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SYSTEMS OF RAYS IN THE PRESENCE OF DISTRIBUTION OF HYPERPLANES

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Abstract. Horizontal systems of rays arise in the study of integral curves of Hamiltonian systems v_H on $T^\star X$, which are tangent to a given distribution V of hyperplanes on X. We investigate the local properties of systems of rays for general pairs (H, V) as well as for Hamiltonians H such that the corresponding Hamiltonian vector fields v_H are horizontal with respect to V. As an example we explicitly calculate the space of horizontal geodesics and the corresponding systems of rays for the canonical distribution on the Heisenberg group. Local stability of systems of horizontal rays based on the standard singularity theory of Lagrangian submanifolds is also considered.

Introduction. Let X be a differentiable manifold. Let ω be a differential 1-form on X, and V a distribution of tangent hyperplanes in $T^\star X$ annihilated by ω. We consider the Hamiltonian systems v_H, with Hamiltonian H on $T^\star X$, which are horizontal with respect to V, i.e. the projections of the bicharacteristics of v_H onto the base of $T^\star X$ are tangent to V. We introduce the notion of the space of geodesics-rays as the reduced symplectic space M of bicharacteristics on $H^{-1}(\frac{1}{2})$ equipped with the reduced symplectic form μ defined by the relation $\omega_X\mid_{H^{-1}(\frac{1}{2})} - \pi^\star\mu = 0$, where ω_X is the canonical Liouville form on $T^\star X$ and π is the projection onto M along bicharacteristics (cf. [9]). Any Lagrangian submanifold L of (M, μ) is called a system of rays. Its counterimage $\pi^{-1}(L)$ represents an optical system of rays in the phase space $(T^\star X, \omega_X)$ of geometric optics. The graph of π is a Lagrangian submanifold in the product symplectic structure $(T^\star X \times M, \pi_2^\star\mu - \pi_1^\star\omega_X)$, where π_i are the projections of the cartesian product $T^\star X \times M$. Then we have a generating function G for *graph*π (cf. [6]), which helps to determine the structure of the counterimage $\pi^{-1}(L)$. Now the stability notion

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for an optical system of rays is defined by the deformation group of Lagrangian submanifolds in M. Then combining the deformed Lagrangian submanifolds with the structure of the fixed (i.e. undeformed) function G we impose on the stability problem the properties of the eikonal equation itself (cf. [4]).

In Section 1 we describe the properties of the function G for horizontal Hamiltonian systems. The integrable and nonintegrable cases of ω are discussed and the relation of G to the distance function (measured along the horizontal curves of V) is described. It is shown how to make use of generating functions and generating families of functions to investigate systems of rays in $T^\star X$ gliding along horizontal curves. For general pairs (V, H), where H is a geodesic Hamiltonian, the space of locally shortest paths (normal geodesics, cf. [14]), horizontal with respect to V is investigated. The differences between integrable and nonintegrable cases of V are described. Section 2 is devoted mostly to computational examples. We calculate explicitly (in contrast to [16]) the space of geodesics and the generating functions G for R^3 endowed with the contact distribution. The same is done for a distribution on R^3 annihilated by the singular contact form $dz + y^2dx$. By this result we obtain the exact form of geodesics—they are liftings of circles and lines in the nonsingular contact case. Finally in Section 3, using the generating function G we introduce the stability notion for optical systems of rays and find stable systems of rays in the above mentioned 3-dimensional example of the Heisenberg group. An extension of the classification of local models of systems of rays to the corresponding classification of their evolutions is discussed.

1. Systems of geodesics. Let X be a smooth manifold. We consider the cotangent bundle $T^\star X$ with the cotangent bundle projection $\pi_X : T^\star X \to X$. We assume $dim X = n + 1$. By $V \subset TX$ we denote a smooth distribution of hyperplanes on X, i.e. a subbundle of TX, $V_q \subset T_qX$. In what follows we restrict our considerations to distributions of codimension one. Locally V is annihilated by a 1-form,

$$\omega = dq^{n+1} + \sum_{i=1}^{n} A_i(q)dq^i,$$

where $\{q^i\}$ are coordinates on X.

A smooth curve $\gamma : (a, b) \to X$ is called *horizontal* if $\frac{d\gamma}{dt}(t) \in V_{\gamma(t)}$ for all $t \in (a, b)$. Let $H : T^\star X \to R$ be a smooth function. We say that the Hamiltonian vector field $v_H = \sum_{i=1}^{n+1}(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i})$, with Hamiltonian H, is horizontal if

$$(\pi_X)_\star v_H\mid_{\bar p} \in V_{\pi_X(\bar p)}$$

for all $\bar p \in T^\star X$.

By $\mathcal{H}_V$ we denote the space of horizontal Hamiltonian vector fields. An easy check shows that if $v_H \in \mathcal{H}_V$, then locally

$$H(q, p) = h(q, p_1', \ldots, p_n')$$

for some smooth function $h : R^{n+1} \times R^n \to R$ and

$$p'_i = p_i - A_i(q)p_{n+1}.$$

Let $v_H \in \mathcal{H}_V$ and H depend on p'_i quadratically, i.e. $H(q,p) = \frac{1}{2}\sum_{i,j=1}^n g^{ij}(q)p'_i p'_j$ for some smooth functions $g^{ij}(q)$, and let V be a nonintegrable distribution. Then the integral curves of v_H are called *sub-Riemannian* geodesics (cf. [13]) and H is called a sub-Riemannian Hamiltonian. The sub-Riemannian Hamiltonian H defines sub-Riemannian geometry provided $g^{ij}(q)$ is nondegenerate on X.

Consider the hypersurface $H^{-1}(\frac{1}{2})$. This is a coisotropic submanifold of $T^\star X$ (cf. [15]). By the canonical symplectic reduction procedure [9] we have the symplectic space of geodesics on $H^{-1}(\frac{1}{2})$, M, equipped with the canonical symplectic form μ. In fact we have the projection π along bicharacteristics of $H^{-1}(\frac{1}{2})$,

$$\pi : H^{-1}\left(\frac{1}{2}\right) \to M,$$

and the symplectic form μ on M is defined uniquely by the Liouville form ω_X on $T^\star X$ and the reduction relation formula,

$$\pi^\star\mu = \omega_X|_{H^{-1}(\frac{1}{2})}. \tag{1}$$

Now we construct the product symplectic space

$$\Theta = (T^\star X \times M, \pi_2^\star\mu - \pi_1^\star\omega_X),$$

where π_i are the canonical projections of the cartesian product $T^\star X \times M$. By (1), $graph\pi$ is obviously a Lagrangian submanifold of Θ. Let (r,s) be Darboux coordinates on M, i.e. M is diffeomorphic to some cotangent bundle $T^\star N$ with $\omega_N = \sum_{i=1}^n ds_i \wedge dr_i$, and $graph\pi$ is locally generated by the generating function $G : X \times N \to R$,

$$graph\pi = \left\{(\bar{p}; r, s) : -\frac{\partial G}{\partial q^i}(q,r) = p_i, \frac{\partial G}{\partial r_j}(q,r) = s_j,\ i = 1,\dots n+1, j = 1,\dots,n\right\}.$$

We notice that G is a complete solution of the Hamilton-Jacobi equation

$$H\left(q, -\frac{\partial G}{\partial q^i}(q,r)\right) \equiv 0.$$

We see that the above introduced generating function G reconstructs the space of geodesics which are horizontal with respect to V; we call this space the space of horizontal geodesics. A smooth function $G : X \times N \to R$ is called a *horizontal genrating function* if it defines the space of horizontal geodesics for some horizontal Hamiltonian system. This Hamiltonian system is determined by the inclusion of the hypersurface $H^{-1}(\frac{1}{2})$ in the following form:

$$X \times N \ni (q,r) \to \left(q, -\frac{\partial G}{\partial q}(q,r)\right) \in T^\star X.$$

Instead of speaking of horizontal Hamiltonian systems we will consider the space of horizontal generating functions.

By $\Delta_j[G](q,r)$ we denote the determinant

$$\Delta_j[G](q,r) = (-1)^{j+1} det \begin{pmatrix} \frac{\partial^2 G}{\partial r_1 \partial q^1} & \dots & \widehat{j} & \dots & \frac{\partial^2 G}{\partial r_1 \partial q^{n+1}} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial^2 G}{\partial r_n \partial q^1} & \dots & \widehat{j} & \dots & \frac{\partial^2 G}{\partial r_n \partial q^{n+1}} \end{pmatrix}.$$

PROPOSITION 1.1. *The smooth function $G : X \times N \to R$ is a horizontal generating function if and only if G satisfies the equation*

$$\Delta_{n+1}[G] + \sum_{j=1}^{n} A_j(q)\Delta_j[G] = 0. \tag{2}$$

Proof. A geodesic $\gamma_{(r,s)}(t) = (q^1(t;r,s), \dots, q^{n+1}(t;r,s))$ as a curve in X is defined by the equations

$$\frac{\partial G}{\partial r_i}(\gamma_{(r,s)}(t), r) - s_i \equiv 0, \qquad i = 1, \dots, n. \tag{3}$$

It has to be horizontal, so it fulfils the equation

$$\dot{q}^{n+1}(t;r,s) + \sum_{i=1}^{n} A_i(\gamma_{(r,s)}(t))\dot{q}^i(t;r,s) \equiv 0. \tag{4}$$

Differentiating (3) with respect to t we obtain a system of equations for the tangent vector to the geodesic:

$$\sum_{j=1}^{n+1} \frac{\partial^2 G}{\partial r_i} q^j(\gamma_{(r,s)}(t), r)\dot{q}^j(t;r,s) \equiv 0, \qquad i = 1, \dots, n.$$

(4) with these equations gives the condition for the vanishing of the determinant of the extended system of equations. The expansion of this determinant gives equation (2). It is straightforward that any solution of (2) is a horizontal generating function. ∎

Remark 1.2. Until now we have not used the property that ω is integrable or not. If ω is integrable then A_i depend on $q' = (q^1, \dots, q^n)$ and there exists a smooth function $q' \to S(q')$ such that $\omega = dF$, where $F(q) = q^{n+1} + S(q')$. Then we take new symplectic coordinates,

$$(q,p) \to (q', q^{n+1} + S(q'), p', p_{n+1}),$$

and reduce the problem to the hypersurface $q^{n+1} + S(q') = C$ (where C is constant) with Hamiltonian

$$\widetilde{H}(Q^1, \dots, Q^n, P_1, \dots, P_n) = H(Q^1, \dots, Q^n, C - S(Q^1, \dots, Q^n), P_1, \dots, P_n).$$

Any Lagrangian subvariety L of $(T^\star N, \mu = \omega_N)$ is called a *system of rays.* The counterimage of L, $\pi^{-1}(L)$, is a Lagrangian subvariety of $T^\star X$ built by geodesics.

Let L be generated by a generating family $F : N \times R^k \to R$, $(r, \lambda) \to F(r, \lambda)$. Then the corresponding generating family for $\pi^{-1}(L)$ has the form

$$\widetilde{R}(q, \nu) = F(\nu_1, \nu_2) - G(q, \nu_1),$$

where $\nu = (\nu_1, \nu_2)$ are the Morse parameters of $\widetilde{R}$ (for the theory of Morse families see e.g. [15]).

Remark 1.3. Let us fix $s = \bar{s}$ in formula (3). We assume $\bar{q} = \gamma_{(r,\bar{s})}(0)$ does not depend on r. Then the family $F(r, \lambda) = \sum_{i=1}^n \bar{s}_i r_i$ generates a bunch of rays through $\bar{q}$. For simplicity we take $F \equiv 0$ (i.e. we put $\bar{s}_i = 0$). Then the corresponding generating family of this bunch of rays in $T^\star X$ is

$$\widetilde{R}(q, \lambda) = -G(q, \lambda).$$

The level sets of the family $\widetilde{R}(q, \lambda)$ form the wave-front evolution corresponding to the system of rays generated by this family. Implicitly, this function represents the distance from the origin (say $\bar{q} = 0$) to the point q along the geodesic. It is obtained by elimination of the parameters (λ) in the equations $\frac{\partial G}{\partial \lambda_i}(q, \lambda) = 0$, $i = 1, \ldots, k$, namely,

$$d(0, q) = -G(q, \lambda)|_{\{\frac{\partial G}{\partial \lambda_i}(q,\lambda)=0,\ i=1,\ldots,k\}}.$$

Now we consider a horizontal curve $\gamma(t)$. We can choose it to be normalized:

$$\dot{\gamma}(t) = \Big(1, \phi^2(t), \ldots, \phi^n(t), -\sum_{i=1}^n A_i(\gamma(t))\phi^i\Big), \qquad \phi^1 \equiv 1.$$

Let us fix $t = t_0$. Then all geodesics $\gamma_{(r,s)}(t)$ in X with the same tangent vector $\dot{\gamma}(t_0)$ passing through $\gamma(t_0)$ satisfy the system of $2n - 1$ equations,

$$\begin{cases} \phi^j(t_0) - \dot{q}^j(t_0; r, s) = 0, \\ \frac{\partial G}{\partial r_i}(\gamma(t_0; r, s), r) - s_i = 0, \end{cases} \tag{5}$$

where $i = 1, \ldots, n$, $j = 2, \ldots, n$ and $\dot{\gamma}_{(r,s)}(t) = (\dot{q}^1(t; r, s), \ldots, \dot{q}^{n+1}(t; r, s))$.

Any solution of (5) is a one-dimensional curve in (r, s) parameters. The corresponding curve in the cotangent space $T^\star_{\gamma(t_0)}X$ is given by the equations

$$p_i = \frac{\partial G}{\partial q^i}(\gamma(t_0), r),$$

where (r, s) satisfies (5).

Let $I \ni t \to \beta(t)$ be a horizontal curve. Then to each $t \in I$ we have attached the one-parameter family of geodesics

$$\tau \to \gamma\bigg(\tau; r, \frac{\partial G}{\partial r_1}(\beta(t), r), \ldots, \frac{\partial G}{\partial r_n}(\beta(t), r)\bigg),$$

gliding along the horizontal curve β. The counterimage of this family in $T^\star X$ appears to be an isotropic subvariety (cf. [8]).

Let H be a general Hamiltonian on $T^\star X$ and let V be a distribution on X as before. By

$$\widetilde{K} = \left\{ (q,p) \in T^\star X; K(q,p) = \frac{\partial H}{\partial p_{n+1}}(q,p) + \sum_{i=1}^{n} A_i(q) \frac{\partial H}{\partial p_i}(q,p) = 0 \right\}$$

we denote the space of tangent directions to V. We write $Y = H^{-1}(0)$ and we assume that Y and $\widetilde{K}$ intersect transversally along the $2n$-dimensional surface $W = Y \cap \widetilde{K}$. Making use of π we have a smooth map

$$\rho = \pi|_W : W \to M$$

into the $2n$-dimensional space of rays defined by H. The image of ρ is called the space of tangent rays. These are integral curves of v_H which are tangent at some point ($\in W$) to the distribution V.

PROPOSITION 1.4. *Generically the spaces of tangent directions are classified by the Whitney stable mappings $R^{2n} \to R^{2n}$, i.e. for a generic pair (H,V), the mapping ρ is locally equivalent* (*right-left equivalent* [10]) *to one of the mappings*

$$(x_1, \ldots, x_{2n}) \to (x_1^{k+1} + x_2 x_1^{k-1} + \ldots x_1 x_k, x_2, \ldots, x_{2n}), \qquad 1 \le k \le 2n.$$

The only difference between this proposition and the result quoted in [2] (p. 4) is that we are a step further in the flag of exceptional submanifolds defined by the subsequent Poisson brackets. The space of singular points of ρ corresponds, in [2], to the set of asymptotic directions (if V is integrable then $dS = \omega$ and $K = \{H, S\}$, where S is a smooth function on X). The set of singular points of ρ is described by

$$H = 0, \quad K = 0, \quad \{H, K\} = 0,$$

and determines the rays of M which are tangent to W.

The biasymptotic directions are described by

$$H = 0, \quad K = 0, \quad \{H, K\} = 0, \quad \{H, \{H, K\}\} = 0,$$

and the triple-asymptotic directions (which correspond to the biasymptotic ones if K is a general hypersurface of X) by

$$H = 0, \quad K = 0, \quad \{H, K\} = 0,$$
$$\{H, \{H, K\}\} = 0, \qquad \{H, \{H, \{H, K\}\}\} = 0.$$

Let us write H in the convenient form $H(q,p) = \bar{H}(q, \bar{p}, \bar{p}_{n+1})$ for some smooth function $\bar{H} : R^{2n+2} \to R$ and $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$, $\bar{p}_i = p_i - A_i(q) p_{n+1}$, $\bar{p}_{n+1} = p_{n+1}$, $i = 1, \ldots, n$. Then the set of critical points of ρ is described by

$$\bar{H} = 0, \quad \frac{\partial \bar{H}}{\partial \bar{p}_{n+1}} = 0, \quad \left\{ \bar{H}, \frac{\partial \bar{H}}{\partial \bar{p}_{n+1}} \right\} = 0.$$

We find

$$\left\{ \bar{H}, \frac{\partial \bar{H}}{\partial \bar{p}_{n+1}} \right\} = p_{n+1} \sum_{i,j=1}^{n} \frac{\partial \bar{H}}{\partial \bar{p}_i} \frac{\partial^2 \bar{H}}{\partial \bar{p}_j \partial \bar{p}_{n+1}} \epsilon_{ij} + \sum_{i=1}^{n} \left(\frac{\partial \bar{H}}{\partial \bar{p}_i} \frac{\partial^2 \bar{H}}{\partial \bar{q}_i \partial \bar{p}_{n+1}} - \frac{\partial \bar{H}}{\partial \bar{q}_i} \frac{\partial^2 \bar{H}}{\partial \bar{p}_i \partial \bar{p}_{n+1}} \right)$$

$$+\, p_{n+1}\sum_{j=1}^{n}\left(\frac{\partial\bar H}{\partial\bar p_j}\frac{\partial^2\bar H}{\partial\bar p_{n+1}\partial\bar p_{n+1}}-\frac{\partial\bar H}{\partial\bar p_{n+1}}\frac{\partial^2\bar H}{\partial\bar p_j\partial\bar p_{n+1}}\right)A_{j,n+1}$$
$$+\sum_{j=1}^{n}\left(\frac{\partial\bar H}{\partial\bar q_{n+1}}\frac{\partial^2\bar H}{\partial\bar p_j\partial\bar p_{n+1}}-\frac{\partial\bar H}{\partial\bar p_j}\frac{\partial^2\bar H}{\partial\bar p_{n+1}\partial\bar q_{n+1}}\right)A_j,$$

where $A_{i,j}=\frac{\partial A_i}{\partial q^j}$,

$$[X_i,X_j]=\epsilon_{ij}\frac{\partial}{\partial q^{n+1}}=(A_{i,j}-A_{j,i}+A_iA_{j,n+1}-A_jA_{i,n+1})\frac{\partial}{\partial q^{n+1}},$$

and the vector fields $X_i=\frac{\partial}{\partial q^i}-A_i\frac{\partial}{\partial q^{n+1}}$, $i=1,\dots,n$, generate V.

Let V be integrable. Then $\epsilon_{ij}=0$, and $\frac{\partial\bar H}{\partial\bar p_{n+1}}=\{\bar H,S\}$ for some smooth function $S:X\to R$, $dS=dq^{n+1}+\sum_{i=1}^n A_i dq^i$. Then $W=\bigcup_t\Omega_t$ is fibered by

$$\Omega_t:\quad H=0,\ S=t,\ \{H,S\}=0.$$

Let $U_{\tilde p}^{\lrcorner}$ denote the symplectic polar of the tangent space $T_{\tilde p}U_t$, where $U_t=\{\widetilde p\in T^\star X;H(\widetilde p)=0,\ S(q)=t\}$, $\widetilde p=(q,p)$, $dim U_{\tilde p}^{\lrcorner}=2$ (we view S as being lifted to $T^\star X$). On each Ω_t of W we have the following vector field of tangent directions (cf. [2]):

$$\widetilde X_{\tilde p}=U_{\tilde p}^{\lrcorner}\cap T_{\tilde p}\Omega_t.$$

If H is a geodesic Hamiltonian, then the integral curves of $\widetilde X$ are locally shortest curves, horizontal with respect to V, so they are surface geodesics on the leaves of the foliation defined by S. If V is nonintegrable, then the corresponding vector field $\widetilde X$ on W is obtained by repeating the above construction for the $\exp_{\pi_X(\tilde p)}(V_{\pi_X(\tilde p)})$-hypersurface of X, at each point $\widetilde p\in W$. $\exp_q$ being defined on the fibre T_qX of the tangent bundle TX (by the geodesic Hamiltonian H).

Let Δ be the set of critical points of ρ. By $\Sigma\subset M$ we denote the corresponding set of critical values of ρ. On the nonsingular part of Δ we have the field of directions

$$\widetilde Y=W_{\tilde p}^{\lrcorner}\cap T_{\tilde p}\Delta.$$

The integral curves of $\widetilde Y$ are called special geodesics of (H,V). These curves project through π onto bicharacteristics of Σ in (M,μ). The special geodesics are integral curves of the Hamiltonian vector field $v_{\widetilde\Sigma}$ with Hamiltonian (see [2], p. 4)

$$\widetilde\Sigma=\bar H\left\{\frac{\partial\bar H}{\partial\bar p_{n+1}},\left\{\bar H,\frac{\partial\bar H}{\partial\bar p_{n+1}}\right\}\right\}+\frac{\partial\bar H}{\partial\bar p_{n+1}}\left\{\bar H,\left\{\frac{\partial\bar H}{\partial\bar p_{n+1}},\bar H\right\}\right\}.$$

This Hamiltonian is not necessarily quadratic with respect to the p-coordinates.

Example 1.5. We consider the contact distribution $dz-xdy=0$, and the Hamiltonian $H(q,p)=\frac12((1+z^2)p_1^2+(p_2+xp_3)^2+p_3^2-1)$, $q=(x,y,z)$. After straightforward calculations we obtain $W=\{(q,p);(1+z^2)p_1^2+p_2^2=1,p_3=0\}$

and

$$\Delta = \{(q,p); (1+z^2)p_1^2 + p_2^2 = 1, p_3 = 0, zp_1 = 0\}.$$

Thus the corresponding Hamiltonian for the space of special geodesics is

$$\widetilde{\Sigma}(q,p) = p_1^2p_3^2 + p_1^2p_3(p_2+xp_3)x - 2zp_1p_3^2(p_2+xp_3) - \frac{1}{2}p_1^2((1+z^2)p_1^2 + (p_2+xp_3)^2 + p_3^2 - 1).$$

Assume that $W = Y \cap \widetilde{K}$ is a coisotropic submanifold of $T^\star X$, i.e. $\{H,K\} = 0$ on W. Then at each point $\widetilde{p} \in W$ the corresponding bicharacteristic passing through $\widetilde{p}$ is tangent to W. Thus W projects, by ρ, into the hypersurface Z of M. In this case systems of horizontal rays are defined by those Lagrangian submanifolds of M which are also submanifolds of Z (if Z is interpreted as an eikonal equation then these systems of rays are called bioptical).

Example 1.6. As an example we consider the geodesic Hamiltonian of the form

$$H(q,p) = \frac{1}{2}\Big(\sum_{i,j=1}^{n} g^{ij}(q)(p_i - A_i(q)p_{n+1})(p_j - A_j(q)p_{n+1}) + p_{n+1}^2\Big) - \frac{1}{2},$$

where g^{ij} do not depend on q_{n+1}. We find easily that $K = p_{n+1}$ and $\{H,K\} = 0$ on

$$W = \Big\{(q,p); \sum_{i,j=1}^{n} g^{ij}(q)p_ip_j = 1,\ p_{n+1} = 0\Big\}.$$

Hence W is coisotropic and the systems of horizontal rays are built by $(n-1)$-parameter families of two-dimensional bicharacteristics of W. They project into Lagrangian submanifolds of the symplectic reduced space of bicharacteristics of Z.

In the above sense of genericity, the pairs (H,V) with horizontal Hamiltonians H are highly nongeneric; however, they are interesting from the point of view of applications in control theory.

2. Some exact calculations. As a representative example of a horizontal Hamiltonian system appearing in the literature (although nongeneric), we consider the 3-dimensional Heisenberg group $\mathbf{H}$ (cf. [16]) endowed with the contact (nonholonomic) distribution annihilated by the 1-form

$$\omega = dz - xdy.$$

We consider the horizontal Hamiltonian system of the form

$$H(q,p) = \frac{1}{2}(p_1^2 + (p_2+xp_3)^2), \qquad g^{ij} = \delta^{ij}.$$

To study the local properties of systems of rays provided by this Hamiltonian we have to cover the space of horizontal geodesics M with four charts $\pi(U_i), i =$

$1,\dots,4$; $U_j=\{p_j-A_jp_3>0\}\cap\{H^{-1}(\frac{1}{2})\}$, $U_{2+j}=\{p_j-A_jp_3<0\}\cap\{H^{-1}(\frac{1}{2})\}$, $j=1,2$; $A_1\equiv 0, A_2=-x$.

Without loss of generality we can work with two representative charts $\Xi_i=\pi(U_i), i=1,2$, where the open set U_1 on $H^{-1}(\frac{1}{2})$ is parameterized by (p_2,p_3,q) and the open set U_2 is parameterized by (p_1,p_3,q). Now we can calculate π in both distinguished charts:

PROPOSITION 2.1. *A. There is a system of Darboux coordinates on Ξ_1 in which $\mu=ds_1\wedge dr_1+ds_2\wedge dr_2$ and $\pi|U_1:(p_2,p_3,q)\to(s_1,s_2,r_1,r_2)$ is given by the equations*

$$\begin{aligned}
r_1&=-p_2,\\
r_2&=-p_3,\\
s_1&=y+\frac{1}{p_3}[(1-(p_2+xp_3)^2)^{\frac{1}{2}}-(1-p_2^2)^{\frac{1}{2}}],\\
s_2&=z+\frac{x}{p_3}(1-(p_2+xp_3)^2)^{\frac{1}{2}}-\frac{1}{2p_3^2}[\arcsin(p_2+xp_3)\\
&\qquad+(p_2+xp_3)(1-(p_2+xp_3)^2)^{\frac{1}{2}}-\arcsin p_2-p_2(1-p_2^2)^{\frac{1}{2}}].
\end{aligned}$$

B. There is a system of Darboux coordinates on Ξ_2 in which $\mu=ds_1\wedge dr_1+ds_2\wedge dr_2$ and $\pi|U_2:(p_1,p_3,q)\to(s_1,s_2,r_1,r_2)$ is defined by the equations

$$\begin{aligned}
r_1&=-p_1-yp_3,\\
r_2&=-p_3,\\
s_1&=x-\frac{1}{p_3}[(1-(p_1+yp_3)^2)^{\frac{1}{2}}-(1-p_1^2)^{\frac{1}{2}}],\\
s_2&=z-xy-\frac{y}{p_3}(1-(p_1)^2)^{\frac{1}{2}}+\frac{1}{2p_3^2}[\arcsin(p_1+yp_3)\\
&\qquad+(p_1+yp_3)(1-(p_1+yp_3)^2)^{\frac{1}{2}}-\arcsin p_1-p_1(1-p_1^2)^{\frac{1}{2}}].
\end{aligned}$$

Now we find the corresponding generating functions of $graph\pi_i\subset(T^\star X\times T^\star\bar{\Xi}_i,\mu\ominus\omega_X)$, where $\Xi_i=T^\star\bar{\Xi}_i$ for some open neighbourhoods $\bar{\Xi}_i\subset X$.

COROLLARY 2.2. *The two Lagrangian submanifolds, $graph\pi|_{U_i}$, $i=1,2$, are generated, up to additive constants, by the following generating functions:*

$$\begin{aligned}
G_1(q,r)=r_1y+r_2z-\frac{1}{2r_2}&[\arcsin(r_1+xr_2)+(r_1+xr_2)(1-(r_1+xr_2)^2)^{\frac{1}{2}}\\
&-\arcsin r_1-r_1(1-r_1^2)^{\frac{1}{2}}],
\end{aligned}$$

and

$$\begin{aligned}
G_2(q,r)=r_1x+r_2(z-xy)-\frac{1}{2r_2}&[\arcsin(r_1-yr_2)+(r_1-yr_2)(1-(r_1-yr_2)^2)^{\frac{1}{2}}\\
&-\arcsin r_1-r_1(1-r_1^2)^{\frac{1}{2}}],
\end{aligned}$$

respectively.

There are two simplest types of systems of rays in $\mathbf{H}$. Now we describe their representative examples.

1. A beam of "parallel" rays L with fixed co-direction $(\hat{p}_1, \hat{p}_2, \hat{p}_3)$ at each point of the hyperplane $\{x = 0\}$ is generated by the family

$$F(r, \lambda) = r_1\lambda_1 + r_2\lambda_2 + \hat{p}_2\lambda_1 + \hat{p}_3\lambda_2.$$

The corresponding Lagrangian submanifold $\pi^{-1}(L) \subset T^\star X$ is generated by

$$\begin{aligned}\widetilde{R}(q,\lambda) = \lambda_1\lambda_3 + \lambda_2\lambda_4 + \hat{p}_2\lambda_1 + \hat{p}_3\lambda_2 - \lambda_3 y - \lambda_4 z + \frac{1}{2\lambda_4}[\arcsin(\lambda_3 + x\lambda_4) \\ + (\lambda_3 + x\lambda_4)(1 - (\lambda_3 + x\lambda_4)^2)^{\frac12} - \arcsin\lambda_3 - l_3(1 - \lambda_3^2)^{\frac12}].\end{aligned}$$

One can directly calculate the corresponding two parameter family of geodesics-rays (parameterized by s_1, s_2):

$$\begin{aligned}&x(t) = t,\\ &y(t) = s_1 - \frac{1}{\hat{p}_3}((1 - (\hat{p}_2 + t\hat{p}_3)^2)^{\frac12} - (1 - (\hat{p}_2)^2)^{\frac12}),\\ &z(t) = s_2 - \frac{1}{\hat{p}_3}t(1 - (\hat{p}_2 + t\hat{p}_3)^2)^{\frac12} + \frac{1}{2(\hat{p}_3)^2}(\arcsin(\hat{p}_2 + t\hat{p}_3)\\ &\qquad + (\hat{p}_2 + t\hat{p}_3)(1 - (\hat{p}_2 + t\hat{p}_3)^2)^{\frac12} - \arcsin\hat{p}_2 - \hat{p}_2(1 - (\hat{p}_2)^2)^{\frac12}).\end{aligned} \tag{6}$$

These rays are not parallel in the metric sense; however, their direction on the plane $\{x = 0\}$ is constant and equal to $(1, \hat{p}_2/(1 - (\hat{p}_2)^2)^{\frac12}, 0)$.

2. A bunch of rays L emanating from the origin is generated by the family $F(r, \lambda) \equiv 0$. Then $\pi^{-1}(L)$ is generated by

$$\widetilde{R}_i(q, \lambda) = -G_i(q, \lambda),$$

in the chart Ξ_i. Now we only write down the bunch of rays around the direction $\frac{\partial}{\partial x}$ (i.e. in the chart Ξ_1). It is enough to put $s_1 = s_2 = 0$ and replace $\hat{p}_2$ and $\hat{p}_3$ by λ_1 and λ_2 respectively, in formula (6). Here λ_1, λ_2 parameterize the geodesics of the bunch. This description is local, i.e. realized in the corresponding chart; however, it is obviously consistent with the well known representation obtained in [16].

One can find that the first and the second formulae of (6) describe the projection of geodesics onto the plane (x, y). They are arcs of circles with center at

$$x_0 = -\frac{p_2}{p_3}, \qquad y_0 = s_1 + \frac{1}{p_3}(1 - p_2^2)^{\frac12}$$

and radius $R = 1/p_3$, provided $p_3 \neq 0$. If $p_3 \to 0$ and say $s_1 = 0$ we obtain the lines $y = p_2x/(1 - p_2^2)^{\frac12}$ through the origin. Making use of the third formula of (6) we obtain the corresponding geodesics—liftings of the above circles and lines (i.e. for the lines the liftings are $z = p_2x^2/2(1 - p_2^2)^{\frac12}$).

EXAMPLE 2.3. An interesting horizontal curve on the Heisenberg group appearing in algebraic K-theory (see [3], Proposition 1.13) gives a well defined, single-valued map $\psi : \mathbf{P} - \{0, 1, \infty\} \to \mathbf{H}(\mathbf{Z})\ \mathbf{H}(\mathbf{C})$ providing an elegant interpretation of the dilogarithm function $\int \log(1-x)\frac{dx}{x}$. We have

$$\psi(x) = \begin{pmatrix} 1 & c\log(1-x) & c^2 \int \log(1-x)\frac{dx}{x} \\ 0 & 1 & c\log x \\ 0 & 0 & 1 \end{pmatrix}$$

and, by our identification with Euclidean space we write the horizontal curve

$$\beta(x) = \left(c\log(1-x), c\log x, c^2 \int \log(1-x)\frac{dx}{x}\right).$$

(We consider a real analog of ψ. In the complex case of [3], $c = \frac{1}{2\pi i}$.) Thus the corresponding dilog-Lagrangian variety of gliding rays is given by the following isotropic map $(t, u) \to (r, s)$:

$$\begin{aligned}
r_1 &= tu - \frac{e^{t/c}}{(2e^{2t/c} - 2e^{t/c} + 1)^{\frac{1}{2}}}, \\
r_2 &= -u, \\
s_1 &= c\log(1 - e^{t/c}) \\
&\quad + \frac{1}{u}\left\{\frac{e^{t/c} - 1}{(2e^{2t/c} - 2e^{t/c} + 1)^{\frac{1}{2}}} - \left(1 - \left(\frac{e^{t/c}}{(2e^{2t/c} - 2e^{t/c} + 1)^{\frac{1}{2}}} - tu\right)^2\right)^{\frac{1}{2}}\right\}, \\
s_2 &= c\int_0^t s\log(1 - e^{s/c})ds + \frac{t}{u}\left(\frac{e^{t/c} - 1}{(2e^{2t/c} - 2e^{t/c} + 1)^{\frac{1}{2}}}\right) \\
&\quad - \frac{1}{2u^2}\left\{\arcsin\frac{e^{t/c}}{(2e^{2t/c} - 2e^{t/c} + 1)^{\frac{1}{2}}} - \arcsin\left(\frac{e^{t/c}}{(2e^{2t/c} - 2e^{t/c} + 1)^{\frac{1}{2}}} - tu\right)\right. \\
&\quad - \left(\frac{e^{t/c}}{(2e^{2t/c} - 2e^{t/c} + 1)^{\frac{1}{2}}} - tu\right)\left(1 - \left(\frac{e^{t/c}}{(2e^{2t/c} - 2e^{t/c} + 1)^{\frac{1}{2}}} - tu\right)^2\right)^{\frac{1}{2}} \\
&\quad \left. + \frac{e^{t/c}}{(2e^{2t/c} - 2e^{t/c} + 1)}\right\}.
\end{aligned}$$

2.1. *Singular distribution.* The situation becomes much more complicated for another distribution on R^3 annihilated by the 1-form

$$\theta = dz + y^2 dx,$$

which is a stable 1-form with singularity of type $\Sigma_{2,0}$ in the sense of Martinet (cf. [11]). We consider the simplest horizontal Hamiltonian

$$H(q, p) = \frac{1}{2}(p_2^2 + (p_1 - y^2 p_3)^2),$$

and take the chart on the space of rays M, $\pi(U)$, $U = \{p_1 - y^2 p_3 > 0\} \cap \{H^{-1}(1/2)\}$. We parameterize U by (p_2, p_3, x, y, z), where $p_1 = (1 - p_2^2)^{\frac{1}{2}} + y^2 p_3$.

As before we are interested in the generating function (distance function) of the graph of the canonical map π along bicharacteristics of v_H. Consider the operator P parameterized by (x, y, r_1, r_2) and acting on smooth functions,

$$P_{(x,y,r_1,r_2)}(\cdot) = r_1 + 2xyr_2 - \int_0^x \frac{\partial}{\partial y}(1-(\cdot)^2)^{\frac{1}{2}}\,dx. \tag{7}$$

PROPOSITION 2.4. *The generating function for the above defined Lagrangian submanifold graph$\pi \in \Xi$ is given by*

$$G(x,y,z,r_1,r_2) = yr_1 + zr_2 - \int_0^x (1 - W(0,x,y,r_1,r_2)^2)^{\frac{1}{2}}\,dx + xy^2r_2,$$

where

$$W(0,x,y,r_1,r_2) = \lim_{n\to\infty}(P_{(x,y,r_1,r_2)})^n(0).$$

The method of proof gives a computational algorithm which allows us to find G by an iteration process. The successive approximations are given by

$$G_n(x,y,z,r_1,r_2) = yr_1 + zr_2 - \int_0^x (1 - (P_{(x,y,r_1,r_2)})^n(0))^2)^{\frac{1}{2}}\,dx + xy^2r_2.$$

By straightforward computation we obtain an equation for W:

$$r_1 + 2xyr_2 - \int_0^x \frac{\partial}{\partial y}(1-W^2)^{\frac{1}{2}}\,dx = W.$$

The canonical projection π in terms of W may be written as follows:

$$\begin{aligned}
-p_1 &= -(1-W^2)^{\frac{1}{2}} + y^2r_2,\\
-p_2 &= r_1 + 2xyr_2 - \int_0^x \frac{\partial}{\partial y}(1-W^2)^{\frac{1}{2}}\,dx = W,\\
-p_3 &= r_2,\\
s_1 &= y + \int_0^x \frac{\partial W/\partial r_1}{(1-W^2)^{\frac{1}{2}}}\,dx,\\
s_2 &= z - xy^2 + \int_0^x \frac{\partial W/\partial r_2}{(1-W^2)^{\frac{1}{2}}}\,dx.
\end{aligned}$$

Abnormal geodesics relate to the global singularity structure of the space of geodesics. In this context we can describe the systems of geodesics close to the singular surface $\{y = 0\}$. The equation

$$G(r_1,r_2) = \int_0^{x_0} \frac{\frac{\partial W}{\partial r_1}W}{(1-W^2)^{\frac{1}{2}}}\,dx\bigg|_{y=0} = 0$$

parameterizes locally the set of geodesics passing through 0 and $x = x_0$, $y = 0$.

3. Stable systems of rays and their caustics. Let W be a smooth hypersurface of X. To W we associate the class of systems of rays which are defined by phase functions on W. Let $\bar{F}: W \to R$ be a smooth function. By $F: X \to R$ we denote its smooth extension to X. We define

$$L_{W,F} = \{\bar{p} \in T^\star X : \langle \bar{p}, u\rangle = \langle dF, u\rangle \text{ for all } u \in TW\},$$

where $\langle \bar{p}, u\rangle$ denotes the evaluation of $\bar{p} \in T^\star X$ at a tangent vector $u \in TX$. We also write $L_{W,0} = L_W$.

Let Φ_t^H be the flow of the Hamiltonian vector field v_H with Hamiltonian H. We let

$$\exp_{(W,F)} : L_{W,F} \to X, \qquad \exp_{(W,F)} = \pi_X \circ \Phi_1^H \,|_{L_{W,F}}$$

be the corresponding exponential map. If F (a phase function on W) is a constant function then W may be treated as a wavefront. The set of critical values of $\exp_W = \exp_{(W,0)}$ is called the *focal set* of W, and this is intuitively the light caustic of the initial wavefront W in general, possibly inhomogeneous media. As in the usual Riemannian case also for the horizontal case the focal set of W is the bifurcation set of the family $d : W \times X \times R^N \to R$ (cf. Remark 1.3) given by the restriction of the extended distance function $d : X \times X \times R^N \to R$ to $W \times X \times R^N$, i.e. it consists of those points $q \in X$ with $d_q : W \times R^N \to R$ having a degenerate critical point at some $(w, \lambda) \in W \times R^N$.

We assume that the distribution V is obtained from a semi-definite bilinear form $\langle\cdot,\cdot\rangle_g$ on the cotangent bundle $T^\star X$, depending smoothly on the base point. Let $h_g : T^\star X \to TX$ be the vector bundle homomorphism, $T_q^\star X \ni p \to u \in T_q X$, where for unique fixed u we have $\langle p, \eta\rangle_g = \eta(u)$ for all $\eta \in T_q^\star X$. We assume h_g is a constant rank map and $V_q = h_g(T_q^\star X)$.

Suppose W is transversal to V. Then at the transversality points the image space $h_g(L_W)$ is normal (transversal) to the induced distribution $V \cap TW$; we call it the normal bundle to W and denote by $N_V W$. We see that in contrast to the Riemannian case, $N_V W$, which exists independently of L_W, does not define any special system of rays on M. If V is integrable, then $\Phi_1^H(C)$, where

$$C = \{\bar{p} \in T^\star X : h_g(\bar{p}) \in N_V W\},$$

is a coisotropic submanifold of $T^\star X$ (cf. [15]). If V is not integrable then only the co-normal bundle $T_W^\star X$ has a symplectic meaning.

Any local system of rays L in $M \cong T^\star N$ is generated by a Morse family $F: N \times R^k \to R$. We recall that the corresponding generating family for $\pi^{-1}(L)$ is written in the form

$$\widetilde{R}(q,\lambda) = -G(q,\lambda_1) + F(\lambda_1,\lambda_2), \tag{8}$$

where $\lambda = (\lambda_1, \lambda_2)$ are Morse parameters of the family $\widetilde{R}$.

If we fix geometry, i.e. an inhomogeneous optical medium, then the function G is given. Any Lagrangian submanifold L of M represents an optical system of rays. Its counterimage $\pi^{-1}(L)$ is an optical Lagrangian submanifold of $T^\star X$, i.e. it fulfills an eikonal equation (cf. [4]). Thus we describe the space of optical systems of rays by the general deformations of Lagrangian submanifolds in M. In this approach (see also [7]) we need a slightly modified notion of Lagrange equivalence and Lagrange stability.

Let $(\bar{L}_1, \bar{p}_1)$, $(\bar{L}_2, \bar{p}_2)$ be two germs of Lagrangian submanifolds in $(T^\star X, \omega_X)$. Following the standard lines of the theory of Lagrangian singularities [1] we say that $(\bar{L}_1, \bar{p}_1)$, $(\bar{L}_2, \bar{p}_2)$ are *equivalent* if there exists a germ of symplectomorphism $\Phi : (T^\star X, \bar{p}_1) \to (T^\star X, \bar{p}_2)$ such that $\Phi(\bar{L}_1) \subset \bar{L}_2$, and Φ preserves the π_X-fiber structure of $T^\star X$.

Let $(\widetilde{R}_1(q,\lambda), (\pi_X(\bar{p}_1), 0))$, $(\widetilde{R}_2(q,\lambda), (\pi_X(\bar{p}_2), 0))$, $(q,\lambda) \in X \times \Lambda$ be two Morse families for $(\bar{L}_1, \bar{p}_1)$ and $(\bar{L}_2, \bar{p}_2)$ respectively. If $(\bar{L}_1, \bar{p}_1)$ and $(\bar{L}_2, \bar{p}_2)$ are equivalent, then there exists a diffeomorphism $\phi : (X, \pi_X(\bar{p}_1)) \to (X, \pi_X(\bar{p}_2))$, a family of diffeomorphisms $\Theta : (X \times \Lambda, (\pi_X(\bar{p}_1), 0)) \to (\Lambda, 0)$ and a smooth function-germ $f : (X, \pi_X(\bar{p}_1)) \to R$ such that

$$\widetilde{R}_2 \circ (\phi \circ \pi_1, \Theta) = \widetilde{R}_1 + f \circ \pi_1,$$

where $\pi_1 : X \times \Lambda \to X$ is the canonical projection. This is the so-called R^+-equivalence of local unfoldings of functions [1].

Now we introduce the notion of stability of local systems of rays in (M, μ). Let $\alpha_1, \alpha_2 \in M$ and $l_i = \pi^{-1}(\alpha_i)$, $i = 1, 2$, be two corresponding rays in $(T^\star X, \omega_X)$.

Definition 3.1. Let $(L, \alpha) \subset (M, \mu)$ be a system of rays. We call it *stable* if there exists an open neighbourhood of L (in the space of Lagrange embeddings endowed with the Whitney C^∞-topology), say $\mathcal{O}_L$, and an open neighbourhood $\mathcal{U}$ of α, such that for every $L' \in \mathcal{O}_L$ and every $\bar{p} \in l = \pi^{-1}(\alpha)$ there exist $\alpha' \in \mathcal{U}$ and $\bar{p}' \in l' = \pi^{-1}(\alpha')$ such that the germs of Lagrangian submanifolds $(\pi^{-1}(L), \bar{p})$ and $(\pi^{-1}(L'), \bar{p}')$ are equivalent.

We see that if the germ $(\pi^{-1}(L), \bar{p})$ is stable (Lagrange stable [1]) for every $\bar{p} \in \pi^{-1}(\alpha)$ then (L, α) is stable in the sense of our definition. We notice that in the formulae (8) we apply only deformations depending on the parameters (λ_1, λ_2).

Let $\lambda \to F(\lambda)$ be a generating function for $(L, 0)$. Using the usual infinitesimal stability condition for local unfoldings (cf. [10]) we have

Proposition 3.2. *$(L, 0)$ is a stable system of rays if and only if*

$$\mathbf{m}_{q_0} \subset \left\langle 1, \frac{\partial G}{\partial q^i} \right\rangle_{\mathcal{E}_{q_0}} + \left\langle \frac{\partial (F - G)}{\partial \lambda_j} \right\rangle_{\mathcal{E}_{q_0,0}},$$

for all $q_0 \in l = \pi^{-1}(0)$, where $\mathcal{E}_{q_0}$ is the local algebra of germs at q_0 of smooth functions on X, and $\mathbf{m}_{q_0}$ denotes the maximal ideal of $\mathcal{E}_{q_0}$. By $\langle 1, \frac{\partial G}{\partial q^i} \rangle_{\mathcal{E}_{q_0}}$ we denote the submodule of $\mathcal{E}_{q_0,0}$ generated by $\frac{\partial G}{\partial q^i}(q,\lambda)$, $i = 1, \ldots, n+1$, and l is defined by the system of equations $\frac{\partial G}{\partial \lambda_j}(q, 0) = 0$, $j = 1, \ldots, n$.

Now we use this condition to the concrete case considered in Section 2. For G_1 we calculate

$$\frac{\partial G_1}{\partial x} = -(1-(r_1+xr_2)^2)^{\frac{1}{2}}, \quad \frac{\partial G_1}{\partial y} = r_1, \quad \frac{\partial G_1}{\partial z} = r_2,$$

and for G_2, respectively (see Corollary 2.2),

$$\frac{\partial G_2}{\partial x} = r_1, \quad \frac{\partial G_2}{\partial y} = xr_2 - (1-r_1^2)^{\frac{1}{2}}, \quad \frac{\partial G_2}{\partial z} = r_2.$$

Then by straightforward check we obtain

COROLLARY 3.3. *Let (M,μ) be the space of rays corresponding to the Heisenberg group endowed with the contact distribution. Then the stable systems of rays $(L,\alpha) \subset (T^\star N,\mu)$ are Lagrange equivalent to those having the following generating functions with simple singularities*: $F(r) = \pm r_1^2 \pm r_2^2$ (*type* A_1); $F(r) = r_1^3 \pm r_2^2$ (*type* A_2); $F(r) = \pm r_1^4 \pm r_2^2$ (*type* A_3); $F(r) = r_1^5 \pm r_2^2$ (*type* A_4); $F(r) = r_1^3 \pm r_2^3$ (*type* D_4).

The similar result is true for a larger class of optical systems, see e.g. [5].

Remark 3.4. Let F be the generating function for a system of rays (L,α). Let $(\bar{L},\bar{p}_0)$ be the corresponding Lagrangian submanifold germ in $(T^\star X,\omega_X)$. We assume that the set of critical points Σ of the Lagrange projection $\pi_X|_{\bar{L}} : \bar{L} \to X$ is described by the parameterization $r \to \sigma(r)$. This is a weak assumption since $\mathrm{rank}(\frac{\partial^2(F-G)}{\partial r}q)(q_0,0) = n$, where $q_0 = \pi_X(\bar{p}_0)$. By means of the equations

$$\frac{\partial(F-G)}{\partial r}(\gamma_r(s),r) \equiv 0,$$

we define the family of rays $r \to \gamma_r(s)$ passing through Σ, i.e. $\gamma_r(0) = \sigma(r)$. Then we deduce that the mapping

$$(r,s) \to (p,q) = \Big(-\frac{\partial G}{\partial q}(\gamma_r(s),r), \gamma_r(s)\Big)$$

has a maximal rank at $(0,0)$. Thus we showed that at any point of Σ the corresponding ray passing through that point on $\bar{L}$ is transversal to Σ (cf. the proof of Theorem 1 in [4]). This fact helps us to prove the existence on Σ of a smooth field of directions. So for smooth Σ the Euler characteristic $\chi(\Sigma)$ has to be zero. This observation was used in [4] to show that the optical realization of the "flying saucers" as optical caustics is not possible. In a different way this fact may be obtained by considering the stable local families of systems of rays (L_t,α) whose deformations preserve the restricted form of generating families

$$-G(q,r) + F_t(r),$$

(t a parameter), excluding the appearance of global structure caustics.

Remark 3.5. In general, systems of rays $(L,\alpha) \subset (M,\mu)$ are generated by Morse families, say $(r,\lambda) \to F(r,\lambda)$. Looking at the stability criterion for F we easily see that F must be a trivial Morse family, i.e. only systems of rays L which are transversal to the cotangent bundle structure $T^\star N$ are stable in M.

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