

TOPOLOGIES DEFINED BY SOME INVARIANT PSEUDODISTANCES

THEODORE J. BARTH

Department of Mathematics, University of California

Riverside, California 92521, U.S.A.

E-mail: tjb@math.ucr.edu

Complex analysis deals with holomorphic mappings between complex spaces. We ought to allow complex spaces to be infinite-dimensional and to have singularities. There are well-developed theories in the finite-dimensional case (see e.g. [GR], [Lo]) and for domains in certain kinds of topological vector spaces (see e.g. [Di], [FV], [He]). The most obvious approach to infinite-dimensional complex spaces is too general to yield sensible results [Do], but there has been progress [Au] in the study of (possibly infinite-dimensional, possibly singular) semi-Fredholm-analytic spaces. In any case, I shall assume that a complex space carries the structure of a Hausdorff space, which I shall call its *underlying topology*; in the finite-dimensional case I shall assume that the underlying topology is paracompact. This note discusses some foundational questions and summarizes what we know about the relationships between the underlying topology and the topologies induced by the classical pseudodistances of Carathéodory and Kobayashi.

Let $\text{Hol}(X, Y)$ denote the set of all holomorphic mappings from the complex space X into the complex space Y ; I shall assume that these mappings are continuous with respect to the underlying topologies of X and Y . Let \mathbb{D} denote the open unit disc in \mathbb{C} . A mapping $f \in \text{Hol}(\mathbb{D}, X)$ is called an *analytic disc* in X . I shall also assume that a complex space has the property that any two points can be connected by (the images of) a finite chain of analytic discs in the space; in the finite-dimensional case this is equivalent to connectedness in the topological sense [K70, pp. 97–98], [La, pp. 15–16].

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1. The Carathéodory pseudodistance. The *Carathéodory pseudodistance* [C, pp. 78–79] between the points p and q of the complex space X is

$$c_X(p, q) = \sup\{\tanh^{-1}(f(q)) : f \in \text{Hol}(X, \mathbb{D}), f(p) = 0, f(q) \geq 0\}.$$

Obviously $0 \leq c_X(p, q) \leq \infty$. Moreover $c_{\mathbb{D}} = \rho$, the Poincaré distance. When a Vitali-Montel Theorem holds for X , one can use a “normal families” argument to show that the supremum is attained, hence that $c_X(p, q)$ is finite. In the finite-dimensional singular case, a suitable Vitali Theorem was proved independently by Gunning [G] and Andreotti and Stoll [AnS, Proposition 4, pp. 326–328]; a Montel Theorem for domains in locally convex spaces was given in [DiTV, Lemme 1.2, pp. 516–517]. Alternatively, for any analytic disc $f : \mathbb{D} \rightarrow X$ it follows from the Schwarz Lemma that $\rho(0, a) \geq c_X(f(0), f(a))$; connecting p and q by a chain of analytic discs shows that $c_X(p, q)$ is finite.

The first topological chore is to check whether $c_X : X \times X \rightarrow [0, \infty)$ is continuous (with respect to the underlying topology). In view of the triangle inequality, this amounts to the continuity of $c_X(p, \cdot)$ at each $p \in X$. The argument given by Carathéodory [C, Satz 6, pp. 85–86] for bounded domains in \mathbb{C}^2 works in great generality as long as X is nonsingular at p : take a balanced convex domain B with center p with respect to a chart of X ; the inequality $c_B(p, q) \geq c_X(p, q)$ and an explicit computation of $c_B(p, q)$ show that $c_X(p, q) \rightarrow 0$ as $q \rightarrow p$. Alternatively, when a Vitali-Montel Theorem holds, one can give a normal families argument. Recently V. Aurich [Au, Lemma 5.4, p. 240] has shown that c_X is continuous when X is a (possibly infinite-dimensional, possibly singular) semi-Fredholm-analytic space.

In an abstract sense the following completely settles all questions about the topology induced by c_X .

PROPOSITION 1.1. *Let X be a complex space. Let $H^\infty(X)$ denote the space of bounded (complex-valued) holomorphic functions on X equipped with the supremum norm $\|\cdot\|$, and let $H^\infty(X)^*$ denote the space of bounded linear functionals on $H^\infty(X)$ equipped with the dual norm $\|\cdot\|^*$. Let $\tau : X \rightarrow H^\infty(X)^*$ be the evaluation map defined by $\tau(p)(f) = f(p)$ for $p \in X, f \in H^\infty(X)$. Then*

- (a) $c_X(p, q) = 2 \tanh^{-1}(\frac{1}{2}\|\tau(p) - \tau(q)\|^*)$ for all $p, q \in X$;
- (b) c_X is a distance if and only if τ is one-to-one;
- (c) c_X is continuous if and only if τ is continuous;
- (d) the topology induced by c_X is $\{\tau^{-1}(U) : U \text{ is open in } H^\infty(X)^*\}$;
- (e) c_X induces the underlying topology if and only if τ is an embedding.

Obviously (b) through (e) follow from (a), though it is easy to check them independently of it. The formula in (a) was stated in [EaH, Lemma 4, p. 65]; its proof relies on a neat argument given by J. Lewittes [Le, p. 1090]:

$$\begin{aligned} \|\tau(p) - \tau(q)\|^* &= \sup\{|f(p) - f(q)| : f \in H^\infty(X), \|f\| \leq 1\} \\ &= \sup\{|f(p) - f(q)| : f \in \text{Hol}(X, \mathbb{D})\} \\ &= \sup\{|f(p) - f(q)| : f \in \text{Hol}(X, \mathbb{D}), f(q) = -f(p) \geq 0\} \end{aligned}$$

$$\begin{aligned}
&= 2 \sup\{f(q) : f \in \text{Hol}(X, \mathbb{D}), f(q) = -f(p) \geq 0\} \\
&= 2 \tanh(\sup\{\rho(0, f(q)) : f \in \text{Hol}(X, \mathbb{D}), f(q) = -f(p) \geq 0\}) \\
&= 2 \tanh(\tfrac{1}{2} \sup\{\rho(f(p), f(q)) : f \in \text{Hol}(X, \mathbb{D}), f(q) = -f(p) \geq 0\}) \\
&= 2 \tanh(\tfrac{1}{2} c_X(p, q)).
\end{aligned}$$

The key point here is the third equality, which says that $|f(p) - f(q)|$ will be maximized by choosing $f(p)$ and $f(q)$ symmetrically about 0.

In connection with Proposition 1.1(e) there are some concrete positive results.

PROPOSITION 1.2. *Each of the following is a sufficient condition for c_X to induce the underlying topology:*

- (a) X is a bounded domain in a Banach space;
- (b) X is a domain in \mathbb{C} and c_X is a distance;
- (c) X is a relatively compact domain in a Stein space;
- (d) X is finite-dimensional, c_X is a distance, and the closed Carathéodory ball $\{q \in X : c_X(p, q) \leq r\}$ is compact (with respect to the underlying topology) for every $p \in X$ and every $r > 0$.

The proof of (a) is easy: if U is a neighborhood of a point p in X , we take a ball B centered at p and containing X , compute $c_B(p, q)$ to find a radius δ for which $\{q \in X : c_B(p, q) < \delta\} \subset U$, and note that $c_X(p, q) \geq c_B(p, q)$. A slick elementary proof of (b) can be found in [JP91, Proposition 1.3, p. 170]. Proofs of (c) and (d) were given by N. Sibony [Sib, pp. 222–223].

Recently Mikihiro Hayashi [Ha] constructed a Riemann surface (a one-dimensional complex manifold) X for which the mapping τ of Proposition 1.1 is one-to-one but not an embedding. In view of parts (b) and (e) of Proposition 1.1, Jean-Pierre Vigué observed that this Riemann surface provides an example of a complex space for which the Carathéodory pseudodistance is a distance but does not induce the underlying topology. Marek Jarnicki, Peter Pflug, and Vigué [JPV91] applied some general theorems to establish the existence of a domain in \mathbb{C}^3 with the same property. Indeed, a Remmert–Narasimhan–Bishop embedding realizes Hayashi’s example as closed complex submanifold of \mathbb{C}^3 ; using Siu’s theorem [Siu], it is easy to construct such a domain containing this submanifold. The experts [EiG, p. 128] seem to believe that every open Riemann surface will embed in \mathbb{C}^2 ; once this has been proved there will be a corresponding example of a domain in \mathbb{C}^2 .

Even when a Carathéodory distance induces the underlying topology, recent examples of Riemann surfaces and bounded pseudoconvex domains [JPV92] show that the closure of an open Carathéodory ball may be a proper subset of the corresponding closed Carathéodory ball.

2. Inner pseudodistances. Kobayashi [K73, p. 484] suggested the idea of applying the metric space theory of inner distances, as codified by Rinow [Ri], to the Carathéodory distance. This theory, which goes back at least to the work of

Menger [M, p. 492], measures distance by the infimum of the lengths of (continuous) curves joining two given points. In view of the Hayashi-Jarnicki-Pflug-Vigué examples, this construction may use curves that do not conform to our intuition of a “curve” in a complex space.

In hopes of clarifying the nature of the difficulties, I shall now introduce some terminology. A *path* in the set X is a mapping $\gamma : [a, b] \rightarrow X$, where a and b are real numbers with $a \leq b$; the values $\gamma(a)$ and $\gamma(b)$ are called the *endpoints* of γ . Let me emphasize that X is only a set and there is no continuity assumption. A *network* on X is a collection \mathfrak{N} of paths in X satisfying:

- whenever p and q are points of X there is a path in \mathfrak{N} with endpoints p, q ;
- if $\gamma : [a, b] \rightarrow X$ and $a \leq t \leq b$, then $\gamma \in \mathfrak{N}$ if and only if $\gamma|_{[a,t]} \in \mathfrak{N}$ and $\gamma|_{[t,b]} \in \mathfrak{N}$;
- if $\gamma \in \mathfrak{N}$ and $A : \mathbb{R} \rightarrow \mathbb{R}$ is a nonconstant affine function, then $\gamma \circ A \in \mathfrak{N}$.

Intuitively, a network is closed under path product, reversal, and restriction. Finally, a *gigadistance* on the set X is a function $D : X \times X \rightarrow [0, \infty]$ satisfying the usual axioms for a pseudodistance:

- (1) $D(p, p) = 0$ for all $p \in X$,
- (2) $D(p, q) = D(q, p)$ for all $p, q \in X$,
- (3) $D(p, q) + D(q, r) \geq D(p, r)$ for all $p, q, r \in X$.

Once we accept these definitions, it is reasonable to define the *length* of the path $\gamma : [a, b] \rightarrow X$ relative to the gigadistance D to be

$$L(\gamma, D) = \sup \left\{ \sum_{j=1}^k D(\gamma(t_{j-1}), \gamma(t_j)) : k \geq 1, a = t_0 \leq t_1 \leq \cdots \leq t_k = b \right\}.$$

Of course $0 \leq L(\gamma, D) \leq \infty$. Setting

$$D^{\mathfrak{N}}(p, q) = \inf \{ L(\gamma, D) : \gamma \in \mathfrak{N} \text{ and } \gamma \text{ has endpoints } p, q \},$$

gives another gigadistance $D^{\mathfrak{N}} \geq D$; it may happen that $D^{\mathfrak{N}}$ is infinite even when D is a (finite) distance and the paths in \mathfrak{N} are continuous with respect to D . We say that the gigadistance D is *inner* for \mathfrak{N} if $D^{\mathfrak{N}} = D$. If D is an arbitrary gigadistance, it is easy to see that $L(\gamma, D^{\mathfrak{N}}) = L(\gamma, D)$ for every $\gamma \in \mathfrak{N}$, so $D^{\mathfrak{N}}$ is inner for \mathfrak{N} . The book [Ri] considers the case where (X, D) is a pathwise connected metric space and \mathfrak{N} consists of all paths that are continuous with respect to D . To conclude the general discussion, here is a simple topological fact.

LEMMA 2.1. *Let D be a continuous distance on the locally compact Hausdorff space X . Let \mathfrak{N} be a network on X consisting of mappings that are continuous with respect to the given topology on X . If D is inner for \mathfrak{N} , then D induces the given topology.*

Now let X be a complex space such that c_X is a continuous pseudodistance (a condition essentially always satisfied—see §1). There are at least four natural networks on X :

- the network \mathfrak{C} of all paths continuous with respect to c_X ,
- the network \mathfrak{U} of all paths continuous with respect to the underlying topology,
- the network \mathfrak{V} of admissible curves of Venturini [Ve, p. 387],
- the network \mathfrak{D} of all piecewise continuously differentiable paths.

All have been used to construct “inner” Carathéodory (pseudo)distances:

- $c_X^{\mathfrak{C}}$, denoted by c'_X in [K73, p. 484];
- $c_X^{\mathfrak{U}}$, denoted by c^i_X in [K76, p. 364], [B77, p. 49], [PS, p. 67];
- $c_X^{\mathfrak{V}}$, denoted by c^i_X in [Ve, p. 390];
- $c_X^{\mathfrak{D}}$, denoted by \tilde{C}_X in [Di, p. 53].

(I have guessed some authors’ intent.) Clearly $\mathfrak{C} \supset \mathfrak{U} \supset \mathfrak{V} \supset \mathfrak{D}$, so $c_X \leq c_X^{\mathfrak{C}} \leq c_X^{\mathfrak{U}} \leq c_X^{\mathfrak{V}} \leq c_X^{\mathfrak{D}}$. It is quite usual to have $c_X \neq c_X^{\mathfrak{C}}$ (see e.g. [B77], [Vi83], [JP90]). Jean-Pierre Vigué has noted that the 2-dimensional complex space constructed in [Vi84] satisfies $c_X^{\mathfrak{C}} \neq c_X^{\mathfrak{U}}$. If I interpret the author(s) correctly, [K76, Theorem 2.6(2), p. 364] (resp. [PS, p. 67]) states that $c_X^{\mathfrak{U}} = c_X^{\mathfrak{D}}$ whenever X is a finite-dimensional complex space (resp. manifold); Jarnicki and Pflug [JP91, p. 170] note that this is true when X is a bounded domain in \mathbb{C}^n , but they point out the lack of a proof in the general case. While it may be counterintuitive for a ragged curve to be short, it is conceivable that a trip along such a curve may encounter little resistance from c_X .

The arguments in §1 show that $c_X^{\mathfrak{D}}(p, q)$ is finite and also that $c_X^{\mathfrak{D}}(p, \cdot)$ is continuous at each nonsingular point p . In the finite-dimensional singular case continuity follows from comparison with the Kobayashi distance (see §3) or by noting that $c_X^{\mathfrak{D}}$ coincides with the integrated form of the Carathéodory-Reiffen metric [Re, Satz 2, p. 321].

THEOREM 2.2. *Each of the following is a sufficient condition for $c_X^{\mathfrak{U}}$ (resp. $c_X^{\mathfrak{D}}$) to induce the underlying topology:*

- X is a bounded domain in a Banach space;
- X is finite-dimensional and $c_X^{\mathfrak{U}}$ (resp. $c_X^{\mathfrak{D}}$) is a distance.

The proof of (a) is the same as for Proposition 1.2(a). Of course (b) follows from Lemma 2.1; the only difficult part is continuity at the singular points.

In view of Theorem 2.2, it is natural to look at unbounded domains in the Hilbert space ℓ^2 of sequences of complex numbers $x = (x_\nu)_{\nu=1}^\infty$ satisfying $\|x\| = \sqrt{\sum |x_\nu|^2} < \infty$. Following [FV, pp. 93–94], set

$$X = \{x \in \ell^2 : \sup_\nu |x_\nu|/\nu < 1\}.$$

Then $c_X^{\mathfrak{U}}$ and $c_X^{\mathfrak{D}}$ are distances, $x^{(n)} = (\delta_\nu^{n+1})_{\nu=1}^\infty \in X$, and

$$c_X^{\mathfrak{U}}(0, x^{(n)}) = c_X^{\mathfrak{D}}(0, x^{(n)}) = \tanh^{-1}\left(\frac{1}{n+1}\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

But $\|x^{(n)}\| = 1$, so these distances cannot induce the underlying (norm) topology.

3. The Kobayashi pseudodistance. The definition of the Kobayashi pseudodistance [K67, p. 462], as explained by Royden [Ro], involves dualizing the Carathéodory construction to obtain an object that may not be a pseudodistance, and then applying a general technique.

At the risk of overdoing the jargon, let us say that a *paradistance* on the set X is a function $D : X \times X \rightarrow [0, \infty]$ satisfying:

- (1) $D(p, p) = 0$ for all $p \in X$,
- (2) $D(p, q) = D(q, p)$ for all $p, q \in X$.

LEMMA 3.1. *Let D be a paradistance on the set X . For $p, q \in X$, $k = 1, 2, 3, \dots$, set*

$$D^{(k)}(p, q) = \inf \left\{ \sum_{j=1}^k D(p_{j-1}, p_j) : p_0 = p, p_k = q \right\}.$$

Then

- (a) $D^{(k)}$ is a paradistance on X ;
- (b) $D = D^{(1)} \geq D^{(2)} \geq D^{(3)} \geq \dots$;
- (c) $D^{(\infty)} = \inf_k D^{(k)} = \lim_{k \rightarrow \infty} D^{(k)}$ is a *gigadistance* on X .

In other words, $D^{(\infty)}$ is the largest gigadistance on X bounded above by D .

Now let X be a complex space. The *one-disc Kobayashi paradistance* between the points p and q of X is

$$\delta_X(p, q) = \inf \{ \tanh^{-1}(a) : 0 \leq a < 1, f \in \text{Hol}(\mathbb{D}, X), f(0) = p, f(a) = q \}.$$

It may happen that the set in this formula is empty, in which case the infimum is defined to be ∞ . But if X is an n -dimensional complex manifold, a theorem of Forneaess and Stout [FS77] (resp. [FS82]) asserts the existence of a holomorphic mapping of \mathbb{D}^n (resp. the n -dimensional ball) onto X , whence δ_X is finite. This is also true if X is a domain in a Banach space [Di, pp. 49–50]. The infimum is not usually attained, but it will be if X is *taut* [W, Definition 1.2, p. 199], [DiT, Definition 6.2, p. 359]. The *Kobayashi pseudodistance* on X is

$$d_X = \delta_X^{(\infty)}.$$

If $f \in \text{Hol}(\mathbb{D}, X)$ and $0 \leq a < 1$ we have

$$d_X(f(0), f(a)) \leq \delta_X(f(0), f(a)) \leq \tanh^{-1}(a) = \rho(0, a).$$

Using a chain of discs one sees that d_X is finite. Moreover the geodesic properties of ρ give $L(f|_{[0,a]}, d_X) \leq \rho(0, a)$. It follows that $d_X^{\mathfrak{A}} \leq \delta_X$, where \mathfrak{A} is the smallest network on X containing $\{f|_{[0,a]} : 0 \leq a < 1, f \in \text{Hol}(\mathbb{D}, X)\}$. Thus $d_X = d_X^{\mathfrak{U}} = d_X^{\mathfrak{V}} = d_X^{\mathfrak{D}} = d_X^{\mathfrak{A}}$.

The argument in §1 shows that $d_X(p, \cdot)$ is continuous at each nonsingular point p . The only simple proof I know for continuity in the finite-dimensional singular case uses a Hironaka resolution of singularities [Hi], [HiR]; see [B72], [La, Proposition 1.7, pp. 17–18]. Possibly the new metric of Kobayashi [K90] could be helpful in this regard.

Let \mathfrak{K} be the network of all paths continuous with respect to d_X . When d_X is continuous, we have $\mathfrak{K} \supset \mathfrak{U}$, so $d_X = d_X^{\mathfrak{K}}$, i.e. d_X is inner for \mathfrak{K} [K73, pp. 483–484].

THEOREM 3.2. *Each of the following is a sufficient condition for d_X to induce the underlying topology:*

- (a) X is a bounded domain in a Banach space;
- (b) X is finite-dimensional and d_X is a distance.

The proof is identical to that of Theorem 2.2. Moreover, for the unbounded domain X in ℓ^2 constructed at the end of §2, d_X is a distance that does not induce the underlying topology [FV, pp. 93–94].

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