_k) of positive integers and to construct processes $x^{n_k}_t$, $z^{n_k}_t$ and $w^{n_k}_t$ defined on a probability space $(\Omega', \mathcal{F}', P')$ such that their joint finite-dimensional distributions coincide for every k with the joint finite-dimensional distribution of the processes $\xi^{n_k}_t$, $\zeta^{n_k}_t$, w_t, and such that $x^{n_k}_t \to \xi'_t$ as $k \to \infty$, and $w^{n_k}_t \to w'_t$ as $z^{n_k}_t \to \zeta'_t$ in probability, where ξ'_t, ζ'_t and w'_t are certain random processes. It can be verified (see [1], p. 65) that there are Poisson measures p'_{n_k} and p' with independent values such that

$$z^n_t = \int_{t_0}^t \int_{|u|\le 1} uq'_{n_k}(ds, du) + \int_{t_0}^t \int_{|u|>1} up'_{n_k}(ds, du)$$

and

$$\zeta_t = \int_{t_0}^t \int_{|u|\le 1} uq'(ds, du) + \int_{t_0}^t \int_{|u|>1} up'(ds, du),$$

where $Ep'_{n_k}(A) = Ep'(A) = \int_A dtdu/|u|^{m+1}$, $q'_{n_k}(A) = p'_{n_k}(A) - Ep'_{n_k}(A)$ and $q'(A) = p'(A) - Ep'(A)$. Finally, as in [1] (pp. 68–73), it may be verified that ξ'_t satisfies (7) if in this equation we replace w, p, q with w', p', q'. On the other hand, for every $t_0 \le s < t \le T$ we have

$$\xi'_t - \xi'_s = \int_s^t a(\tau, \xi'_\tau)d\tau + \int_s^t b(\tau, \xi'_\tau)dw'_\tau + \int_s^t \int_{|u|\le 1} f(\tau, \xi'_\tau, u)q'(d\tau, du) + \int_s^t \int_{|u|>1} f(\tau, \xi'_\tau, u)p'(ds, du) \in \int_s^t F_\tau(\xi'_\tau)d\tau + \int_s^t G_\tau(\xi'_\tau)dw'_\tau + \int_s^t \int_{|u|\le 1} H_{\tau,u}(\xi'_\tau)q'(d\tau, du) + \int_s^t \int_{|u|>1} H_{\tau,u}(\xi'_\tau)p'(ds, du).$$

Therefore,

$$[(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\in[t_0,T]}, P'), (\xi'_t)_{t\in[t_0,T]}, (w'_t)_{t\in[t_0,T]}, p', q']$$

is a weak solution to (5). ∎

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