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ON THE CHARACTERISTIC PROPERTIES OF CERTAIN OPTIMIZATION PROBLEMS IN COMPLEX ANALYSIS

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Abstract. We shall be concerned in this paper with an optimization problem of the form: $J(f) \to \min(\max)$ subject to $f \in \mathcal{F}$ where $\mathcal{F}$ is some family of complex functions that are analytic in the unit disc. For this problem, the question about its characteristic properties is considered. The possibilities of applications of the results of general optimization theory to such a problem are also examined.

1. Introduction. Denote by $\mathbb{C}$, K, A, respectively, the complex plane, the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and the set of all complex functions which are analytic in the disc K. Let further $\mathcal{F} \subset A$ be some family of complex functions that are analytic in K.

We are going to consider the following extremal problem:

$$\begin{cases} J(f) \to \min(\max) \\ f \in \mathcal{F} \end{cases} \tag{1}$$

where $J(\cdot)$ is a real-valued functional.

It seems interesting to ask the question about the characteristic properties of problem (1) in the light of other optimization problems and about the possibility of applications of the results of the general theory of optimization to this problem.

The variety of problems of the form (1) is large. In view of the question formulated above, one can distinguish three factors which prejudge the properties of the problems under consideration as well as the possibilities of applications of the general theory of optimization to them. The factors are:

- the topological properties of the space A,
- the way the family $\mathcal{F}$ is defined,

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- the regularity of the functional J.

We shall discuss successively the importance of the above factors.

2. On the influence of the space A. The space A of complex functions analytic in the disc K bears a specific character from the point of view of topology and functional analysis. This influences both positively and negatively the character of problem (1) in comparison with other optimization problems.

Since K is open, the natural topology of A is the topology of uniform convergence on compact subsets of K. By the results of general topology and functional analysis (see e.g. [5], [7]), A is a locally convex linear-topological space with the topology determined by a complete and translation-invariant metric. This means that A is a Fréchet space. It turns out that A is not locally bounded and thus it is *not normable*. Hence, although the space of analytic functions is metrizable, it is not normable.

It is very interesting that the space A possesses the Heine-Borel property. This is rather unusual since the Heine-Borel property characterizes finite-dimensional normed spaces. Worth stressing are the "analyticity" features prejudging the above properties. They are as follows:

- "almost uniform convergence preserves analyticity" (the Weierstrass theorem),
- "the almost uniform boundedness of the family $\mathcal{F}$ is equivalent, in the topology considered, to the local compactness of this family" (the Montel theorem),
- "the modulus of an analytic function does not attain its maximum inside the circle of convergence" (the maximum principle).

It follows from the Heine-Borel property that, for problems of the form (1), it is relatively easy to obtain existence theorems because it is easy to get the closedness and the almost uniform boundedness (i.e. the boundedness in the topology under consideration) of the family $\mathcal{F}$ as well as the continuity of the functional J. The relative ease in the obtaining of theorems on the existence of an optimal solution may be regarded as a positive influence of the space A upon the character of problem (1). The lack of the norming of this space has a negative influence and restricts, in the essential way, the possibility of applying the results of the general theory of optimization to problems of the form (1) since it does not allow us to apply the results of smooth optimization. The lack of a norm in the space A does not allow us to speak of the strong Fréchet differentiability of functionals defined on the family $\mathcal{F}$. We may only use Gateaux-differentiable functionals as, for instance, in [2] and the directional derivatives of the functional J. When solving problems of the form (1), we may also make use of the results concerning differentiability in the generalized sense as in [1]. For more details concerning the regularity of the functional J, see section 4 of this paper.

3. On the influence of the family $\mathcal{F}$. The character of problem (1) strongly depends on the way the family $\mathcal{F}$ is defined. We shall comment on two opposite cases: the easiest and the most difficult ones.

It is easiest to apply the general theory in those situations where the membership in $\mathcal{F}$ is expressed in an analytic way, that is, when the family of functions possesses the so-called structural representation. Such a structural formula establishes the correspondence (in the case of certain families, it is one-to-one) between functions of $\mathcal{F}$ and probability measures on the boundary of K. Many structural representations were obtained. We mention one of such general representations obtained in 1971 by L. Brickman, T. H. Mac Gregor and D. R. Wiken [4].

Let us denote by X any compact Hausdorff space and consider the mapping $q : K \times X \to \mathbb{C}$ with the following properties:

(i) for each $t \in X$, the mapping $z \to q(z,t)$ is analytic in K;

(ii) for each $z \in K$, the mapping $t \to q(z,t)$ is continuous on X;

(iii) for each r, $0 < r < 1$, there exists a number $M_r > 0$ such that $|q(z,t)| \le M_r$ for $|z| \le r$ and for $t \in X$.

Denote by $\mathcal{P}$ the set of probability measures defined on Borel subsets of the space X. For $\mu \in \mathcal{P}$, let

$$f_\mu(z) = \int_X q(z,t)\,d\mu(t), \qquad z \in K. \tag{2}$$

The family $\mathcal{F}$ defined with the use of structural formula (2) is now determined as

$$\mathcal{F} = \{f_\mu : \mu \in \mathcal{P}\}. \tag{3}$$

The best-known example of a representation of the form (2) is the Herglotz formula (see e.g. [11])

$$p(z) = \int_{|x|=1} \frac{1+xz}{1-xz}\,d\mu(x), \qquad |z| < 1, \tag{4}$$

for $p(z)$ analytic in the disc K and satisfying $p(0) = 1$, re $p(z) > 0$, $z \in K$ ($\mu(x)$ is the probability measure on the boundary of K, corresponding to the function $p(z)$). In this case, the mapping $\mu \to f_\mu$ is one-to-one (cf. [11], p. 40). Many interesting families of analytic functions possess a representation of the form (2), where $X = [a,b]$, $a < b$, while $\mathcal{P} = \mathcal{P}(a,b)$ is the set of measures supported on the segment $[a,b]$, such that

$$\int_a^b d\mu(t) = 1.$$

Applying general optimization theory to problems concerning the family (3), one uses the following properties of this family ([4]):

(a) each function of $\mathcal{F}$ is analytic in K,

(b) the mapping $\mu \to f_\mu$ is continuous (under the weak-*-topology induced from $C(X)^*$ and under the topology of almost uniform convergence on $\mathcal{F}$),

(c) the set $\mathcal{F}$ is compact and it is a closed convex hull of the set of functions $\{z \to q(z,t) : t \in X\}$,

(d) the functions $z \to q(z,t)$, $t \in X$, are the only possible vertical points of $\mathcal{F}$. If $t_0 \in X$ and the condition

$$q(z,t_0) = \int_X q(z,t)\,d\mu(t), \qquad z \in K,$$

holds only for $\mu = \delta_{t_0}$, where δ_{t_0} denotes the Dirac measure at t_0, then the function $z \to q(z,t_0)$ is a vertical point of the set $\mathcal{F}$. In particular, if $\mu \to f_\mu$ is one-to-one, each function $z \to q(z,t)$, $t \in X$, is some vertical point of $\mathcal{F}$.

Using the above properties when applying some general optimization method, say the extremum principle of [6], to problem (1), one comes to the inequality of the form

$$\int_X w(t)\,d\mu(t) \le \int_X w(t)\,d\mu_*(t) \tag{5}$$

valid for any measure $\mu \in \mathcal{P}$ and some measure $\mu_* \in \mathcal{P}$. L. Mikołajczyk and S. Walczak ([9], th. 2, p. 150) proved that inequality (5) characterizes completely the *extremal measure* μ_*, namely, μ_* must be supported on the set E where

$$E = \{\tau \in X : w(\tau) = \max_{t\in X} w(t)\}.$$

The function f which is a solution of problem (1) is the one that corresponds to the measure μ_* via formula (2). In most extremal problems considered in the theory of complex functions, the set E contains only a finite number of points. This enables us to obtain a form of the optimal solution f_*.

The reasoning presented above seems the simplest and most efficient way of applying the results of general optimization theory to extremal problems regarding families $\mathcal{F}$ of the form (3), see, for instance, [1], [3], [9].

Let us now turn to the case of families $\mathcal{F}$ of univalent functions. This is the most difficult case from our point of view. Univalence is a property strongly connected with the domain of definition of a function; the larger the domain is, the fewer univalent functions on it. Locally, any analytic function $f(z)$ for which $f'(z_0) \ne 0$ is univalent, whereas in the whole plane only functions of the form $az+b$, $a \ne 0$, are univalent. Univalence strongly influences the behaviour of a function on the boundary of the disc K. The influence of this property upon the coefficients of an analytic function is also very essential. It should also be emphasized that one does not know the analytic conditions, equivalent to univalence, which could be included in general optimization considerations. The basic algebraic operations, such as the sum, the convex combination, do not preserve univalence. The structure of the majority of families of univalent functions is therefore *highly nonlinear*. In other words, linear disturbances are not admissible in families of univalent functions.

These properties of families of univalent functions make it impossible or very difficult to apply the general optimization methods to problems of form (1).

Let us mention that, in the case of families of univalent functions, the so-called *variational methods* are widely applied (see, for instance, [8]), consisting in the construction of a sufficiently general *disturbance* which preserves univalence and in obtaining some information about the extremal function by the comparison with functions that are "close" to it. However, the variational methods fall out of the general theory of optimization and are treated as special extremal methods.

We can summarize our remarks concerning families of univalent functions by the statement: "The question of how to apply optimization theory to extremal problems in univalent families still remains open".

4. On the influence of the functional J**.** As we have already mentioned in Section 2, because of the lack of a norm in the space A, we cannot speak of the Fréchet differentiability of the functional J. What we may only use is a sufficiently regular directional derivative $J'(f_0; f_1)$ of the functional $J : A \to \mathbb{R}$, defined as the limit

$$J'(f_0; f_1) = \lim_{\lambda \downarrow 0} \frac{J(f_0 + \lambda f_1) - J(f_0)}{\lambda}.$$

If the function $J'(f_0; \cdot) : A \to \mathbb{R}$ is linear, i.e. if the functional J is Gateaux-differentiable, then one may apply certain general optimization methods, for instance, the extremum principle or the Dubovitskiĭ–Milyutin method (cf. e.g. [10], [2], [3]). From the point of view of applications, the class of Gateaux-differentiable functionals is too narrow, for instance, the functional

$$J(f) = \max[\operatorname{re} a_1(f), \operatorname{re} a_2(f)], \tag{6}$$

where $f(z) = a_0(f) + a_1(f)z + a_2(f)z^2 + \cdots$, $z \in K$, is not differentiable in the sense of Gateaux.

In our opinion, the most appropriate class of functionals to be considered in problem (1) is the essentially wider class of *regularly locally convex* functions, introduced in [6]. We recall the corresponding notions.

1) A functional $J : A \to \mathbb{R}$ is said to be *uniformly differentiable* at the point f_0 in the direction f_1 if, for any $\varepsilon > 0$, there are a neighbourhood U of f_1 and a number $\lambda_0 > 0$, such that

$$\left| \frac{J(f_0 + \lambda f) - J(f_0)}{\lambda} - J'(f_0; f_1) \right| < \varepsilon$$

for any $f \in U$ and $0 < \lambda < \lambda_0$.

2) A functional $J : A \to \mathbb{R}$ is said to be *regularly locally convex* at the point f_0 if it is uniformly differentiable at the point f_0 in any direction and the function $J'(f_0; \cdot) : A \to \mathbb{R}$ is convex.

3) If a functional $G : A \to \mathbb{R}$ is convex, then the set

$$\partial G(f_0) = \{x^* \in (A)^* : G(f) - G(f_0) \geq \langle x^*, f - f_0\rangle \ \forall f \in A\}$$

is called the *subdifferential* of G at f_0, whereas the subdifferential of a functional $J : A \to \mathbb{R}$ regularly locally convex at the point f_0 is defined as follows:

$$\partial J(f_0) = \partial G(f_0) \qquad \text{where} \quad G(f) = J'(f_0; f).$$

A regularly locally convex functional at the point f_0 has a nonempty subdifferential at this point (see [6], 4.2, Proposition 3).

The local behaviour of a regularly locally convex functional is completely determined by its directional derivative (see [6] and [12]).

Paper [1] which deals with regularly locally convex functionals constitues an example of an application of the methods of nonsmooth optimization to problem (1) since it concerns functionals that are merely subdifferentiable. The necessary optimality condition obtained in [1] (Th. 4.1.) contains subdifferentials of the corresponding functionals, hence the possible applications are limited by subdifferential calculus. It was possible to calculate subdifferentials in the following problem solved in [1]:

$$\begin{cases} J(f) := \max\{\operatorname{re} a_1(f), \operatorname{re} a_2(f)\} \to \min \\ f \in \mathcal{F} \end{cases} \tag{7}$$

where $\mathcal{F}$ is the family of all Carathéodory functions in K, that is, of all analytic functions f such that $f(0) = 1$ and $\operatorname{re} f(z) > 0$ for all $z \in K$: $a_1(f) = f'(0)$, $a_2(f) = \frac{1}{2} f''(0)$. Let us remark that problems like (7) are comparatively difficult to investigate by means of other extremal methods, for instance, variational ones.

In the case when the functional J is linear or convex, we may apply the methods of convex analysis, for example, the *method of extreme points* (cf. e.g. [5]). This method is based on the Krein–Milman theorem stating that a compact and convex subset of a locally convex linear-topological space is determined by the set of its extreme points. Since most families of functions considered in the theory of extremal problems are not convex, the Krein–Milman theorem is applied to the closed convex hulls of these families. Consequently, for a compact family $\mathcal{F}$, we have $H\mathcal{F} = H(E\mathcal{F})$ where $H\mathcal{F}$ denotes the closed convex hull, and $E\mathcal{F}$ the set of extreme points of the family $\mathcal{F}$. For a complex, continuous, linear functional defined on $\mathcal{F}$, we then get (cf. [5])

$$\begin{aligned} \max\{\operatorname{Re} J(f) : f \in H\mathcal{F}\} &= \max\{\operatorname{Re} J(f) : f \in \mathcal{F}\} \\ &= \max\{\operatorname{Re} J(f) : f \in EH\mathcal{F}\}. \end{aligned} \tag{8}$$

An analogous result is obtained for the functional $|J(f)|$ and for a real, continuous and convex functional J. Note that equality (8) is useful since $EH\mathcal{F} \subset \mathcal{F} \subset H\mathcal{F}$. Consequently, in order to solve the extremal problem, it suffices to solve it for the smaller class $EH\mathcal{F}$. Moreover, the solutions are good even for the wider class $H\mathcal{F}$. It turns out that, for many families under consideration, the set $EH\mathcal{F}$ is essentially smaller than $\mathcal{F}$, and the optimization problems can be solved relatively easily. Note that the success in applying this method depends on the possibility of determining the extreme points of the family $\mathcal{F}$. In the case of families determined by a structural formula, the extreme points are the functions corresponding to the Dirac measures on the boundary of the disc K. For families possessing no structure representations, the problems of determining the extreme points and their convex hulls are comparatively difficult and still open. And so, for instance, the problems of describing the sets EHS and ES are open, where S denotes the family of functions univalent in the disc K, with the expansion $f(z) = z + a_1 z^2 + \ldots$, $z \in K$. It is known that functions of the form $z/(1 - xz)^2$, $|x| = 1$ (the Koebe functions) belong to EHS, thus to ES, because $EHS \subset ES$. One also knows other functions belonging to EHS. One also obtained some necessary conditions for a function to belong to ES, but the general problem is still open. It follows from the above observations that the method of extreme points, like other general optimization methods, does not yield satisfactory results when applied to extremal problems concerning families of univalent functions.

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